Ordinary and Partial Differential Equations and Applications
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Lecture - 32
Origin and Classification of First Order PDE

Hello friends. Welcome to my lecture on origin and classification of first order partial differential equations which we write in short as PDE. First, we define a partial differential equation. An equation which involves partial derivatives of an unknown function of two or more independent variables is called a partial differential equation. These equations arise in connection with numerous physical end geometrical problems.

The independent variables involved in such problems may be the time variable and/or one or several coordinates in a space.

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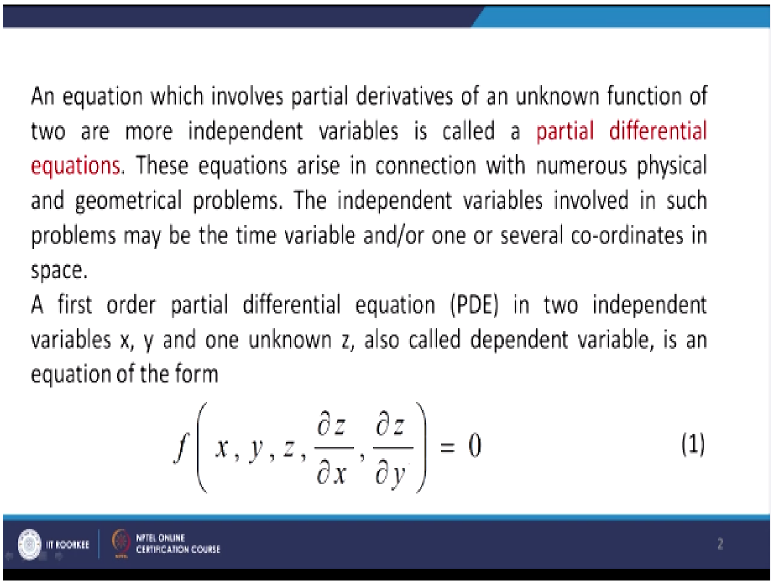

A first order partial differential equation in two independent variables x, y and one unknown z also called dependent variable is an equation of the form f x, y, z, partial derivative of z with respect to x then partial derivative of z with respect to y=0.

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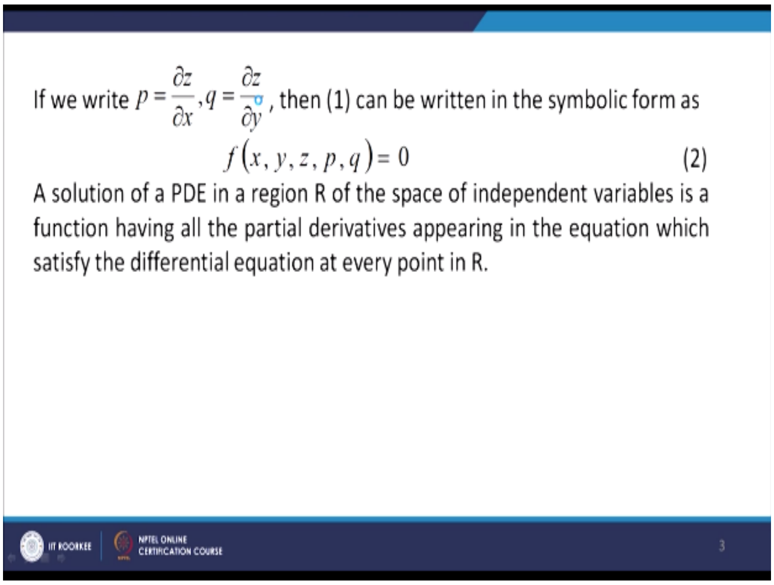

If we write p=partial derivative of z with respect to x and q=partial derivative of z with respect to y. Then equation 1 can be written as f x, y, z p, q=0. A solution of a partial differential equation in a region R of the space of independent variables, here we are taking them as x, y is a function having all the partial derivatives appearing in the equation which satisfy the differential equation at every point in R.

