Ordinary and Partial Differential Equations and Applications

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Lecture – 35

Surfaces Orthogonal to a given system of surfaces

Hello friends! Welcome to my lecture on surfaces orthogonal to a given system of surfaces. We know that angle between any 2 surfaces at a point of intersection is the angle between their respective tangent planes at that point. Now let us say suppose we are given a 1 parameter family of surfaces we can write its equation as f(x, y, z) = C, where C is a parameter. Our aim is to find a system of surfaces which cut each of these surfaces at right angles or you can say which cut each of the given surfaces orthogonally.

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Definition: Angle between two surfaces at a point of intersection is the angle between their respective tangent planes.

Suppose a one parameter family of surfaces is given by the equation

$$f(x, y, z) = c, \tag{1}$$

'c' being a parameter.

We shall find a system of surfaces which cut each of these given surfaces at right angles.

The normal at any point (x, y, z) to the surface (1) has direction ratios

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

$\text{grad} f = \nabla f = \hat{i} f_x + \hat{j} f_y + \hat{k} f_z$

Now we know that if we are given a surface f(x, y, z) = C, then gradient of f and gradient of f is a vector normal to the surface. Okay a gradient of f is ifx + jfy + kfz. Okay. So the normal at any point (x, y, z) to the surface f(x, y, z) = C has direction ratios fx, fy and fz.

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Assume that the surface $z = g(x, y)$ cuts each surface of the given system orthogonally. At the point (x, y, z), its normal has direction ratios $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$.

Since both the surfaces intersect orthogonally, at the point of intersection (x, y, z) their respective normal are perpendicular.
Therefore

$$\frac{\partial f}{\partial x}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial z}{\partial y} - \frac{\partial f}{\partial z} = 0 \qquad (2)$$

which is a quasi-linear PDE.

$\frac{\partial f}{\partial x}p + \frac{\partial f}{\partial y}q = \frac{\partial f}{\partial z}$

$f_x(x,y,z)p + f_y(x,y,z)q = f_z(x,y,z)$

$z = g(x,y)$

$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial g}{\partial x}, \frac{\partial z}{\partial y} = \frac{\partial g}{\partial y}$

$z = g(x,y)$

or $F(x,y,z) = g(x,y) - z = 0$

$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$

now,

$\frac{\partial F}{\partial x} = \frac{\partial g}{\partial x}, \frac{\partial F}{\partial y} = \frac{\partial g}{\partial y}, \frac{\partial F}{\partial z} = -1$

Hence

$\frac{\partial F}{\partial x} = \frac{\partial z}{\partial x} = z_x, \frac{\partial F}{\partial y} = \frac{\partial z}{\partial y} = z_y, \frac{\partial F}{\partial z} = -1$

Let us assume that the surface z = g(x, y) cuts each surface of the given system orthogonally. We can write z = g(x, y) as let us write f(x, y, z) = g(x, y) - z = 0. Then the surface f(x, y, z) = 0, has normal with direction ratios. The direction ratios of the normal to the surface f(x, y, z) = 0 are given by f(x), f(y), f(z). Now f(x) is equal to the derivative of g with respect to x, f(y) is equal to derivative of z derivative of g with respect to y and partial derivative of f with respect to z is - 1.

Now let us note further that z = g(x, y) gives the partial derivative of z with respect of x as, g(x) and partial derivative of z with respect to y as, g(y). Okay so hence from these equations we get, the partial derivative of f with respective of x as, derivative of g with respect to x which is z(x) and similarly partial derivative of f with respect to y, which is derivative of g with respect to y and derivative of g with respect to y is z(y) and partial derivative of f with respect to z is - 1 Okay.

Now let as look at this equation since both the surfaces z = g(x, y) and given surface intersect orthogonally at the point of intersection (x, y, z) the respective normal are perpendicular. So we must have f(x) * z(x) + f(y) * z(y) – f(z) = 0. And this equation is a quasilinear PDE because we can write it as this equation.

This equation can be written as * p, * p derivative of f with respect to x * p derivative of f with respect to y * q = derivative of f with respect to z. So f(x) is the partial derivative of f(x, y, z) with respect to x which we can find from the given equation of the surface and this is f(y) (x, y, z) *q = then this is partial derivative of f(x, y, z) with respect to z. So this equation can be interpreted as a first order PDE f(x) * p + f(y) * q = f(z) * f(z).

