**Ordinary and Partial Differential Equations and Applications**
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**Lecture - 34**
**Existence and Uniqueness of Solutions**

Hello friends. Welcome to my lecture on existence and uniqueness of solutions. First, we shall consider a quasi-linear partial differential equation of first order.

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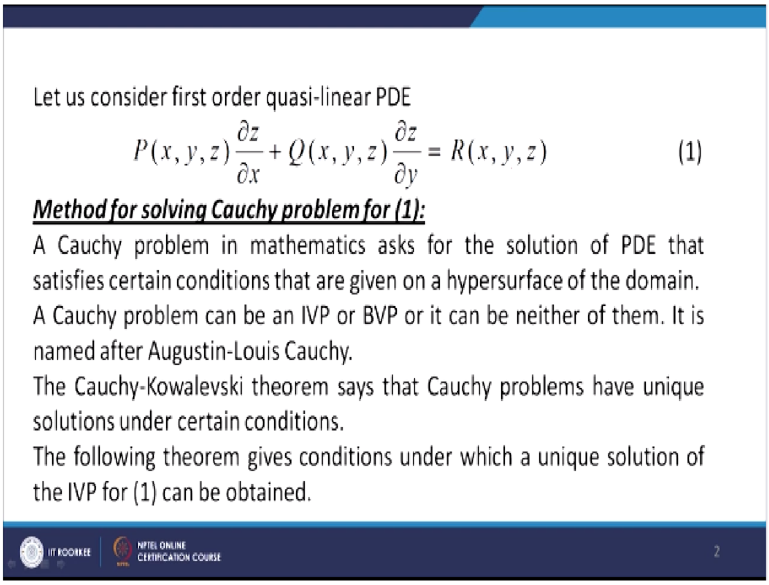


Let us consider the equation as P x, y, z*the partial derivative of z with respect to x+Q x, y, z*the partial derivative of z with respect to y=R x, y, z. You can see that it is a quasi-linear PDE of first order. Now we shall be discussing the method for solving the Cauchy problem for this quasi-linear PDE. A Cauchy problem is what, a Cauchy problem in mathematics asks for the solution of partial differential equation that satisfy certain conditions that are given on a hypersurface of the domain.

A Cauchy problem can be an initial value problem or boundary value problem or it can be neither of them, it is named after Augustin-Louis Cauchy. The Cauchy-Kowalevski theorem says that the Cauchy problems have unique solutions under certain conditions. Let us discuss the theorem which gives the conditions under which a unique solution for the initial value problem for the first order quasi-linear PDE given by equation 1 can be obtained.

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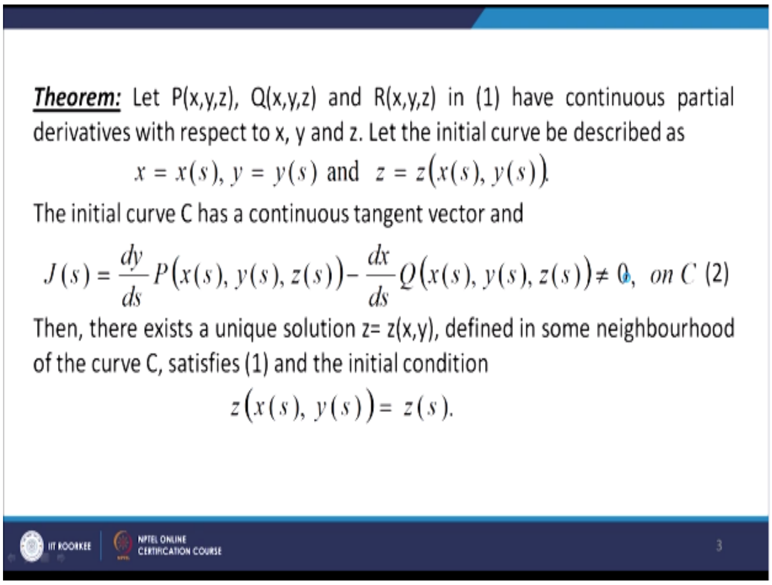


So let us discuss the theorem, it says that let P x, y, z; Q x, y, z and R x, y, z in the equation 1 have continuous partial derivatives with respect to x, y, and z and let us assume that the initial curve which we denote by C we described as x=x s, y=y s and z=z x s, y s. The initial curve C we assume that has a continuous tangent vector and J s which is given by dy/ds*P-dx/ds*Q x s, y s, z s is !=0 on the curve C.

