Ordinary and Partial Differential Equations and Applications

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Lecture – 36
Nonlinear PDE of First Order

Hello friends! Welcome to my lecture on nonlinear partial differential equations of first order. Let us consider the problem of findings the solutions of the partial differential equation.

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Let us consider the problem of finding the solutions of the partial differential equation

$$f(x, y, z, p, q) = 0, \quad p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y} \qquad (1)$$

in which the function f is not necessarily linear in p and q.

We know that the partial differential equation of two parameter system

$$F(x, y, z, a, b) = 0, \qquad (2)$$

is of this form.

Proof: The equations

$$F_x + pF_z = 0,\ F_y + qF_z = 0$$

and $\qquad F(x, y, z, a, b) = 0$

lead to a relation of the form (1) after eliminating the constants a and b.

f(x, y, z, p, q) = 0. P is partial derivative of z with respect to x. Q is partial derivative of z with respect to y where the function f is not necessarily linear in p and q. We know that the partial differential equation of 2 parameter system f(x, y, z, a, b) = 0 is of this form. This we can easily prove if you differentiate f(x, y, z, a, b) = 0 with respect to partially with respect to x, what we get is?

Partial derivative of f with respect to x + partial derivative of f with respect to z * partial derivative of z with respect to x so we have Fx + pFz = 0 and then we differentiate F(x, y, z, a, b) = 0 with respect to y we get partial derivative of f with respect to y + partial derivative of f with respect to z * partial derivative of z with respect to y which is q. So Fy + qFz = 0. Now the 2 equations Fx + pFz= 0 and Fy + qFz = 0.

And the given relation F(x, y, z, a, b) = 0 will lead us to a relation of the form 1 okay when we eliminate the constants a and b. So generally it is possible to eliminate arbitrary constants a and b from these 3 equations to arrive at a partial differential equation of first order that is f(x, y, z, p, q) = 0.

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The converse of the result is also true that is any PDE of the type (1) has a solution of the type (2).

Now we show that any envelope of the system (2) touches at each of its points a member of the system.

Envelope: The envelope of a family of surfaces is tangent to each surface in the family along the characteristic curve in that family.

The converse of the result is also true that is any PDE of the type (1) f(x, y, z, p, q) = 0 has a solution of this type F(x, y, z, a, b) = 0. Later on in this course, you will be taught the methods by which we can find the solution of f(x, y, z, p, q) = 0 in the form 2 that is F(x, y, z, a, b) = 0 so the converse also holds 2.

Let us now show that any envelope of the system (2) touches at each of its points a member of the system. What do we mean by an envelope? let us see. The envelope of the family of surfaces is tangent to each surface in the family along the characteristics curve in that family. Now let us make it more clear.

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A ***one parameter system*** of surfaces is given by

$f(x,y,z,a)=0,$
$f(x,y,z,a+\delta a)=0$

$$f(x,y,z,a)=0 \tag{3}$$

Let us consider a member of the above system corresponding to the value $a+\delta a$.

Then
$$f(x,y,z,a+\delta a)=0 \tag{4}$$

The surfaces (3) and (4) will intersect in a curve. This curve is also the intersection of the surface given by (3) and the surface whose equation is

$$\frac{1}{\delta a}\{f(x,y,z,a)-f(x,y,z,a+\delta a)\}=0$$

As the parameter $\delta a \to 0$, this curve of intersection tends to a limiting position given by the equations

$$f(x,y,z,a)=0, \quad \frac{\partial}{\partial a}f(x,y,z,a)=0 \tag{5}$$

Suppose we have 1 parameter. Let us begin with 1 parameter system of surfaces. One parameter system of surface is given by f(x, y, z, a) = 0 where a is a parameter. Let us consider another member of this family by taking a as a + delta a. Then f(x, y, z, a + delta a) = 0. The surfaces (3) and (4) will therefore intersect in a curve. This curve is also the intersection of the surface (3) with the surface whose equation is 1/delta a * f(x, y, z, a) - f(x, y, z, a, + delta a) = 0 why?

Because the curve which intersects which is obtained by the intersection of the surfaces (3) and (4) okay we will satisfy f(x, y, z, a) = 0 will be this curve f(x, y, z, a) = 0 and f(x, y, z) okay. So the curve of intersection of the surfaces (3) and (4) will be given by these 2 equations and when these 2 equations hold this equation 1/delta a f(x, y, z, a) - f(x, y, z, a + delta a) will also hold okay.

So, the curve of intersection of the surfaces (3) and (4) will also be the curve of intersection of the surface 3 and the surface given by this equation. Now as the parameter delta a tends to 0, this curve of intersection tends to a limiting position which is given by the partial derivative of a f(x, y, z, a) here f(x, y, z, a) = 0. So when your delta a goes to 0 f(x, y, z, a) - f(x, y, z, a + delta a)/delta a will tend to the partial derivative of f(x, y, z, a) with respect to a, okay.

So the curve of intersection tends to a limiting position which is given by this equation 3 that is f(x, y, z, a) = 0 and the equation partial derivative of f(x, y, z, a) with respect to a = 0.

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This limiting curve is called the characteristic curve of the system on the surface (3) or the characteristic curve of (3). Geometrically, it is the curve on the surface of (3) approached by the intersection of (3) and (4) as $\delta a \to 0$.
As the parameter a varies, the characteristic curve (5) traces out a surface whose equation $g(x, y, z) = 0$, is obtained by eliminating a between the equations (5). This surface is known as the envelope of the one parameter system.
For example, consider

$$x^2 + y^2 + (z-a)^2 = 1 \quad (6)$$

The characteristic curve to the surface (6) is the circle

$$x^2 + y^2 + (z-a)^2 = 1, \ z = a.$$

The envelope of this family is the cylinder $x^2 + y^2 = 1$. (7)

Now this limiting curve is called the characteristic curve of the system on the surface (3) or the characteristic curve of (3). Geometrically, it is the curve on the surface of (3) which is approached by the intersection of (3) and (4) as delta a goes to 0. As the parameter a varies, the characteristic curve (5) this one given by these 2 equations traces out a surface whose equation g(x, y, z) = 0 is obtained by eliminating a between the equations (5).