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Hence integral surfaces of PDE (2) are orthogonal to the given system of surfaces (1) i.e. the integral surfaces orthogonal to (1) are generated by integral curves of the equations

$$\frac{dx}{\partial f/\partial x} = \frac{dy}{\partial f/\partial y} = \frac{dz}{\partial f/\partial z}.$$

Hence the integral surfaces of the equation 2 are orthogonal to the given system of surfaces that is they are given by equation 1. Now thus the we can say the integral (()) (06:04) curves orthogonal to 1 are generated by the integral surfaces orthogonal to 1 are generated by the integral curves of the characteristic equations dx/fx, dy/fy, dz/fz. This follows from the (()) (06:23) this equation is a quasilinear PDE.

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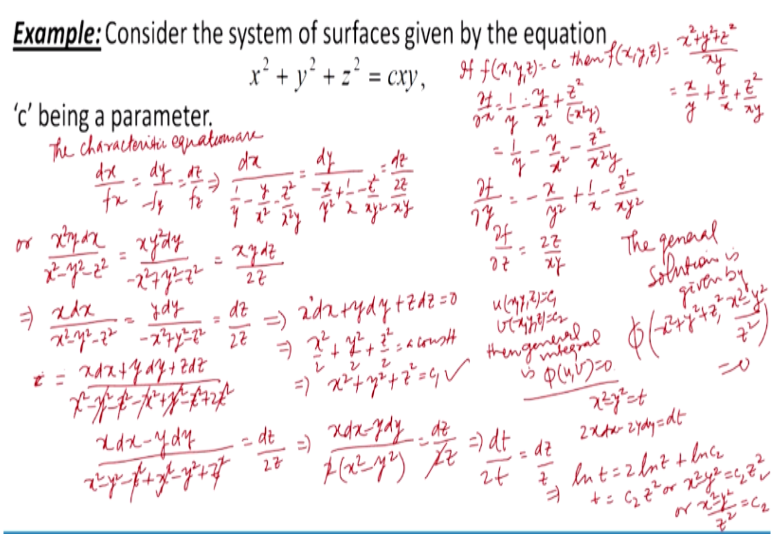

Now, let us consider so we then, then we will solve this characteristic system to arrive at the integral surfaces that are orthogonal to 1. So let us now consider a system of surfaces given by x square + y square + z square=cxy, where, c is a parameter. So we shall first write it in the standard form, f(x, y, z) = c. Okay so if you want to write it in that form, then if f(x, y, z) = c, then f(x, y, z) will be = x square + y square + z square/xy. Okay now let us find the partial derivatives of f with respect to x, y, z.

So f(x) is equal to now this is what? X square/xy. I can write it as x/y. Then y square/xy, I can write it as y/x and z square/xy I will write just like that. Okay. So what I will get when I differentiate it partially with respect to x. I get 1/y and then partial derivative with respect to x will give – y/x square. Here partial derivative with respect to x will give z square/1/x gives – 1/x square so – x square y. So this is 1/y – y/x square – z square/x square y. This is partial derivative of f with respect to x.

Then partial derivative of f with respect to y, we can write so – x/y square and then here we get 1/x here we get with respect to y so –z square/xy square. Partial derivative of f with respect to z is = this, this is 0, this is 0 here we get 2z/xy. So the characteristic equations are dx/fx, dy/fy, dz/fz, which will give as fx = this. So dx/1/y –y/x square – z square/x square y = dy 4 fy. So – x/square + 1/x – z square/xy square = dz/2z/xy. This is what we get okay.

Now we can simplify this or so this here if you take LCM it is x square y, x square y will go up there. I can write x square y of dx/x square y when you multiply here you get x square, x square y so this is – y square and x square y is in – z square = here xy square we multiply in the numerator and denominator so xy square dy/xy square gives – x square here we get + y square and here we get – z square and here what get xy dz/2z okay.

Or I can write it as so canceling xy we get x dx/x square – y square – z square is = y dy/ – x square + y square – z square = dz/2z. Okay from here what do you notice? This gives you, we multiply in the numerator by x in the numerator here by y here by z. We get x okay we write we multiply here y1, here y1 okay and then add z times here the time dz.

So x dx + y dy okay + z dz is = 0. Because this is this is what? You can see this give this is = x dx + y dy + z dz this is = x square – y square – z square – x square + y square – z square we are multiplying here by z so we get 2z square and the whole things cancels. So x dx + y dy + z dz = 0.