Then, we can find a unique solution z=z x, y defined in some neighborhood of the curve C which satisfies the equation 1 and the initial condition z x s, y s= z s. So let us discuss the proof of this theorem.

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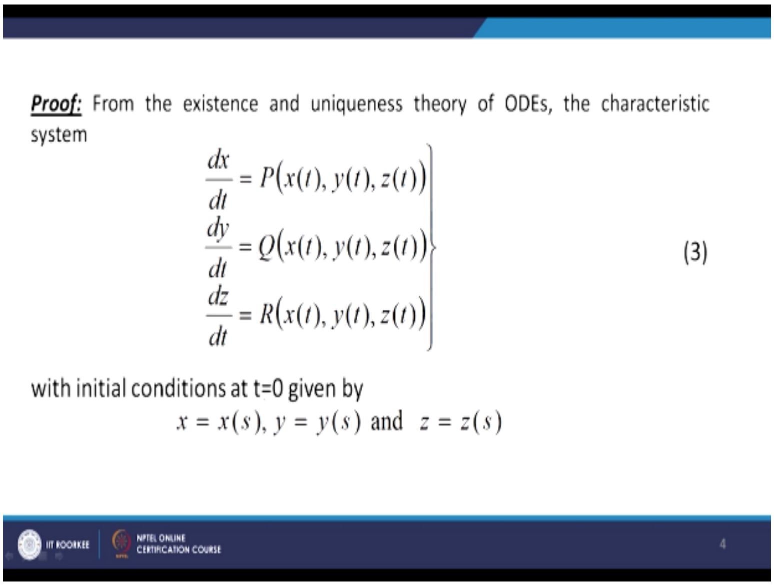


From the existence and uniqueness theorem of the ordinary differential equations, we have already discussed that the existence and uniqueness theory of ordinary differential equations, the characteristic system dx/dt=P x t, y t, z t; dy/dt=Q x t, y t, z t and dz/dt=R x t, y t, z t with initial conditions at t=0.

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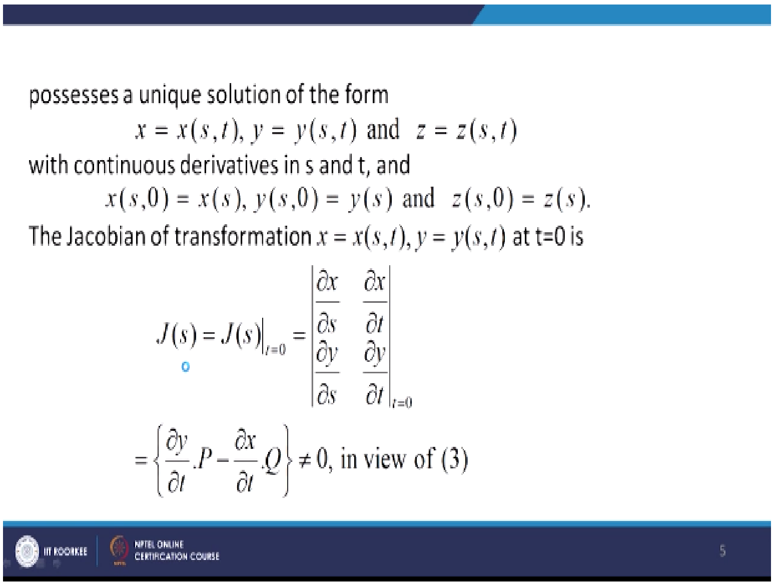


At t=0, the initial conditions are x=x s, y=y s and z=z s possesses a unique solution of the form x=x s, t; y=y s, t and z=z s, t with continuous derivatives in s and t and at t=0 x=x s, y becomes y s and z=z s. The Jacobian of the transformation x=x s, t; y=y s, t at t=0 is given by J s at t=0 which is=J s that is partial derivative of x with respect to s, partial derivative of x with respect to t then partial derivative of y with respect to s and then partial derivative of y with respect to t.

