Ordinary and Partial Differential Equations and Applications
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Lecture - 33
Initial Value Problem for Quasi-Linear First Order Equations

Hello friends. Welcome to my lecture on initial value problem for quasi-linear first order equations.

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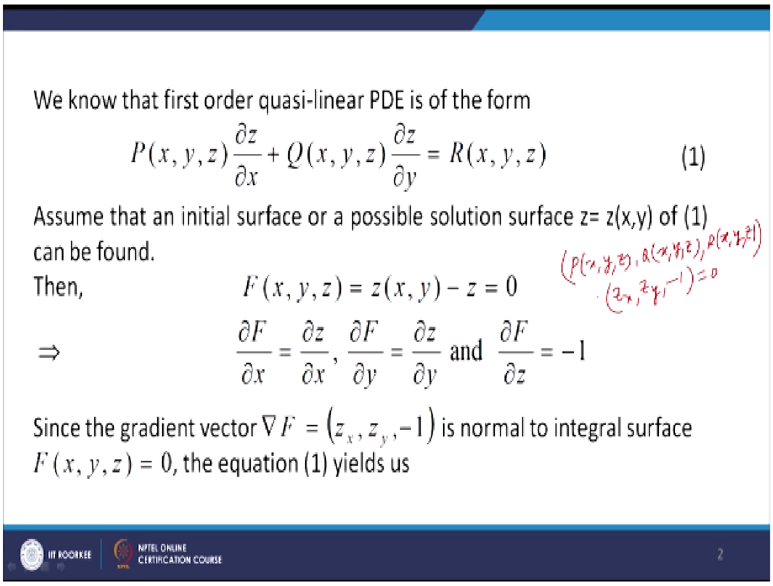

We know that a first order quasi-linear partial differential equation is of the form P x, y, z*partial derivative of z with respect to x which we have denoted by p earlier and then +Q x, y, z*partial derivative of z with respect to y which we have denoted by q earlier is=R x, y, z. Now let us assume that an initial surface or a possible solution surface z= z x, y of the equation 1 can be found.

Then, we can write this equation z=z x, y as z x, y-z=0, let us write z x, y-z as a function of x, y, z. So F x, y, z will be=0 then the partial derivatives of F with respect to x, y and z can be determined, partial derivative of F with respect to x is derivative of z with respect to x, partial derivative of F with respect to y is partial derivative of z with respect to y and partial derivative of F with respect to z is -1.

Now we know that F x, y, z=0 represents a surface and del F is a vector which is normal to the surface. So here the components of del F which are the partial derivatives of F with respect to x, y, z are zx, zy and -1, so the gradient vector del F=zx, zy, -1 is normal to the integral surface F x, y, z=0. The equation 1 can be written as P x, y, z; Q x, y, z; R x, y, z dot zx zy -1=0.

Let us take the dot product of the vectors P x, y, z; Q x, y, z; R x, y, z with zx, xy, -1=0 then what we get is this equation.

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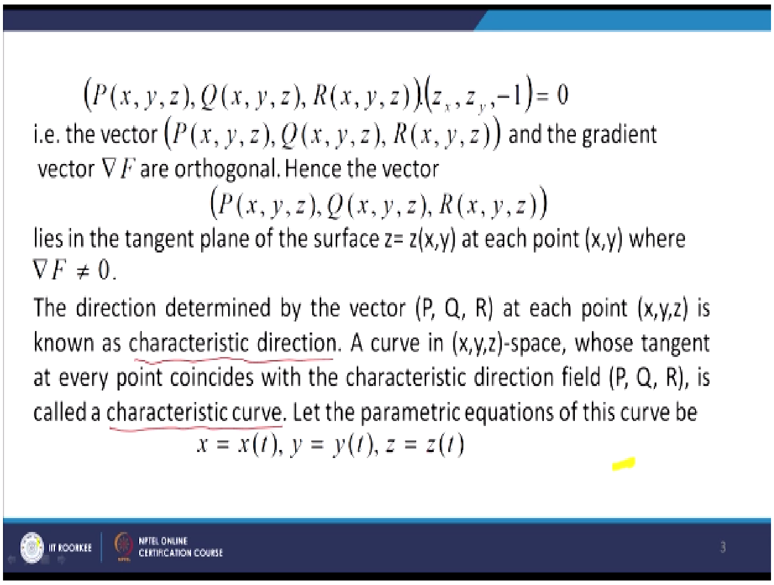