So we eliminate the parameter f from the 2 equations f(x, y, z, a) = 0 and partial derivative of f(x, y, z, a) with respect to a = 0. This surface is known as the envelope of the 1 parameter system. For example, let us consider x square + y square + (z - a) whole square = 1. The characteristic curve to the surface (6) is the circle x square + y square + (z - a) whole square = 1, z = a. Let us see how we get this.

So let us find the partial derivative of this is let us write f(x, y, z, a) = x square + y square + z - a whole square - 1 = 0 okay and when we differentiate it with respect to a what we get 2 times z - a = 0 or * - 1. 2 * z - a * - 1 = 0 or z - a = 0 okay. So the curve of intersection of the surfaces f(x, y, z, a) and the surface this when delta tends to 0 is given by f(x, y, z, a) = 0 and the partial derivative of f(x, y, z, a) with respect to a = 0.

So what we have? z - a so the curve of intersection that is the characteristic curve is x square + y square + (z - a) whole square = 1 and z = a. Now you can see x square + y square + z - a whole square = 1. It is the family of spheres with radius 1 and centers on the z axis 0, 0, a. So it is a family of spheres with unit radius and having their centers on the z axis and the characteristics curve of this surface is the circle given by x square + y square + z - a whole square = 1, z = a.

So the circle lies in the plane z = a, and when we want to find the envelope of this family then we eliminate the parameter a between the quotients f(x, y, z, a) = 0 and partial derivative of f with respect to a = 0. So partial derivative of f with respect to a is n - a = 0. So z = a. So eliminating a between f(x, y, z, a) = 0.

And partial derivative of f(x, y, z, a) we with respect to a = 0, we get x square + because we are eliminating a from these 2 equations means we replace z by a here and that gives you x square + y square = 1. So we get x square + y square = 1 which you know that it represents a right circular cylinder.

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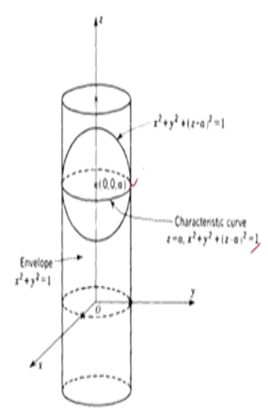
(It is clear that the envelope touches the each member of the family along the characteristic curve of the system on that member).



So it is clear that now let us see this figure. This is your family of spheres x square + y square + z - a the whole square = 1 with center at the point 0, 0, 1 and having unit radius. Now, this is the characteristic curve x = a, x square + y square + z - a whole square = 1. This is the characteristic curve which is the circle lying in the plane z = a having the center 0, 0, a and the envelope, envelope is this 1.

The right circular cylinder x square + y square = 1. It is clear that the envelope x square + y square = 1 touches each member of the family. Let us consider this member x square + y square + z - a whole square = 1. Let us fix the value of a so that a member is fixed so this member is the member of the family of spheres with the unit radius and having their centers on the z axis. So this is 1 particular member.

The envelope touches the member of this family along the characteristics curve you can see here okay. So the envelope touches each member of the family along the characteristics curve of the system on that member.

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Two Parameter systems: Let us now consider two parameter system of surfaces defined by the equation

$$f(x, y, z, a, b) = 0 \qquad (8)$$

where a and b are parameters. First let us consider the case where b is a prescribed function of 'a' e.g. $b = \phi(a)$.

Then $f(x, y, z, a, \phi(a)) = 0$ is a one parameter family of the surfaces.

Then its envelope is obtained by eliminating 'a' from

$$f(x, y, z, a, \phi(a)) = 0 \quad \text{and} \quad \frac{\partial}{\partial a} f(x, y, z, a, \phi(a)) = 0$$

$$\text{or} \qquad \frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}\frac{db}{da} = 0, \quad \text{where } b = \phi(a).$$

Now let us consider 2 parameter systems. So let us consider 2 parameter system of surfaces defined by the equation f(x, y, z, a, b) = 0. So let us take an equation where f is a function of f(x, y, z) and a, b are 2 parameters. First let us consider the case where b is a prescribed function of a. So we are given b = phi(a) where phi is a known function of a. Then the 2 parameter system reduces to a single 1 parameter system f(x, y, z, a, phi(a)) = 0.

It is a 1 parameter family of surfaces. Its envelope is obtained by eliminating a. Now take a 1 parameter system of surfaces. So its envelope is obtained by the equation f(x, y, z, a, phi(a)) = 0 and its partial derivative with respect to a = 0. Now partial derivative of this f(x, y, z, a, phi(a)) = 0 with respect to a can also be written as partial derivative of f with respect to a + partial derivative of f with respect to b, b is phi(a) * db/da = 0.

So this is another way. This equation is another way of writing the partial derivative of f(x, y, z, a, phi(a)) with respect to a = 0.

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Note that for every form of the function $\phi(a)$, the characteristic curve of subsystem on (8) passes through the point defined by the equations

$$f = 0, \; f_a = 0, \; f_b = 0. \qquad (9)$$

This point is called the characteristic point of the two parameter system (8) on the particular surface (8).

As the parameters a and b vary, this point generates a surface known as the envelope of the surface (8).