Now this is equal to partial derivative of y with respect to t*P-partial derivative of x with respect to t*Q and we have assumed that J s is !=0 on C, so in view of our assumption this Jacobian does not vanish and we have also assumed that C has a continuous tangent vector.

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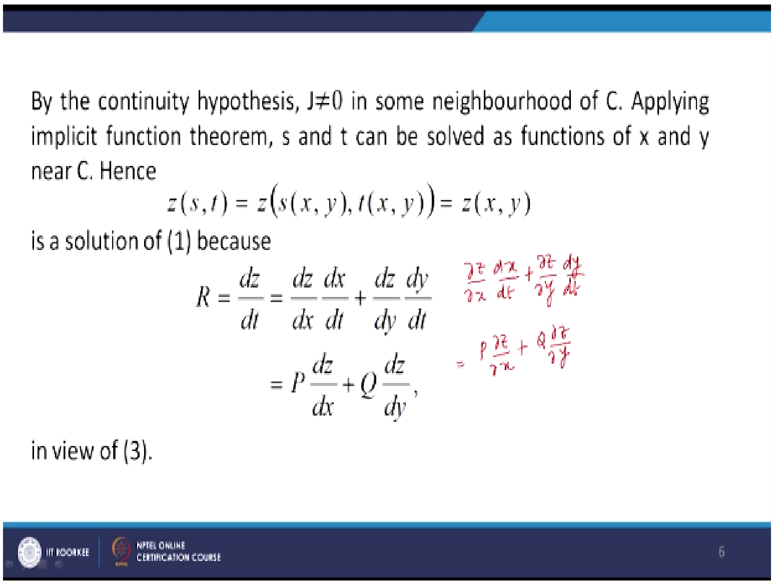


So by the continuity hypothesis J !=0, this J!=0 in some neighborhood of the curve C. Now let us apply the implicit function theorem and when we apply the implicit function theorem it follows that s and t can be solved as functions of x and y near C okay. So z s, t is now z of s x, y and t x, y, s and t are now functions of x and y because that follows from the implicit function theorem.

So z is also a function of x and y, z s, t becomes a function of x and y, so this is a solution of 1. Now this z s, t=z x, y is the solution of 1 because R=dz/dt from the characteristic theorem R=dz/dt and dz/dt now can be written as dz/dx*dx/dt because z is a function of x and y, so dz/dx*dx/dt+dz/dy*dy/dt and from the characteristic system we know that dx/dt=P and dy/dt=Q so P*dz/dx+Q*dz/dy.

There should not be actually the partial derivatives, we should be writing like this. This should be replaced by this and then this should be = dx/dt is P, so P partial derivative of z with respect to x+Q times partial derivative of z with respect to y okay. So when z s, t is expressed as a function of x and y okay we find that it is a solution of 1 because R=when we find dz/dt it turns out that it is=P partial derivative of z with respect to x+Q partial derivative of z with respect to y which is=R.

So R=this and therefore z is a solution of z=z x, y is a solution of the given quasi-linear partial differential equation of first order.

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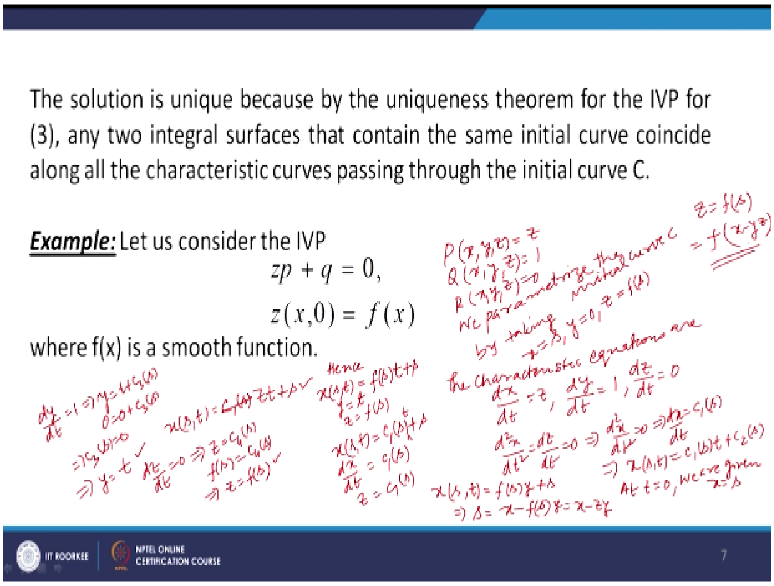