So what do we obtain, the vector P x, y, z; Q x, y, z; R x, y, z is orthogonal to the vector zx, zy, -1 but zx, zy, -1 is a vector along the normal to the surface F x, y, z=0 so the vector P x, y, z; Q x, y, z; R x, y, z is a vector which lies in the tangent plane to the surface z=z x, y at each point x, y where gradient of F is not 0. Now the direction determined by the vector P, Q, R that is P x, y, z; Q x, y, z; R x, y, z which we are writing in short as P, Q, R.

The direction determined by the vector P x, y, z; Q x, y, z; R x, y, z at each point x, y, z is known as the characteristic direction. A curve in x, y, z space whose tangent at every point coincides with the characteristic direction field P, Q, R is called as a characteristic curve. At each point of this characteristic curve, the tangent coincides with the characteristic direction that is the direction given by the vector P, Q, R.

Now the parametric equations of this characteristic curve let be x=x t, y= y t, z= z t. Then, the tangent vector to this curve we know if the curve in the space is given by x=x t, y=y t, z=z t.

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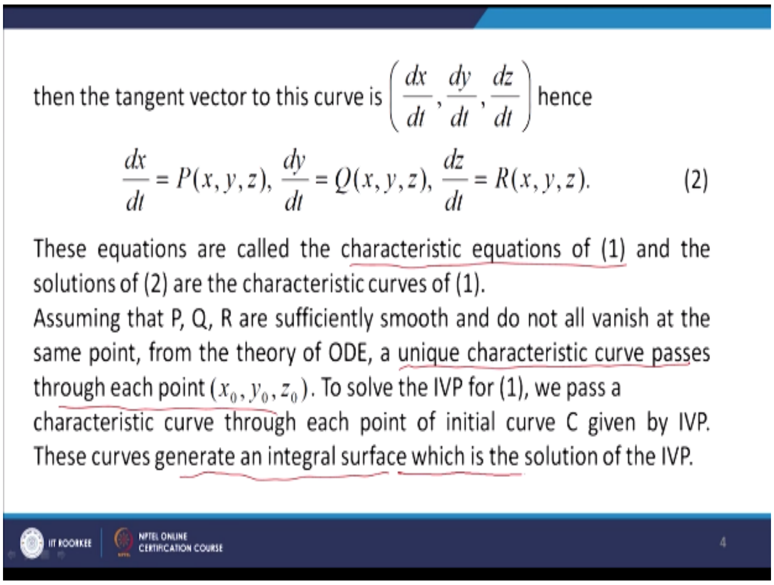

Then, the tangent vector to this curve be given by dx/dt, dy/dt, dz/dt and therefore dx/dt will be=P x, y, z; dy/dt=Q x, y, z; dz/dt=R x, y, z because of this because this curve is a characteristic curve. So at each point of this curve the tangent to this curve will coincide with the characteristic direction and so dx/dt will be P x, y, z; dy/dy will be Q x, y, z; dz/dt will be R x, y, z. These equations are called as the characteristic equations of the quasi-linear PDE.

And the solutions of 2 are called the characteristic curves of 1, so the solutions of this equation dx/dt=P x, y, z; dy/dt=Q x, y, z; dz/dt=R x, y, z are called the characteristic curves of the quasi-linear PDE. Now assuming that P, Q, R are sufficiently smooth and do not all vanish at the same point then from the theory of ordinary differential equation a unique characteristic curve passes through each point x0, y0, z0.

So this is to be noted if P, Q, R are not of I mean do not vanish at the same point then a unique characteristic curve passes through each point x0, y0, z0. Now to solve the initial value problem for the quasi-linear PDE we pass a characteristic curve through each point of the initial curve C given by the IVP. In the initial value problem, you will be given an initial curve at t=0.

So to solve the initial value problem, we will have to pass the characteristic curve through each point of the curve C. These curves generate an integral surface which is the solution of the IVP.