The envelope is obtained by eliminating a and b from the equation (9).

Now note that for every form of the function phi(a) okay. Let us note that whatever mean function phi(a) the characteristic curve of the subsystem on 8 okay the characteristic curve. The characteristic curve is obtained by this equation and this equation okay. The 2 equations together determine the characteristic curve. So the characteristic curve of the subsystem okay on 8 passes through the point defined by the equations f = 0, fa = 0, fb = 0.

Let us see that. Characteristic curve of this f(x, y, z, a, phi(a)) = 0 this subsystem f(x, y, z, a, phi(a)) = 0 is obtained from this equation and this equation. Now it passes through the point which is determined by fa = 0, f = 0, fa = 0, fb= 0 because if fa = 0 and fb = 0 then obviously this partial derivative = 0 okay. The partial derivative is if fa = 0 and fb = 0 then fa + fb * db/da = 0. So the point determined by f = 0, fa= 0, and fb = 0.

Okay will always lie on the characteristic curve okay given by f(x, y, z, a, phi(a)) = 0 and this equation partial derivative of f with respect to a = 0 so that is what we are saying. For every form of the function phi(a) the characteristic curve of the subsystem passes through the point given by the equations f = 0. fa = 0, fb = 0. This point is called the characteristic point of the 2 parameter system on the particular surface (8) okay.

This is a 2 parameter system okay we are taking the particular subsystem by taking b = phi(a). So this point is called the characteristic point of the 2 parameter system on the particular surface (8). As the parameters a and b vary, this point generates a surface known as the envelope of the surface (8). The envelope is obtained by eliminating a and b from the equations f = 0. fa = 0, fb = 0.

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For example, let us consider

$$(x-a)^2+(y-b)^2+z^2=1 \qquad (10)$$

Then

$$f(x,y,z,a,b)=(x-a)^2+(y-b)^2+z^2-1=0$$

Now,

$$\frac{\partial f}{\partial a}=-2(x-a),\ \frac{\partial f}{\partial b}=-2(y-b)$$

x = a, y = b

z = ±1

Hence

$$f(x,y,z,a,b)=0,\ f_a=0 \text{ and } f_b=0$$

The characteristic points of the two parameter system on the surface (10) are $(a,b,\pm 1)$.

Now let us consider for example (x- a) whole square + (y - b) whole square + z square = 1. Then we can write this is a 2 parameter system of surfaces you can see. So here we consider f(x, y, z, a, b) = (x - a) whole square + (y - b) whole square + z square - 1 = 0. Then we can see that partial derivative of f with respect to a is 2 times (x - a) * - 1 and partial derivative of f with respect to these 2 *(x - b) * - 1 hence f(x, y, z, a, b) = 0 and fa = 0, fb = 0.

Given as the characteristics points as you see f - a = 0 so this is = 0 partial derivative of f with respect to b when we find it should be y okay. So - 2 * y - b. so y - b okay. So x - a = 0 means x = a, y = b okay. When we put this in this equation we get z = + - 1. So the characteristic points of the 2 parameter system on the surface (10) okay on the surface (10) are a, b, + - 1.

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Let us consider a subsystem of the system (10) by taking $b = 2a = \phi(a)$.
Hence the envelope is obtained as

$$4x^2 + y^2 + 5z^2 - 4xy = 5$$

which is a right circular cylinder with axis $\frac{x}{1} = \frac{y}{2} = \frac{z}{2}$ and unit radius.

since $b = 2a$, we get

$(x-a)^2 + (y-2a)^2 + z^2 = 1$, one parameter subsystem

$f(x, y, z, a, \phi(a)) = f(x, y, z, a, 2a)$

$= (x-a)^2 + (y-2a)^2 + z^2 - 1 = 0$ ✓

$\frac{\partial f}{\partial a} = 2(x-a)(-1) + 2(y-2a)(-2) = 0$

or $(x-a) + (y-2a)2 = 0 \Rightarrow x + 2y = 5a$

or $a = \frac{x+2y}{5}$ ✓

$\left(x - \frac{x+2y}{5}\right)^2 + \left(y - \frac{2x+4y}{5}\right)^2 + z^2 - 1 = 0$

$4x^2 + y^2 + 5z^2 - 4xy = 5$

Let us consider a subsystem of the equation of the subsystem (10). Let us consider a subsystem of this system by taking b = 2a. Then let us find the envelope of that surface. So let us see we have (x - a) whole square + (y - b) whole square + z square = 1 and we take b equal to a prescribed function of a say b = 2a. So what we do is since b = 2a we get x - a whole square + y - 2a whole square + z square = 1. This is 1 parameter subsystem term.

Now in order to obtain its envelope okay let us put f(x, y, z, a, phi(a)) phi(a) is 2a here = (x - a) whole square + (y - 2a) whole square + z square - 1 = 0 okay. Now let us differentiate it with respect to a. So we get 2 *(x - a) * - 1 + 2 * (y - 2a) * - 1 = 0 or we can say we can divide this equation by - 2 so (x - a) + y - 2a = 0. Here, we have - 2 okay because when we differentiate it with respect to a we get 2 *(x - a) * - 1 then 2 * (y - 2a) * - 2 okay.

So we will have x - y - 2 we are dividing by - 2 this equation so this will be 2 times okay. So this gives you x + 2y = 5a or a = x + 2y/5 okay. Now eliminating a, the parameter a, between this equation and this equation okay. We will arrive it (x - x + 2y/5) whole square + (y - 2x + 4y/5) whole square + z square - 1 = 0 and when we simplify we will get 4 x square + y square + 5 z square - 4 xy = 5 okay.