Which implies x square/2 + y square/2 + z square/2 is a constant, a constant. So we can write x square + y square + z square = some constant C1. Okay now let us find another solution of this and for that what we will do x dx - y dy, what do we get? x dx - y dy will give you x square – y square – z square and we are subtracting this one so we get + x square – y square + z square = dz/2z okay. So what you get x square x square okay and this z we will get z square z square cancel or this will give x dx – y dy/2 * (x square – y square) = dz/2z okay.

So this cancel with this and what do you get if you take x square – y square = t. Then you have 2x dx – 2y dy = dt. So this is = this gives you dt/2t okay. x dx – y dy dt/2. So dy/2t = dz/z okay or you can say ln t this 2 y I can multiply here so ln t is 2 ln z okay + some constant ln C2. So t is = C2 z square or x square – y square = some constant C2 times z square okay so we get one solution as x square + y square + z square = C1 and other solution as x square – y square = C2 * Z2 square or we can say x square – y square/z square = C2.

And therefore the general system therefore the general solution is given by some function phi (x square + y square + z square and x square – y square/z square) = 0. Because we know that, if u(x, y, z) = constant and v(x, y, z) = constant are 2 independence surfaces of the characteristic equations, then general solution is phi(u, v) = 0. Then general integral is phi(u, v) = 0.

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Then the general solution is

$$\phi\left(\frac{x^2 - y^2}{z^2}, x^2 + y^2 + z^2\right) = 0$$

So using this result okay the general solution of the given system is phi(x square + y square + z square and x square – y square/z square) = 0. So this is what we get, the solution okay.

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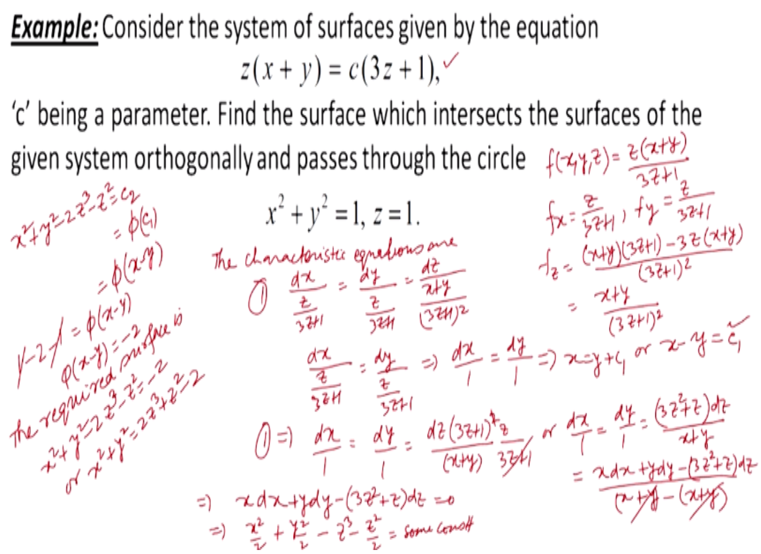

Now let us go to another question where we take system surfaces given by the equation z* (x + y) = c * (3z + 1). C is a parameter. Now here we want to find particular surface okay, which intersects the surfaces of the given system orthogonally and passes through this curve okay. This curve is a circle which is given by the equations x square + y square = 1, z = 1. So this circle lies in the plane z = 1. So here let us write this equation in the standard form f(x, y, z) = c. So f(x, y, z) = z * (x + y)/3z + 1.

So f(x) let us find first here f(x) = z/3z + 1, f(y) let us find which will also z/3z + 1 and when we find f(z), what is f(z)? In the numerator, we have z, in the denominator also we have z. So let us differentiate in the y the quotient rule. So the derivative of this with respect to the z is (x + y)*(3z + 1) – derivative the denominator is 3 * 3 and then (()) (17:18) the numerator divided by (3z + 1) whole square. So what do we get? So this is (x + y) * (3z + 1) – 3z so we get (x + y)/(3z + 1) whole square okay.

So what do we get here? The characteristic equations are dx/fx that is dx/z/(3z + 1), dy/fy which is z/(3z + 1) and dz/fz which is (x + y)/(3z + 1) whole square okay. Now taking the ratio first ratio and second ratio okay we have dx/z * (3z + 1) = dy/z/ (3z + 1) this gives you dx/1 = dy/1 when we take the first and second ratios. So this gives you x = y + c1 or x – y = c1 okay. Now let us from the characteristic equations 1, 1 imply that dx/1 = dy/1 = dz (3z + 1) whole square/(x + y) and I multiplying by z/(3z + 1). So I get this okay.