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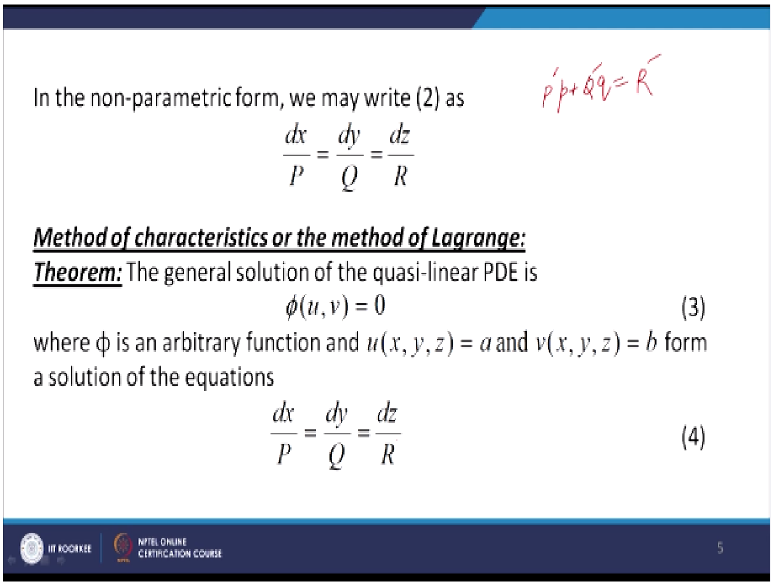

In the non-parametric form, we can write the equation 2 as dx/P=dy/Q=dz/R. Now let us look at the method of characteristics or the method of Lagrange. So the general solution of the quasi-linear PDE is the theorem says that the general solution of the quasi-linear PDE Pp+Qq=R where P, Q, and R they are functions of x, y, z.

So the general solution of this quasi-linear PDE is phi u, v=0 where phi is an arbitrary function and u x, y, z=some constant a and v x, y, z=some constant b they form a solution of the characteristic equations in the non-parametric that is dx/P, dy/Q, dz/R. Now let us prove this theorem. So we have to prove that the general solution of the quasi-linear PDE is given by phi u, v=0 if u x, y, z=a, v x, y, z=b form a solution of the characteristic equations dx/P=dy/Q=dz/R and phi is an arbitrary function.

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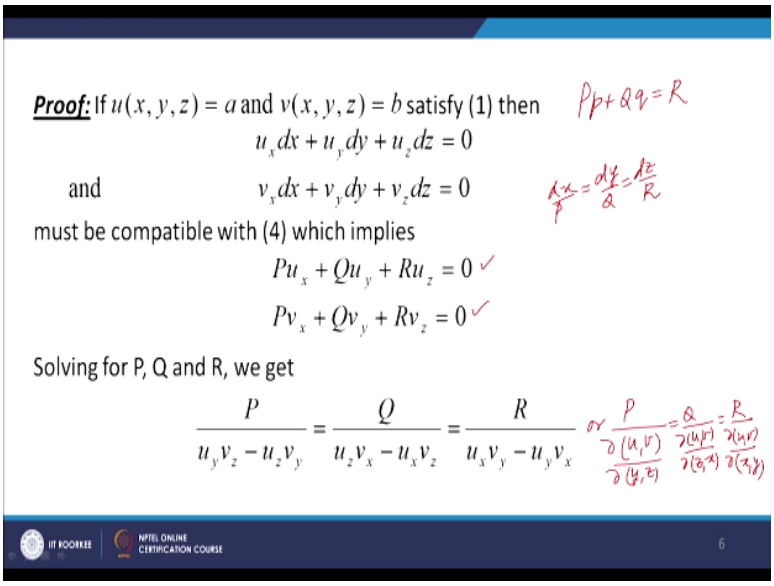

So let us see how we prove this, so if u x, y, z=some constant a and v x, y, z=some constant b they satisfy the quasi-linear PDE that is Pp+Qq=R then from u x, y, z=a we see that the partial derivative of u with respect to x that is ux*dx+ partial derivative of u with respect to y that is uy*dy+partial derivative of u with respect to z that is uz*dz is 0. Similarly, v x, y, z=b gives you partial derivative of v with respect to x*dx+partial derivative of v with respect to y*dy+partial derivative of v with respect to z*dz=0.