Now let us look at the unicity of this solution. The solution is unique because by the uniqueness theorem for the initial value problem for the characteristic system 3, for this system the uniqueness follows because any 2 integral surfaces that contain the same initial curve coincide along all the characteristic curves which pass through the initial curve C, so this is how we prove the existence and uniqueness of the solution of quasi-linear PDE.

So we have discussed the method for solving the Cauchy problem for this quasi-linear PDE. Now let us apply this result, so let us consider the initial value problem zp+q=0 and where z x, 0 is given=so initial condition is given z x, 0=f x where f x is a smooth function. Now what we do, so here when we compare this with the given form 1 then what do we notice, let us write the values of P, Q and R.

So P x, y, z=z; Q x, y, z=1 and R x, y, z=0, what we do is to solve this initial value problem let us first parameterize the curve C okay. So let us first parameterize the initial curve, so we parameterize the initial curve C by taking x=so let us take x=s, y=0 then z=f s okay. Now the characteristic equations are dx/dt=P so we get z, dy/dt=Q so we get 1, dz/dt=R so we get 0 okay.

Now what do we get dx/dt=z, so this gives you d square x/dt square, we have to solve this characteristic equation. So d square x/dt square=dz/dt and dz/dt=0, so what do we get, d square x/dt square=0, x is a function of s and t. So when we integrate d square x/dt square twice with respect to t what do we get, if you integrate it once we get x=some dx/dt=c1 s okay, we are integrating with respect to t keeping s as a fixed quantity.

So dx/dt=c1 s and when we integrate it once more we get x s, t=c1 s*t+c2 s okay. Now we are given that t=0, at t=0 x=s okay, so what do we do, at t=0 we are given x=s and when we put t=0 in this x s, t becomes s, so what do we get c2 s=s okay, so we get x s, t=c1 s+s okay and when we differentiate it dx/dt=what we get c1 s*t+c2 s, c2 s=s because at t=0 x s, t=s and then dx/dt=c1 s from here so what do we get, at dx/dt=z so z=c1 s okay.

We are given dx/dt=z, so this we get x s, t=c1 s=z so z*t+s. This is x s, t and dy/dt=1 gives you y=t+some constant c3 s. Now at t=0, y=0, so what do we get, 0=0+c3 s so c3 s=0 gives you y=t okay, so we have x s, t=zt+s y=t and dz/dt=0 gives you z=c4 s. Now at t=0, z=f s, so what do we get f s=c4 s. So we get z=f s okay. So we have solved this system of characteristic system.

We have got x=zt+s, y=t, z=f s okay so what do we get now then. We can write x s, t okay in terms of s and t we want to write. So z we will put as f s so f s*t+s, y=t and z=f s okay. Now using y=t we can write x s, t=f s*y+s, so we need to solve this system for the values of s and t in terms of x and y okay. We have to write s as x, y s x, y and t as t x, y. So what do we get we can write s=x-f s*y or x-zy.

So we get s=x-zy and z=f s so what do we get z=f s gives you f of s=x-yz, so what do we get is the solution of the given initial value problem is then obtained in an implicit form z=f of x-yz.