We know that it represents a right circular cylinder whose axis is given by x/1 = y/2 = z/2 and its radius is unity from our 3 dimensional geometry.

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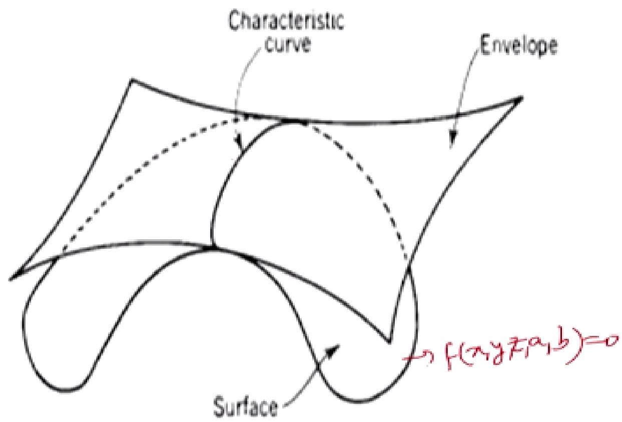

Let us look at this. So this is a surface 2 parameters system of surface f(x, y, z, a, b) = 0 with a 2 parameter system okay. f(x, y, z, a, b) = 0 we have this characteristic curve okay. On this, let us look at this.

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The envelope of a two parameter system is touched at each of its points P by the surface of which P is the characteristic point. Therefore it possesses the same set of values (x, y, z, p, q) as the particular surface, so that it must also be a solution of a differential equation.

Thus we have three classes of integrals of a PDE of type (1):

a) Two parameter systems of surfaces (2), such an integral is called a complete integral. [$F(x,y,z,a,b)=0$ → complete integral]

b) If in the complete solution (2), b is replaced by a prescribed function $\phi(a)$ and its envelope is obtained, we obtain a solution of (1). The totality of solutions obtained by varying $\phi(a)$ is called the general integral of (1) corresponding to the complete integral (2). [$F(x,y,z,a,\phi(a))=0$]

The envelope of a 2 parameter system is touched at each of its points P by the surface of which P is the characteristic point. So this is the envelope, okay. This is the envelope. It is touched at each of its points by the characteristic curve okay. The characteristic curve and you can see the envelope is touching at each of its point where the surface of which P is the characteristic point. Therefore, it possesses the same set of values (x, y, z, p, q) as the particular surface and therefore it must also be a solution of the differential equation.

So we have 3 classes of integrals of partial differential equation of type (1). Type (1) means the question of the form f(x, y, z, p, q) = 0 of this form okay. So there are 3 types of integrals okay. So there are 3 types of integrals of the partial differential equation of type (1). One is the 2 parameter system of surfaces okay (2) where the equation a will be of the form some function of x, y, z and parameters a and b = 0.

So 2 parameter systems of surfaces (2) such an integral is called complete integral okay that means 2 parameter systems of surfaces will be given by f(x, y, z, a, b) = 0 okay. This solution is called as a complete integral. If in the complete solution (2) okay if in this complete solution b is replaced by a prescribed function and we have. If b is replaced by a prescribed function phi(a) and we obtain its envelope.

Envelope is obtained by this equation and its derivative with respect to a = 0 by eliminating a between the 2 equations and its envelope is obtained. We obtain a solution of the nonlinear PDE F(x, y, z, p, q) = 0. The totality of solutions obtained by varying this function phi is called the general solution of equation 1 corresponding to the complete integral (2) F(x, y, z, a, b) = 0.

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c) If the envelope of the two parameter system (2) exists, it is also a solution of (1). It is called the singular integral of (1).

Example: Consider the PDE

$$z^2(1+p^2+q^2)=1 \qquad (11)$$

Then

$$(x-a)^2+(y-b)^2+z^2=1 \qquad (12)$$

is a solution of this equation with arbitrary a and b.

$f(x,y,z,a,b)=0$
$f_a=0,\ f_b=0$

$f(x,y,z,a,b) = (x-a)^2+(y-b)^2+z^2-1=0$

$\frac{\partial f}{\partial x} = 2(x-a)+2z\frac{\partial z}{\partial x}=0$

$(x-a)=-zp$

$2(y-b)+2z\frac{\partial z}{\partial y}=0$

$2(y-b)+2zq=0 \Rightarrow (y-b)+zq=0$

$z^2p^2=(x-a)^2$
$z^2q^2=(y-b)^2$
we get $z^2+(x-a)^2+(y-b)^2=1$

And the third is if the envelope of the 2 parameter system is obtained by suppose your equation is f(x, y, z, a, b) = 0 then if you want to find the envelope you eliminate a and b from these 3 equations f(x, y, z, a, b) = 0 fa= 0, fb = 0. After eliminating a and b, here the solution that you get it is called as the singular integral of the equation (1). So for example, consider z square (1 + p square + q square) = 1.

Then (x - a) whole square + (y - b) whole square + z square = 1 is a solution of this equation with arbitrary a and b. So here you can see in this equation relation there are 2 parameters a and b and therefore this solution will be a complete integral of the given PDE z square * 1 + p square + q square = 1. Let us show that this is the solution of this equation. So you have let me write f(x, y, z, a, b) = (x - a) whole square + (y - b) whole square + z square - 1 = 0.

Let us differentiate it with respect to x okay. So then partial derivative of f with respect to x then we find what we get 2 * (x - a) we get + 2z * partial derivative of z with respect to x okay. So we get 2 *. So we get (x - a) = - z * p. Similarly, when we differentiate this equation with respect to y we get 2 * (y - b) + 2z * partial derivative of z with respect to y = 0 or 2 * y - b + 2 zq = 0 which will give you y - b + zq = 0 okay.