Now these equations must be compatible with 4 this characteristic equation because u x, y, z=a and v x, y, z=b form a solution of this characteristic equation so they must be compatible with 4 which implies that now 4 gives what 4 is dx/P=dy/Q=dz/R okay so if these equations are compatible with this equation then what we will have P*dux+Q*uy+R*uz=0 and then P*vx+Q*vy+R*vz=0.

Now we solve these two equations this and this for P, Q, and R what we will get P/uy vz-uz vy=Q/uz vx-ux vz and then R/ux vy-uy vx. So this is what we get when we solve these two equations for P, Q and R. Now we can write them in the form of the Jacobian. This gives you P/Jacobian of u, v with respect to y, z okay and then we will get the Jacobian of here u, v again with respect to z and x.

And we get here Jacobian of u, v with respect to x, y okay. So we can write these ratios in the form of Jacobian P/delta u, v/del y, z and then del u, v/del z, x and then del u, v/del x, y.

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or

$$\frac{P}{\partial(u,v)/\partial(y,z)} = \frac{Q}{\partial(u,v)/\partial(z,x)} = \frac{R}{\partial(u,v)/\partial(x,y)} \qquad (5)$$

$\phi(u,v)=0$, $u=u(x,y,z)=c_1$, $v=v(x,y,z)=c_2$

Differentiating (3), partially with respect to x and y we have

$$\frac{\partial \phi}{\partial u}\left\{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right\} + \frac{\partial \phi}{\partial v}\left\{\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right\} = 0$$

$$\frac{\partial \phi}{\partial u}\left\{\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right\} + \frac{\partial \phi}{\partial v}\left\{\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right\} = 0.$$

So we get this now differentiating 3. Let us differentiate 3, 3 is it is phi u, v=0 so let us see when we differentiate phi u, v=0 where u=u x, y, z and v= v x, y, z; u x, y, z=c1 and v x, y, z=c2. So when we differentiate this equation phi u, v=0 partially with respect to x what we get phi u*ux+uz*zx+phi v*vx+vz*zx=0 and similarly when we differentiate it with respect to y we get phi u*uy+uz*q+phi v*vy+vz*q=0. So this is what we get when we differentiate phi u, v=0 with respect to x and y.

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Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$, we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} & \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y} \end{vmatrix} = 0$$

or $$p\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial z}\right) + q\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial z}\frac{\partial u}{\partial x}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right)$$

or $$p\frac{\partial(u,v)}{\partial(y,z)} + q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)}$$

or $Pp + Qq = R$, in view of (5).

And then when we eliminate phi u and phi v from these two equations what we will get, we will get the determinant of ux+uz*p uy+uz*q and vx+vz*p vy+vz*q=0. So this determinant will be equal to 0 and when we solve this determinant we get this equal to 0 so when we put p for zx and q for zy and solve okay simplify this equation, we get p times uy vz-vy uz+q times uz vx-vz ux and then ux vy-vx uy and which we can write in the form of Jacobian as p*del u, v/del y, z+q del u, v/del z, x=del u, v/del x, y.

And then we can see in view of 5 this equation okay, let us use this equation 4 then what do we notice this equation will be replaced by Pp+Qq=R in view of 5.

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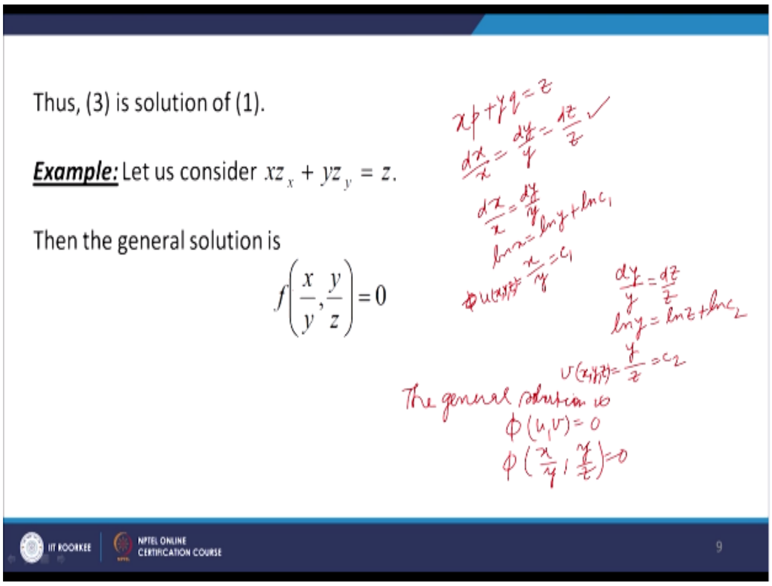

So we see that phi u, v=0 is the solution of equation 1 if u x, y, z=constant and v x, y, z=constant satisfy the characteristic equations. Now let us look at this equation xzx+yzy=z. So this is of the form xp+yq=z. So it is quasi-linear first order PDE, so we can write the characteristic equations dx/x=dy/y=dz/z/ Now solving the relation dx/x, there are two independent relations.