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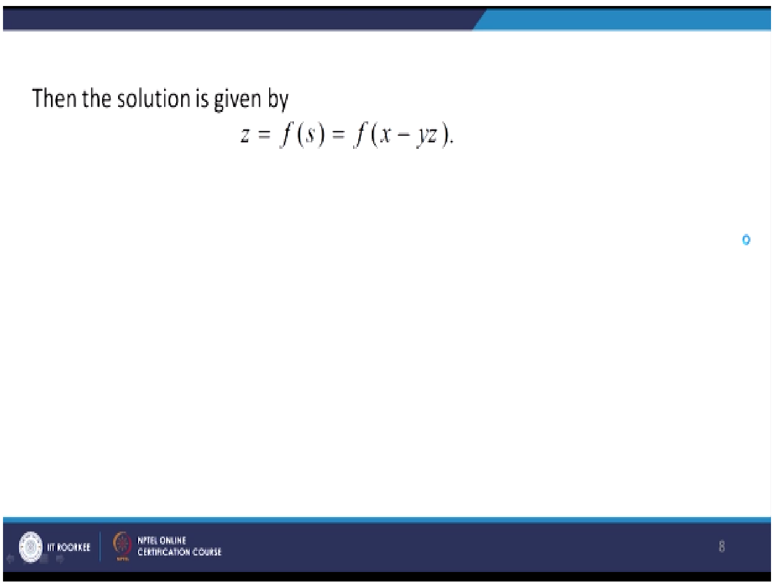

So the solution is z= f s and s is x-yz, so the solution is obtained in the form of an implicit equation.

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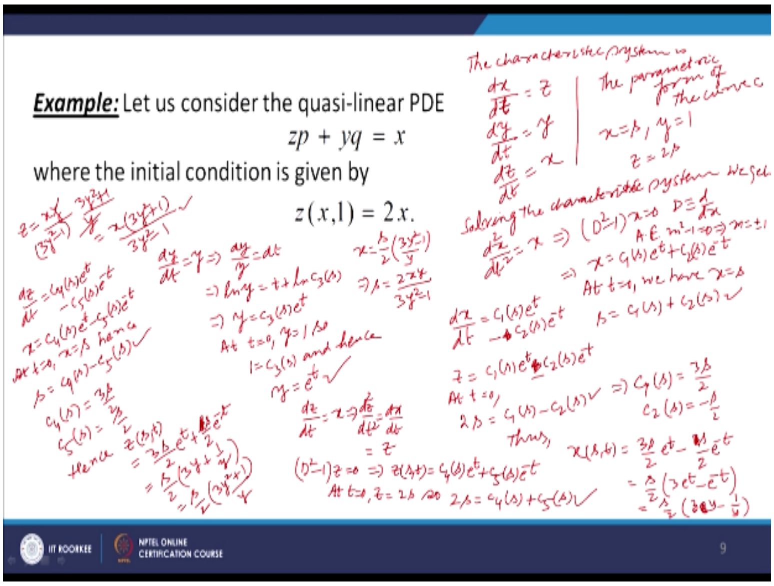


Now let us consider another problem suppose we have the quasi-linear PDE, zp+yq=x where the initial condition is given by z x, 1=2x. So let us write the characteristic system here. The characteristic system is dx/P okay let us write like this dx/dt=P that is z, dy/dt=Q which is y and then dz/dt=x, so we get the characteristic equations dx/dt=P x, y, z P x, y, z is z; dy/dt=Q x, y, z which is y, dz/dt which is = x.

And let us parameterize the parametric form of the curve C, so what we do, let us put x=s, y=1 and z=2s okay. So if we do this what we will get then let us solve this characteristic system. Solving the characteristic system, we get dx/dt=z, see dx/dt=z so we get d square x/dt square=dz/dt and dz/dt=x, so we get d square x/dt square=x and this is nothing but d square-1x=0 where d denotes d/dx.

So solving this second order linear differential equation in x what we get is the general solution x is equal to because the auxiliary equation is m square-1=0, so this gives 2 distinct roots m=+-1 and corresponding to 2 distinct roots of the auxiliary equation we have the solutions e to the power t and e to power –t of this equation so c1 s*e to the power t+c2 s*e to the power –t we can write for x, this x is x s, t.

Now what we have at t=0 we are given x=s, so we have s=c1 s+c2 s. Now we need to find one more condition in order to determine c1 and c2. So we use the condition that dx/dt=z, so dx/dt=z and dx/dt here is what c1 s*e to the power t+c2 s*e to the power –t. Now dx/dt is z, z=c1 s e to the power t+c2 s e to the power –t, put t=0 at t=0 z=2s, so 2s=c1 s, when we differentiate we should have got a negative sign here, so this is negative okay.