So what we have? Z square * p square = (x - a) whole square * z square * q square = (y - b) whole square okay and we have the equation (x - a) whole square + (y - b) whole square + z square - 1 = 0 okay. So replacing (x - a) whole square by x square * p square (y - b) whole square by z square * q square we get. So this is a solution of means it must satisfy this equation okay. So z square + z square p square is (x - a) whole square, z square * q square is (y - b) whole square and we know that it is = 1 okay. So this is the solution of the equation 11.

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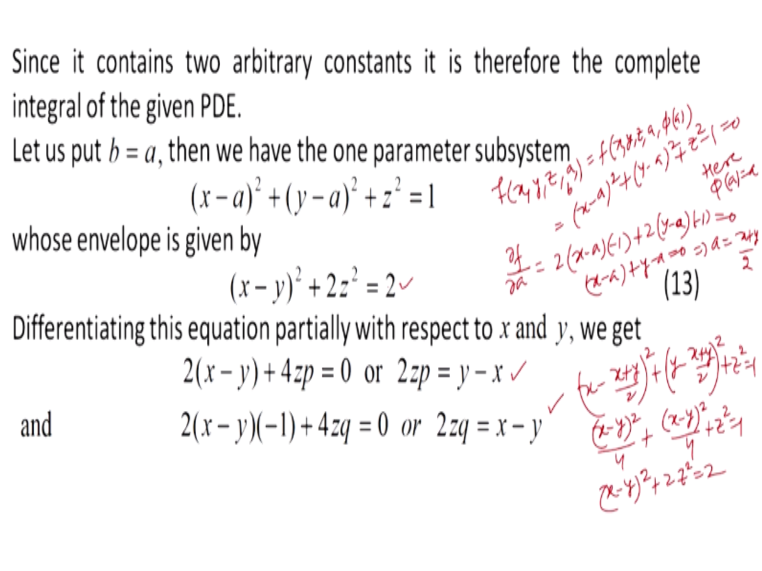

Since it contains 2 arbitrary constants it is therefore the complete integral of the given PDE. Now let us put b = a here. Then we have 1 parameter subsystem (x - a) whole square + (y - a) whole square + z square = 1. Its envelope is given by (x - y) whole square + 2 * z square = 2. So this we can find. Let us write it as f(x, y, z, a, b) which is f(x, y, z, a, phi(a)) this = (x - a) whole square + (y - a) whole square + z square - 1 = 0. Here phi(a) = a.

Now we differentiate with respect to a we will get 2(x - a) (- 1) 2 (y - a) * (- 1) = 0 okay. So (x - a) - 2 we can divide (x - a) + (y - a) = 0. So a = x + y/2. Let us place the value of a in this equation. So x - x + y/2 whole square + y - x + y/2 whole square + z square = 1. What does it give 2x - x - y whole square/4 + 2y - z - y. So y - x whole square so again I can write x - y whole square/4 + z square = 1 and this is x - y whole square/2 + z square. So z square = 2.

So this is the envelope of the 1 parameter subsystem. Now differentiating this equation x - a whole square + y - a whole square + z square = 1 what do we notice? We notice that this is 2 * x no we are differentiating this equation differentiating the envelope okay. So 2 * (x - y) 4z * p = 0 if we differentiate with respect to x. So to get p = y - x and when we differentiate this with respect to y what we get 2 * x - y * - 1 + 4zq = 0. So what do we get 2zp = y - x and 2zq = x - y okay. These 2 equations will get by differentiating the equation number 3 with 13 with respect to x and y.

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which implies that

$$= z^2(1+p^2+q^2) = z^2 + \left(\frac{y-x}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2$$

$$= z^2 + \frac{2(x-y)^2}{4} = z^2 + \frac{(x-y)^2}{2} = 1$$

Hence (13) is an integral surface of the given PDE. It is a solution of type (b), where a definite function $\phi(a)$ has been used so we obtain a particular case of the general integral.

When $b = \phi(a)$, φ arbitrary, is used, the elimination of 'a' between

$$f(x, y, z, a, \phi(a)) = 0 \text{ and } \frac{\partial f}{\partial a} = 0$$

is not possible.

And then z square * 1 + p square + q square will be = z square + zp = y - x/2 so y - x/2 whole square, zq = x - y/2 + x - y/2 whole square and what we get is z square + x - y whole square/2, but x - y whole square/2 + z square = 1. Hence the envelope which we obtain from the 1 parameter subsystem is an integral surface of the given PDE. It is therefore a solution of type (b) okay in type (b) we said what? Let us look at that again.

If in the complete integral (2) okay in the 2 parameter subsystem b is replaced by a prescribed function of a that is phi(a) and its envelope is obtained, we obtain a solution of (1). So envelope is equal to this one what we got x - y whole square + 2z square = 1 and we have seen that it is a solution of this equation 1.

The totality of solutions obtained by varying phi(a) is called the general integral of (1) corresponding to the complete integral (2) okay Now here we found we took a particular value of phi that is phi(a) = a and we found that envelope, but when b = phi(a) and phi is arbitrary the elimination of a between f(x, y, z, a, phi(a)) = 0 and partial derivative of f with respect to a is != 0 is not possible okay.