So dx/x=dy/y gives us lnx=lny+lnc1, so we get x/y=c1 so this is one solution. We can take another independent relation dy/y=dz/z which gives us lny=lnz+lnc2 and we can write y/z=c2 so we get this is your u x, y, z and this is our v x, y, z. So we got two solutions u x, y, z=constant, v x, y, z=constant which are the solutions of the characteristic equation dx/x=dy/y=dz/z.

Hence, the general solution is given by phi u, v=0. So that means phi of x/y and y/z=0. We can instead of taking dy/y=dz/z one can take the dx/x=dz/z also. So there are two independent relations, third one is dependent on the other two. So we can write the general solution of this quasi-linear PDE as an arbitrary function phi of x/y, y/z=0 here we have written f instead of phi.

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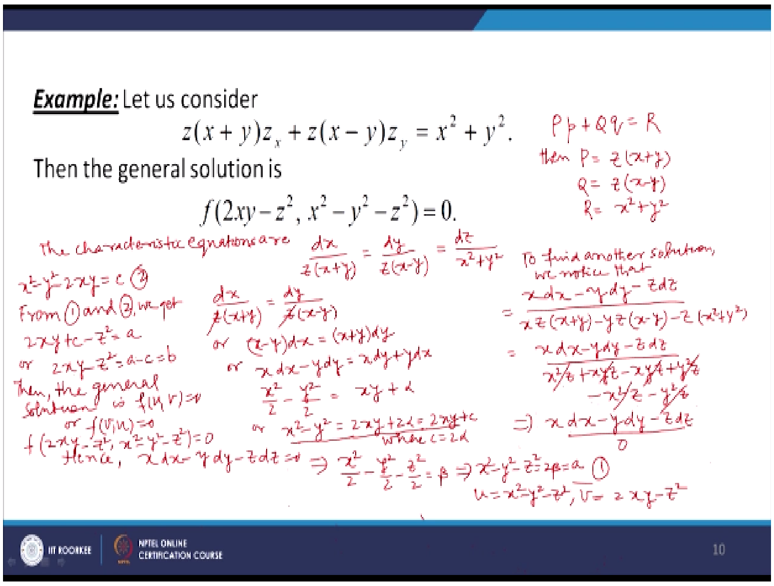

Now let us go to another problem, so we have a quasi-linear PDE of first order. Let us compare this given partial differential equation with the standard form Pp+Qq=R then P=z*x+y or Q=z*x-y, R=x square+y square. The characteristic equations are dx/P that is z*x+y=dy/Q that is z*x-y and we have dz/R which is x square+y square. Now let us first consider dx/z times x+y=dy/z times x-y.

Then, we can write it as x-y*dx=x+y dy or I can write it as x dx-y dy=x dy+y dx. Integrating both sides I have x square/2-y square/2=xy+some constant say alpha okay or I can write x square-y square=2xy+some constant+2 alpha which I can write as 2xy+some constant c where c is 2 alpha okay so one solution we have found x square-y square-2xy=constant. Let us find another solution.

So we notice that xdx-ydy-zdz if you consider then this is x*z x+y-yz*x-y-z times x square+y square okay. So these ratios are equal to this, the characteristic equations are dx/P, dy/Q, dz/R which is also equal to this, so this is equal to xdx-ydy-zdz/x square z+xyz-xyz+y square z-x square z-y square z. So x square z cancel, y square z also cancels and xyz xyz cancel so this implies that xdx-ydy-dzdz/0 okay.

So this equal to this and this implies that hence xdx-ydy-zdz=0 when we integrate this we get x square/2-y square/2-z square/2=some constant let us say beta okay. So this gives you x square-y square-z square=2 beta which we can take as say a okay. So x square-y square-z square is=a, the other solution we have got x square-y square=2xy+c. So I can also write this solution this one okay this solution.