So this cancel with this and we have or dx/1 = dy/1 = (3z square + z) *dz/(x + y). This is what we get. Now this is further equal to multiply this by x, x dx. I multiply this by y, y dy and then multiply by it - 1. So – (3z square + z) *dz and in the denominator what we get (x + y) – (x + y) so this cancel with this and what we get is? X dx + y dy = or – (3z square + z)dz = 0. Now this will give you when we integrate this will give you x square/2 + y square/2 and then - z cube – z square/2 is equal to some constant okay.

Multiplying by 2 I get another solution x square + y square - 2z cube - 2z cube – z square = some constant. Let us say c2 okay. Now c1 is, so c2 is phi c1. Some constant c2 is phi c1 where c1 is an arbitrary function. So I can write it as phi of (x - y) because (x - y) = c1. Now let us use the initial condition that is the curve we have to find the surface which passes through the circle so when the surface passes through the circle x square + y square = 1.

So 1 and z = 1. So we get 1 - 2 - 1 = phi of (x - y) okay. So this is phi (x - y) = - 2 and therefore I get this required surfaces x square + y square - 2z - 2z cube - z square = - 2 or x square + y square = 2z cube + z square - 2 that is the surface which passes through the circle x square + y square = 1 and the plane z = 1 and intersects the given surfaces orthogonally.

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Then the required surface is given by

$$x^2 + y^2 = 2z^3 + z^2 - 2.$$

So that is the answer.

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Example: Find the surface which is orthogonal to the one parameter system

$$z = cxy(x^2 + y^2),$$

and which passes through the hyperbola

$$x^2 - y^2 = a^2,\ z = 0.$$

Solution: We have

$$f(x,y,z) = \frac{z}{xy(x^2 + y^2)} = c.$$

$$f(x,y,z) = \frac{z}{xy(x^2+y^2)} = c$$

$$f_x = -\frac{z}{x^2y^2(x^2+y^2)^2}(3x^2y+y^3)$$

$$f_y = -\frac{z}{x^2y^2(x^2+y^2)^2}(x^3+3xy^2)$$

$$f_z = \frac{1}{xy(x^2+y^2)}$$

The characteristic equations are

$$\frac{dx}{\frac{-z(3x^2y+y^3)}{x^2y^2(x^2+y^2)^2}} = \frac{dy}{\frac{-z(x^3+3xy^2)}{x^2y^2(x^2+y^2)^2}} = \frac{dz}{\frac{1}{xy(x^2+y^2)}}$$

or

$$\frac{x^2y^2(x^2+y^2)^2dx}{-z(3x^2y+y^3)} = \frac{x^2y^2(x^2+y^2)^2dy}{-z(x^3+3xy^2)} = \frac{xy(x^2+y^2)dz}{1}$$

$$\frac{dx}{-yz(3x^2+y^2)} = \frac{dy}{-xz(x^2+3y^2)} = \frac{dz}{xy(x^2+y^2)}$$

$$\frac{x\,dx}{3x^2+y^2} = \frac{y\,dy}{x^2+3y^2} = \frac{-2dz}{x^2+y^2}$$

Now 1 more question we can consider where we have taken this system of surfaces family of surfaces z = cxy(x square + y square) and this family of surfaces, we have to find that surface which is orthogonal to this family and passes through the hyperbola x square - y square = a square, z = 0. So here again f(x, y, z) we can write f(x, y, z) = z/(x * y)(x square + y square) = c. So let us find fx derivative with respect to x.

When we differentiate with respect to x we get – z upon (x square * y square) (x square + y square) whole square and then we have x cube y. So we are differentiating with respect to x. So 3x square y and then x y cube so we get y cube okay and then we have fy similarly so – z/(x square * y square)(x square + y square) whole square okay and then we have derivative with respect to y so we have x cube derivative this side so x cube and then xy cube so 3xy square.

And fz similarly fz can be written as 1/xy * (x square + y square). So let us write the characteristic equations are dx/fx so we have – z/(x square * y square)(x square + y square) whole square (3x square y + y cube). Dy/ – z/(x square * y square)(x square + y square) whole square and then we have x cube + 3xy square. And then we have dz/fz which is 1/xy * (x square + y square).