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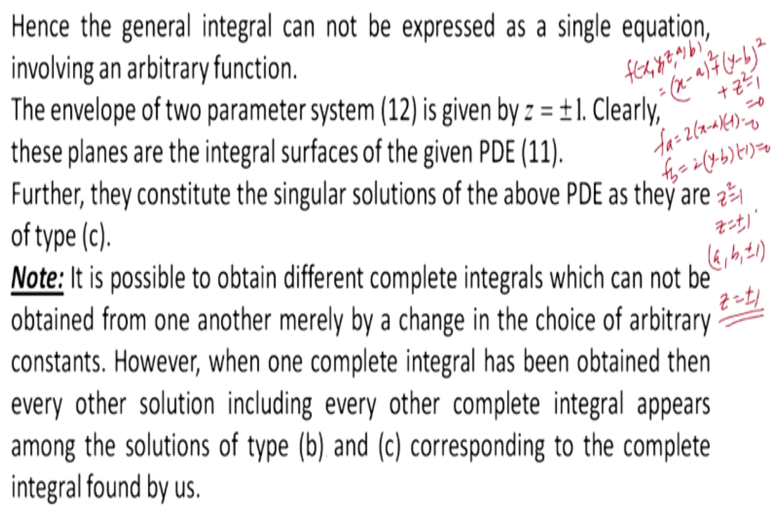

So, general integral can never be explained as a single equation involving an arbitrary function. The envelope of 2 parameter of system (12) is given by now let us look at this equation number 12 okay. So x - a whole square + y - b whole square + z square = 1. When we find the envelope of the 2 parameter system, envelope of the 2 parameter system means we have x - a whole square + y - b whole square + z square = 1 so I write z square - 1 = 0 so this is f(x, y, z, a, b) okay.

So this is the equation and then fa= 2* (x - a) * - 1 = 0 and fb = 2 * (y - b) * -1 = 0. Solving these 3 equations what we get? Z square = 1 okay. So z = + - 1. Eliminating a and b okay. Eliminating a and b from f = 0, fa = 0, fb = 0, we get z = + - 1. So the characteristic points are a, b, + - 1 the envelope is given by eliminating a and b from these 3 equations f = 0, fa= 0, fb = 0 which gives us z = + - 1 and which represents a pair of parallel planes okay.

So the envelope of the 2 parameter system is given by z = + - 1. These planes are the integral surfaces of the given PDE, given PDE 11 this one okay. Because z = + - 1 means when we differentiate z with respect to x, p = 0 when the differentiate z with respect to y, q = 0 so (()) (32:45) z = + - 1 satisfies 11 so these planes are the integral surfaces of the equation 11. Further, we constitute the similar solutions okay. We described the third case with the envelope of the 2 parameter systems exist it is also a solution of 1, it is called the singular integral.

So z = + - 1 are the singular solutions of PDE z square * 1 + p square + q square = 1. But it is possible to obtain different complete integrals which cannot be obtained from one another nearly by a change in the choice of arbitrary constants however one complete integral has been obtained when every other solution including every other complete integral appears among the solutions of the type (b) and (c) corresponding to the complete integral found by us. Let us see how it happens.

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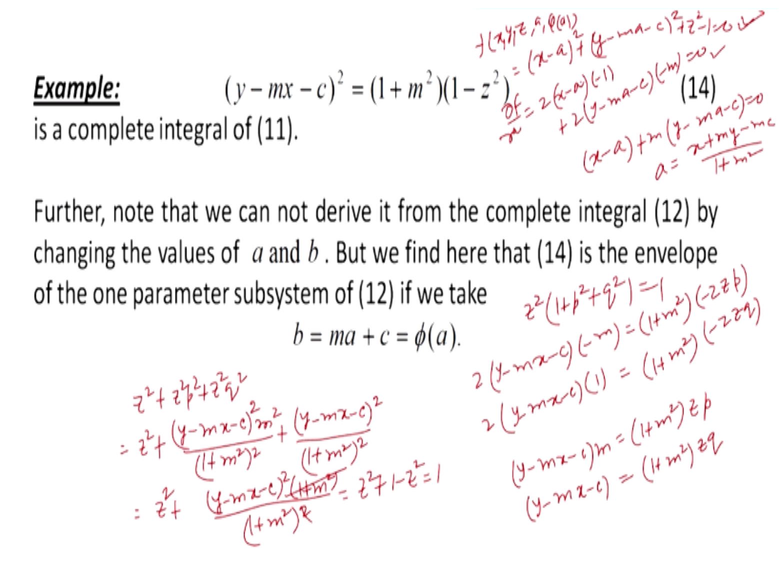

(y - mx - c) whole square = (1 + m square) * (1 - z square). It is a complete integral of equation (11). Equation 11 is z square * 1 + p square + q square = 1 okay. So that this is the equation for t okay gives us a solution of this equation. So 2 times we differentiate with respect to x. 2 * (y - mx - c) (- m) = (1 + m square) * (- 2z) partial derivative of z with respect to x that is - 2zp. Similarly we differentiate with respect to y 2(y - mx - c) when we differentiate with respect to y we get 1 = (1 + m square) (- 2 zq).

So what we get here? We can cancel - 2 on both sides so (y - mx- c) * m = (1 + m square) * zp okay and (y - mx - c) we can also cancel - 2 here = (1 + m square) * zq. So (y - mx - c) * m = (1 + m square) zq. Now e have z square + z square * p square + z square * q square. Let us put the value here. So this is z square + zp what (y - mx - c) whole square * m square/(1 + m square) whole square + z square q square is (y - mx - c) whole square/(1 + m square) whole square sorry z square * q square.

Now this is what z square + we can take (1 + m) square whole square. So what we get (y - mx - c) whole square (1 + m square)/(1 + m square) whole square. So this 1+ power m is cancel with this here. Now (y - mx- c) whole square/(1 + m square) is 1 - z square. So this is z square here. So z square + 1 - z square and this gives you 1. So this is a complete integral of equation (11) and you can see that we cannot obtain this complete integral from the complete integral (12) early by a change of variables this one complete integral 12 okay.