I can also write as x square-y square-2xy=c okay. Now from this solution let us say this is 1, this is 2 so from 1 and 2, if we write x square-y square=2xy+c then what we get from 1 and 2. I get 2xy+c-z square=a or I can say 2xy-z square=a-c which I can write as another constant b okay. So one solution is x square-y square-z square=u and the other solution we can take as 2xy-z square=b.

So then the general solution is f u, v=0 say u is x square-y square-z square and v we can take as 2xy-z square. So I can write f u, v=0 or I can write as f v, u=0. So we get the solution as f of 2xy-z square and then x square-y square-z square=0 so this is how we get the general solution of this quasi-linear first order PDE.

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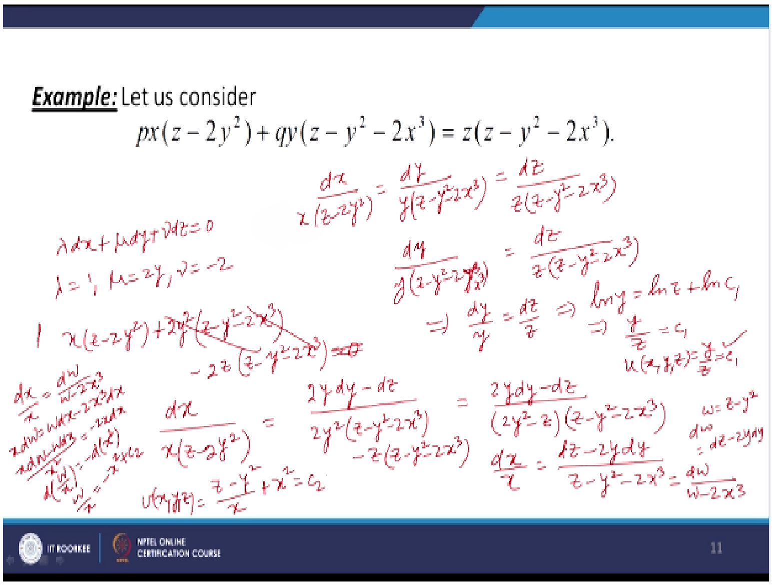

Now let us look at this equation. This is again a quasi-linear PDE of first order, so dx/x times z-2y square=dy/y*z-y square-2x cube and then dz/z*z-y square 2x cube. Now let us see these characteristic equations carefully. This second ratio dy/y times z-y square-2x cube=dz/z*z-y square-2x cube, if we take first this relation then we will easily able to get the first solution. So dy/y times z-y square-2y cube=dz/z*z-y square-2x cube okay.

So from this what we get this vector will cancel, we will get dy/y=dz/z and this will give you lny=lnz+lnc1. So we get one solution y/z=c1, so one solution we have got, we can take it as u x, y, z=y/z=c1, so one solution we have got. Now let us look at the other solution. So we have to choose lambda, mu, nu in such a way that lambda dx+mu dy+nu dz becomes 0. So what I do is I choose lambda as say 1, mu as 2y and nu as -2.

Then what I will get, so x times z-2y square we are multiplying by 1 so this+mu we are choosing 2y, so 2y by times this so 2y square z-y square-2x cube, what we have to do, what I will get -2z*z-y square-2x cube=0, will I get that 0? We should not try this. Let me see dx/x z-2y square is=and here what we will take 2ydy-dz let us try this okay. So here we take 2y so 2y square*z-y square-2x cube-z times z-y square-2x cube, let us do this okay.

So this is what 2ydy-dz/2y square-z and what I will get z-y square-2x cube okay. So what I will do, this z-2y square will cancel with z-2y square, we will get dx/x=yeah z-2y square will cancel with this dz-2ydy/z-y square-2x cube, we get this okay. Now let us take some w=z-y square, then dw will be=dz-2ydy. So this will be=dw/w-2x cube and let us solve this okay. So what I will get, dx/x=dw/w-2x cube and what I get x dw=w dx-2x cube dx which I can write as x dw-w dx/x square=-2x dx okay.