So –c2 s okay, now let us solve this equation and this equation for the values of c1 and c2. So when we add them we get 2c1 s=3s so c1 s=3s/2 and c1 s=3s/2 means c2 s=-s/2 okay. Thus, we have solved this equation d square x/dt square=s and we got x s, t=3s/2 e to the power t-s/2 e to the power –t okay. Now we have the second equation dy/dt=y, so dy/dt=y gives you dy/y=dt which gives lny=t+ln, let us say c3 s. So what we get y=c3 s times e to the power t.

At t=0 we are given y=1, so 1=c3 s and hence we get y=e to the power t. Now let us solve for z. We have dz/dt=x. So let us differentiate once more we get d square z/dt square=dx/dt which is = z. So we get again d square-1 z=0 which gives you z s, t=some constant c4 s*e to power t+c5 s e to the power–t. We put t=0, at t=0 z=we are given z=2s so 2s=c4 s+c5 s. So we have got one condition involving c4 and c5.

We need one more condition in c4 and c5 to determine their values, so let us find dz/dt, dz/dt=c4 s e to the power t-c5 s e to the power–t and dz/dt=x. So we get at t=0, x is given to be = s, so s=c4 s-c5 s okay. Now let us take this equation and this equation and then determine the values of c4 and c5. So when we add this equation with this we get c4 s=3s, so c4 s then=3s/2 and c5 s will then be=s/2.

And so z s, t will be = 3s/2 e to the power t+s/2 e to the power–t okay, so we have got the values of x s, t; y s, t; z s, t as functions of s and t. Let us now solve them to determine s as a function of x and y and t also as a function of x and y. So from this equation let us see, this is equal to we can write s/2 times 3 e to the power t-e to the power –t okay. Now we use this equation y=e to power t so s=s/2 3y-1/y okay.

Ad this gives you what we get x=3s/2 and we get 3y square-1/y okay. So s=we can find s in terms of x and y here, 2xy/3y square-1. Now what we do here s=s/2 3e to the power t+e to the power –t so 3y+1/y okay and this gives you s/2 3y square+1/y okay. Now we can use the value of s as 2xy/3y square-1 here and get the value of z in terms of x and y. So z=s/2, s/2 is xy/3y square-1*3y square+1/y.

So this I can cancel and I get x times 3y square+1/3y square-1. So we have got z=x times 3y square+1/3y square-1 which is in explicit solution of the given quasi-linear PDE z=3y square+1*x/3y square-1.

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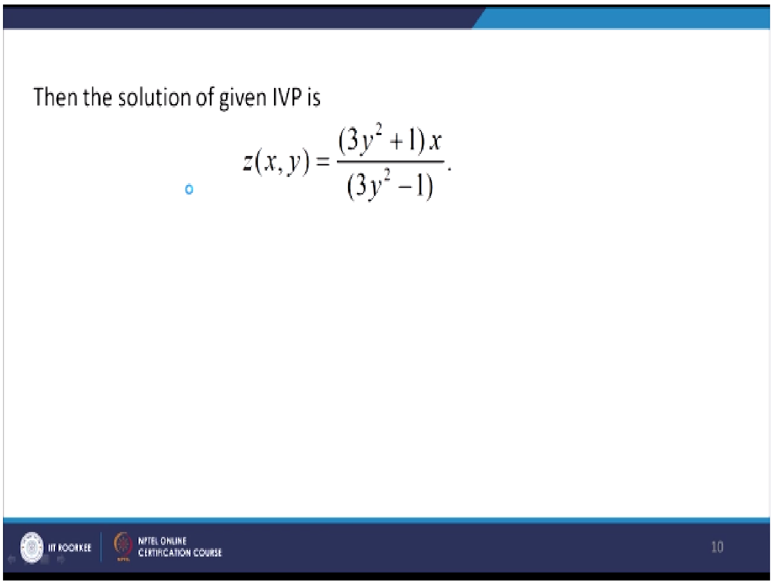


So this is how we solve the quasi-linear PDE zp+yq=x where the initial condition is given by z x, 1=2x. So with this I would like to end my lecture. Thank you very much for your attention.