(x - a) whole square + (y - b) whole square + z square = 1 nearly by changing in the arbitrary constants a and b will never give us this solution this one (y - mx - c) whole square = (1 + m square) (1 - z square), but we find here where the (14) is the envelope of the 1 parameter subsystem of (12) if we take b = ma + c. So let us consider that, the equation (12). This equation number 12 okay. Here if you take particular value function b - phi (a) then we will be able to get that solution. Let us see how we get that.

So we will take b = ma + c. So (x - a) whole square + (y - ma - c) whole square + z square - 1 = 0. This is f(x, y, z, a, phi(a)) okay when we differentiated with respect to a we get 2(x - a) (- 1) + 2(y - ma - c) (- m) = 0 okay. Now we can eliminate a between this equation and this equation (x - a) whole square + (y – ma - c) whole square + z square - 1 = 0. We will arrive it by we will arrive at this solution (y - mx - c) whole square = (1+ m square) (1 - z square).

So this complete integral can be found from that complete integral given by equation (12) if we take if we consider the 1 parameter subsystem of (12) b = ma + c. So here what we will get if you divide this equation by - 2 you get (x - a) + m * (y - ma - c) = 0. So we can find the value of a from here okay a will be = how much? (x + my - mc)/1 + m square. So this value of a when we put in this equation okay in this equation we will arrive at the equation (14). Let us now consider another example.

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Example: Verify that $z = ax + by + a + b - ab$ is a complete integral of the PDE is a complete integral of the PDE $z = px + qy + p + q - pq$, where a and b are arbitrary constants. Show that the envelope of the all planes corresponding to complete integrals provides a singular solution of the equation and determine a general solution by finding the envelope of those planes that pass through the origin.

$z = px + qy + p + q - pq$ ✓

$z = ax + by + a + b - ab$ ✓

$p = \frac{\partial z}{\partial x} = a, \quad q = \frac{\partial z}{\partial y} = b$

$z = ax + by + a + b - ab$

Let us write $f(x, y, z, a, b) = z - ax - by - a - b + ab = 0$ ✓ (i)

$\frac{\partial f}{\partial a} = -x - 1 + b$ then $f_a = 0 \Rightarrow b = x + 1$ ✓ — (ii)

$\frac{\partial f}{\partial b} = -y - 1 + a$ & $f_b = 0 \Rightarrow a = y + 1$ ✓ — (iii)

Then $a + b - ab = 0$, $b(a-1) = a \Rightarrow b = \frac{a}{a-1}$

So Eliminating a and b between (i)-(iii) we get

$z = (y+1)x + (x+1)y - (y+1) - (x+1) + (x+1)(y+1)$

or $z = xy + x + xy + y - y - 1 - x - 1 + xy + x + y + 1$

or Envelope: $z = 3xy + x + y - 1$

We have a first order nonlinear PDE. z = px + qy + p + q - pq. First we show that z = ax + by + a + b - ab is a complete integral of this nonlinear PDE so z = px + qy + p+ q - pq. This is the PDE given to us. Let us differentiate it okay. So we have and z = ax + by + a + b - ab okay. Let us show that it is a complete integral of this nonlinear PDE. So let us differentiate this with respect to x partial derivative let us find p = a and when we find q what we get is b okay.

So replacing a and b by p and q what we have you see here replacing p and qy where we get z = ax + by + a + b - a okay. So we can see that this is a complete integral of this partial differential equation. Now let us show that the envelope of all planes corresponding to the complete integrals provides a singular solution of the equation. So for that we have to find let us write f(x, y, z, a, b) = z - ax - by - a - b + ab = 0 and find its partial derivatives with respect to a and b.

So let us find partial derivative of f with respect to a what we get - x - 1 and then we get here + b okay and similarly partial derivative of f with respect to b if you find what you get - y - 1 + a okay. So then fa = 0 gives b = x + 1 and fb = 0 gives you a = y + 1. The complete the envelope of all planes corresponding to the complete integral this plane corresponding to the complete integral provides a singular solution. So let us find the singular solution.

The singular solution is obtained by eliminating a and b between these 3 equations okay. So let us put a = y + 1 here and b = x + 1. So then eliminating a and b okay. Let me call it 1. This is 2, this is 3 okay between we get z = we have ax means (y + 1) * x okay and then b = x + 1. So (x + 1) * y - a - b so - (y + 1) - (x + 1) and then we get + (x + 1) (y + 1). So let me simplify z = xy + x okay. Then we have xy + y okay we have - y - 1 - x - 1 then let us multiply x + 1 and y + 1.

We have xy + x okay and y + 1. xy + x then we multiply, we are multiplying x to y + 1 so xy + x then we multiply 1 to y + 1 so we get y + 1 okay. So what we get this x we can cancel with this x, y we can cancel with this y and we have the envelope as this is given by you see z = 1 xy, this is 2, this is 3. So, we have 3xy and then we have x + y. We can cancel with 1 here okay. So we have 1, 2, 3 xy and then x + y and we have - 1. So this is the envelope of all planes corresponding to the complete integral.

These are similar solution of the equation. Now let us determine a general solution by finding the envelope of those planes that pass through the origin. So if this plane z = ax + by + a + b - ab okay passes through the origin then what we will have. 0 = 0 + 0a + b - ab = 0 okay. So then a + b - ab = 0 and what we will get b * if we take it together the other side b(a - 1) = a. So we will get b = a/a - 1 okay so thus in other 2 we get the general solution. What we will do is now let us consider the equation.