And this is d of w/x=-d of x square, so when we integrate we get w/x=-x square+a constant c2 okay and w is what z-y square, so we get z-y square/x+x square=c2. So this is another solution. We can write it as u x, y, z=this. So what we get, one solution is this y/z=c1 and the other solution is x square+zyx-y square/x=c2.

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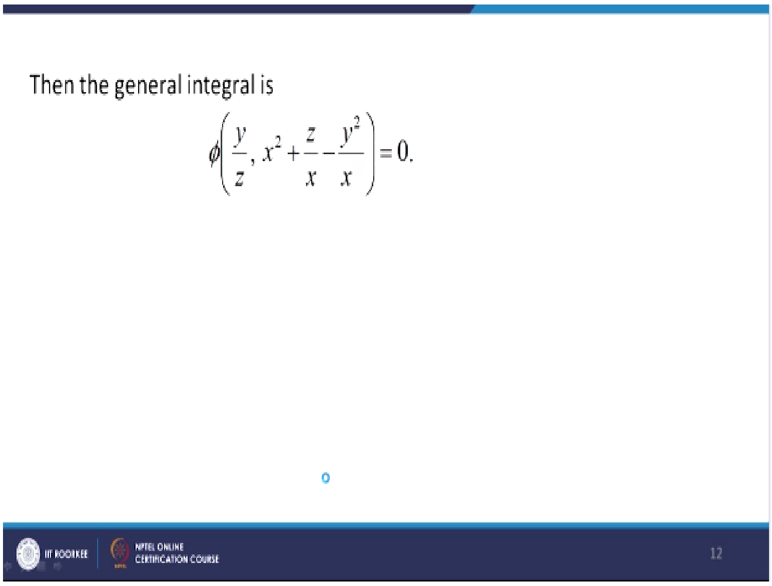

And we therefore have this general solution, phi y/z, x square+z/x-y square/x=0. So this is how we get the general integral for this equation.

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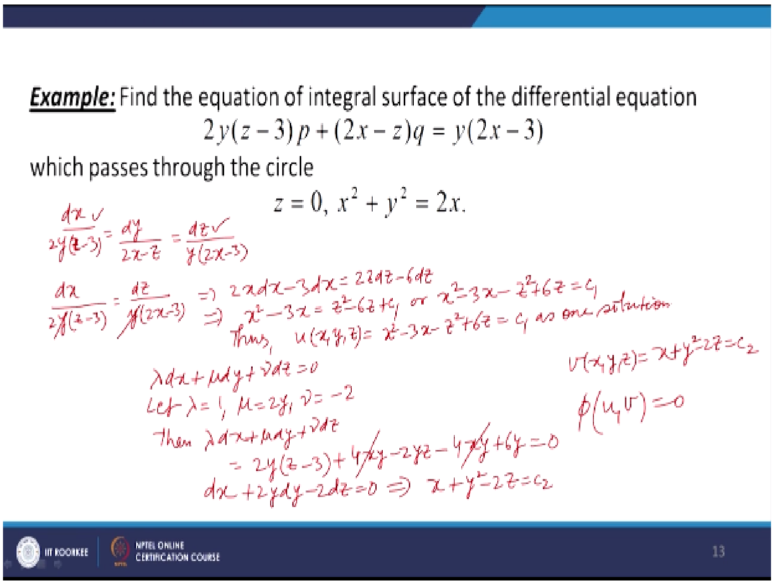

Now let us look at last problem of this lecture. Find the equation of integral surface of the differential equation first order quasi-linear PDE 2y z-3 p+2x-z q=y times 2x-3 which passes through the circle. So let us first solve this quasi-linear PDE. So we have the characteristic equations as dx/2y times z-3=dy/2x-z=dz/y times 2x-3. So these are the characteristic equations for this quasi-linear first order PDE.

So you can see we take dx/2yz-3 and dz/y times 2x-3 then y will cancel and we will be able to integrate easily. So dx/2yz-3 let us take dz/y times 2x-3. Then this y will cancel with y and we shall get 2xdx-3dx=2zdz-6dz which gives us on integration we get x square-3x then we get z square-6z+constant c1 or we can say x square-3x-z square+6z=c1. So thus we have u x, y, z=x square-3x-z square+6z=c1 is one solution.