So what we will do is I can write it as or x square * y square (x square + y square) whole square it will go up x square * y square (x square + y square) whole square/ - z times (3x square y + y cube). Let us write like this dx and then we have x square * y square (x square + y square) whole square/ – z * (x cube + 3xy square). And here what we have this is dy and here we have xy * (x square + y square) * dz/1 okay.

So what we do first we divide by we first/ x square * y square (x square + y square) whole square let us divide that okay. So we get dx/ – z and here also I can take y common so - yz and then we have (3x square + y square) okay and here x square * y square (x square + y square) whole square we divide so we get – dy/ - xz * (x square + 3y square) and we are dividing by x square * y square (x square + y square) whole square so we get dz/xy * (x square + y square) okay. Now what we do?

We multiply by xyz in the numerator so if we do that we have xdx divided by we are multiplying by - xyz so xdx/(3x square + y square) and then we have dy/x square + ydy/(x square + 3 y square) and here we get xyz, - xyz so - z dz/(x square + y square) okay. Now what we do is when I add x this one and this one I get 4 times okay so this gives you okay we let goes to the next page.

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The characteristic equations are

$$\frac{dx}{\partial f/\partial x} = \frac{dy}{\partial f/\partial y} = \frac{dz}{\partial f/\partial z}$$

which implies

$$\frac{x\,dx}{3x^2 + y^2} = \frac{y\,dy}{x^2 + 3y^2} = \frac{-z\,dz}{x^2 + y^2}.$$

So we get this x dx/(3x square + y square) = y dy/(x square + 3 y square) = –z dz/(x square + y square). Now from here what do you notice x dx + ydy/(4x square + 4y square) so 4 * (x square + y square) = - zdz/(x square + y square). So these 2 cancel out and we get xdx + ydy = - 4z dz. This gives you an integration x square/2 + y square/2 = - 2z square + some constant.

But you multiply by 2 and you get x square + y square + 4z square = some constant let us say c1 okay. This is 1 solution another solution, let us find so we subtract now x dx - ydy let us consider this and you see that 3x square - x square is 2x square, y square - 3y square is - 2y square so we get 2 times (x square - y square) and what we do here –zdz/(x square + y square) you can put from here (x square + y square) is - c1 - 4z square.

So let us put that c1 - 4z square here. Now we can integrate easily. So this is 1/2 times now xdx – ydy/(x square - y square) is 1/2 ln (x square - y square) and here what do we get c1 - 4z square if you put = t, and then c1 - 4z square we put = t and then - 8z dz = dt. So we get z - zdz as dt/8 and so we get dt/1 so this is 1/8 ln (c1 - 4z square) + some constant okay. So we can multiply by 8 and that will give you 2 * ln (x square – y square) = ln (c1 - 4z square) + ln c2.

And this gives you x so we get (x square - y square) whole square c2 * (c1 - 4z square). Now (c1 - 4z square) is (x square + y square) so c2 times (x square + y square) okay. So this is another 1 solution. Now let us find the required surface x square - y square = a square, z = 0. So (x square -

y square) we are given that given that (x square – y square) = a square, and z = 0 okay z = 0. In the plane xy we are given a hyperbola.

So here what we get? c2 will be = (x square + y square), x square this c2 I can write as phi c1, let c2 be phi c1 okay. So we get (x square – y square) a4 (x square - y square) a4 = phi c1 * (x square + y square). Now this is what okay. So (x square + y square) and c1 is what x square + y square + 4z square that is c1 okay. So what we get? phi c1 = a4/(x square + y square) okay. We have c2 = this okay.

We can write c1 = x square + y square + 4z square, c2 = what is the relation between c2 and c1? Let us find that. So c2 = a4/c1, - c1 okay you can see. c2 = a4/(x square + y square) and x square + y square is c1 - 4z square. So c1 * c2 = a4 okay. And let us now put the value of c1 and c2. So we get it the following okay. c1 * c2 = (x square + y square + 4z square) * c1 * c2 is what? (x square - y square) whole square/(x square + y square) = a4.

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Hence the required surface is given by

$$x^2 + y^2 + 4z^2 = \frac{a^4(x^2 + y^2)}{(x^2 - y^2)^2}.$$

So that is the surface okay x4 x so we have this as the required surface. With this I would like to end my lecture. Thank you very much for your attention.