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Then $z = ax + \frac{a}{a-1} y$ $\qquad$ $b = \phi(a) = \frac{a}{a-1} = 1 + \frac{1}{a-1}$

Thus, $f(x, y, z, a, b(a)) = ax + \frac{a}{a-1} y - z = 0$ ✓ (iv)

$$\frac{\partial f}{\partial a} = x + \left(-\frac{1}{(a-1)^2}\right) y = 0 \Rightarrow \frac{x}{y} = \frac{1}{(a-1)^2} \text{ or } (a-1)^2 = \frac{y}{x}$$

$$\Rightarrow a = 1 \pm \sqrt{\frac{y}{x}} \quad \text{✓ (v)}$$

Eliminating 'a' between (iv) and (v), we get

$$ax + \left(1 + \frac{1}{a-1}\right) y = z$$

$$\left(1 \pm \sqrt{\frac{y}{x}}\right) x + \left\{1 \pm \sqrt{\frac{x}{y}}\right\} y = z$$

$$x \pm \sqrt{xy} + y \pm \sqrt{xy} = z$$

$$\text{or} \quad x + y - z = \mp 2\sqrt{xy} \Rightarrow (x + y - z)^2 = 4xy.$$

The general solution is $(x + y - z)^2 = 4xy$.

Z = ax, b is a/a - 1 * y. Z = ax + by + a + b - ab is 0 which gave us the value of v = a/a - 1 so substituting v = a/a - 1 we get z = ax + a/a - 1 y okay. Now let us consider so then thus f(x, y, z, a, phi(a)). Phi(a) is b = phi(a) which is a/a - 1 okay so this is ax + a/a - 1 y - z = 0. So let us now determine partial derivative of f with respect to a and we get x and this I can write a/a - 1 can be written as 1 + 1/a - 1 okay.

So when we differentiate a/a - 1 with respect to a what I will get is - 1/a - 1 whole square * y = 0. So let us put this = 0. Then what we will get x/y = 1/(a - 1) whole square or (a - 1) whole square = y/x. So this gives you a = 1 + - square root y/x. Now let us eliminate parameter a between this equation and this okay. So eliminating this we call it as IV and this as V okay so between IV and V we get okay so a is 1 + - ax + (1 + 1/a - 1) okay * y = z.

So a is 1 + - square root y/x this is a * x and then we have (1 + 1/a - 1 + - square root x/y). So + - square root x/y * y = z. This is what we get and this is x + - under root xy and what we get here + y + - square root xy = z. So I can also write this as x + y - z = - + 2 under root xy. Now squaring this both sides I get x + y - z whole square = 4 xy so that is the general solution.

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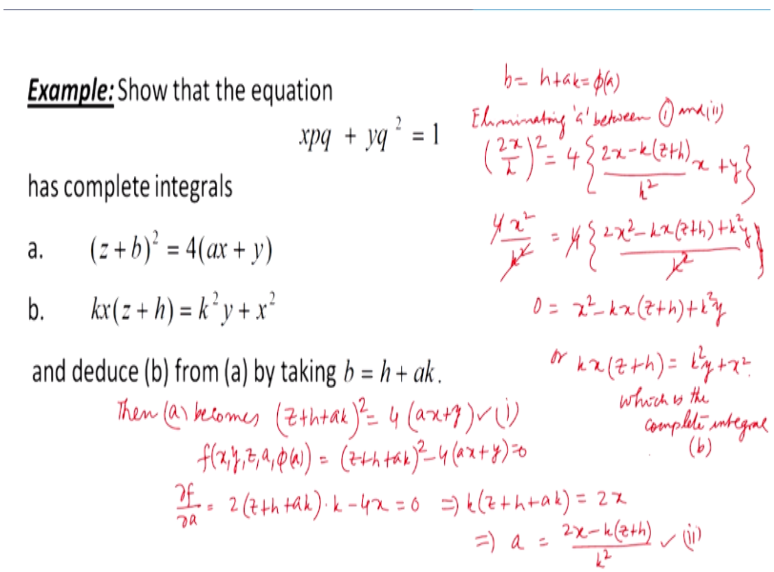

Now let us consider 1 more problem first order nonlinear PDE. xpq + yq square = 1 by Charpit method you can show that there are 2 complete integrals 1 is (z + b) whole square = 4(ax + y) the other one is kx(z + h) = k square y + x square. What we have to do here. We want to show that the complete integral b can be obtained from the complete integral a by taking b = h + ak. So let us take b = h + ak and then a becomes z + h + ak whole square = 4 * ax + y.

This is what we get. z + h + ak whole square = 4(ax + y). Now we can write f(x, y, z, a, phi(a)) because we are writing b as this is phi(a). b = h + ak = phi(a). So f(x, y, z, a, phi(a)) will become z + h + ak whole square - 4 * ax + y = 0 and let us differentiate this partially with respect to a. So partial derivative of f with respect to a is 2(z + h + ak) * k - 4x. We put it = 0 okay and determine the value of a. So this gives you z + h + ak * k = 2x. So what we get?

This implies a = 2x - k(z + h)/k square okay. Now let us eliminate a between this equation and this equation okay. So let me call it as I and this as II. So eliminating a between I and II okay. I and II. What we will get z + h + ak is 2x/k okay. So 2x/k whole square = we have 4(ax + y) so a is 2x - k(z + h)/k square * x + y okay or I can write it as 4 x square/k square = 4 *(2x square - kx * (z + h) + k square y/k square okay.

This k square will cancel with k square. 4 will cancel with 4 and what we will get. We will have x square = 2 x square - kx * z + h + k square y. So I can say that 0 = x square - kx * z + h + k square y or we can say that kx * (z + h) = x square k square y + x square and which is the complete integral b okay given by b okay. So with this I would like to end my lectures. Thank you very much for your attention.