Now let us try to find the other solution. So for that let us again try lambda, mu, nu such that lambda dx+mu dy+nu dz=0 where lambda, mu, nu are functions of x, y, z. So what I will do is let us take lambda=1, mu=2y and nu=-2 let us see this. So what we will get, then lambda dx+mu dy+nu dz, this will be giving you, we are multiplying by 1 so 2y z-3, mu is 2y so 2y*2x-z.

So we get 4xy-2yz and then nu is -2 so -4xy+6y and we can see that this cancels with this and then here we get 2yz which will cancel with this 2yz and -6y will cancel with 6y, so we get 0 and thus dx+2ydy-2dz=0 which gives us on integration x+y square-2z=a constant c2. So we got another solution v x, y, z=x+y square-2z=c2 and therefore the general solution is phi u, v=0.

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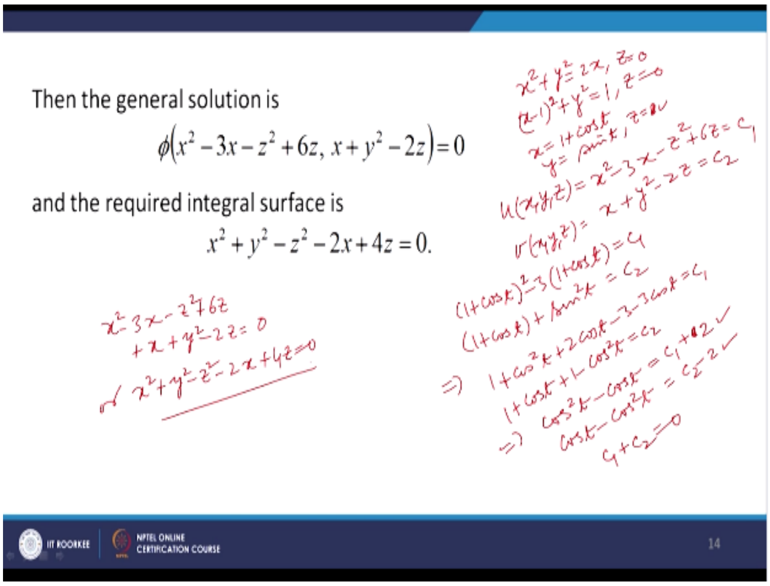

So we can write the general solution as phi x square-3x-z square+6z x+y square-2z, x+y square-2z x square-3x-z square+6z=0. Now let us find the integral surface which passes through the circle z=0, x square+y square=2x. So the circle is given by x square+y square=2x z=0 and this equation can be written as x-1 whole square+y square=1 z=0. Now let us write the equations of the circle in the parametric form.

So I can write x=1+cos t and y=sin t and z=t=0 so z=0 we get. Now what we will get, we have two solution u x, y, z=x square-3x-z square+6z=c1 okay so this is c1 and then v x, y, z=x+y square-2z=c2 okay. So we have to find the integral surface that passes through this curve. Then, let us use the parametric form of the curve that is the circle through which the general solution or the solution of this has to pass.

So x square-3x so 1+cos t whole square-3 times 1+cos t-z, z is 0 so we get 0 here, here also 0 so this is equal to c1 and then we have x is 1+cos t the second equation and then we get +sin square t and we get z=0 so this is equal to c2. Now let us square 1+cos t so this gives you 1+cos square t+2cos t-3-3cos t=c1 and here what we get 1+cos t+1-cos square t=c2. So simplifying we get cos square t, then we have here –cos t and then (()) (38:22) so we get c1+2 and here we get cos t-cos square t=c2-2.

So adding these two equations, this equation and this equation what do you notice, cos square t will cancel with cos square t, cos t will cancel with cos t, 2 will cancel with 2 and we will get c1+c2=0 okay. So c1+c2=0, if eliminating c1, c2 what we get, x square-3x-z square+6z that is c1 and c2 is x+y square-2z=0 okay c1+c2=0, so this can be written as x square+y square-z square and then we get -2x and we get 4z=0, so this is the required integral surface.

So in this lecture, we have looked at the initial value problem for the first order quasi-linear PDE. In the next lecture, we shall look at the existence and uniqueness of the solution of the quasi-linear PDE and we will discuss the Cauchy method to determine the solution of the first order quasi-linear PDE, uniqueness and existence problem we shall consider. Thank you very much for your attention.