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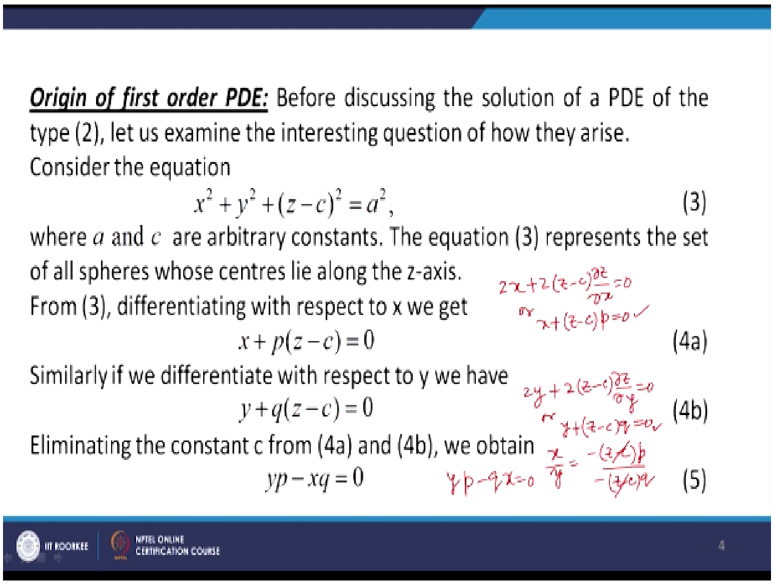

Let us see how the first order partial differential equation originate, so before discussing the solution of a partial differential equation of the type 2 that is f x, y, z, p, q=0 let us see how the partial differential equation of first order arise. So consider the equation x square+y square+z-c whole square=a square where a and c are arbitrary constants. You can see that this equation represents the set of all spheres whose centers 0, 0 c lie along the z-axis and the radius is a.

Now from this equation when we differentiate it partially with respect x we get 2x+2 times z-c*partial derivative of z with respect to x=0 or we can write it as x+z-c*p=0. Similarly, when we differentiate this equation x square+y square+z-c whole square=a square with respect to y we get 2y+2 times z-c*partial derivative of z with respect to y=0 or we can say y+z-c*q=0. Now if we eliminate from this equation x+z-c*p=0, y+z-c*q=0 the constant c then we can write it as x/y=z-c*p/z-c*q.

So we can say we get qx=yp or yp-qx=0, so eliminating the constant c from the equation 4a and 4b we obtain the first order partial differential equation by p-xq=0 which is a partial differential equation of first order.

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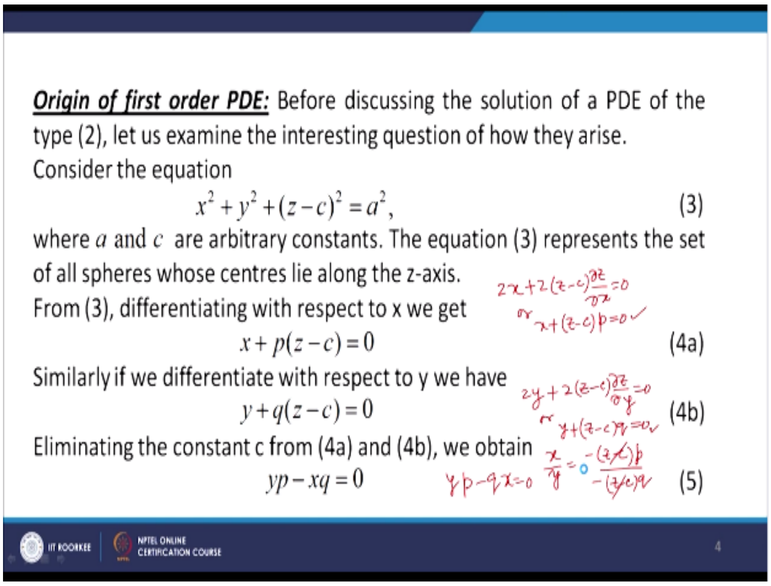

Now in some sense then the set of all spheres whose centers lie on the z-axis, here we are taking them as 0, 0 c is characterized by the partial differential equation yp-xq=0. Now we shall see that the other geometrical entities like x square+y square+z-c whole square tan square alpha which represents the set of all right circular cones whose axes coincide with z-axis.

So here you can see that the vertex of the cone lies at the point 0, 0 c, the semi-vertical angle of the cone is alpha and the z-axis is the axis of the right circular cone. So this equation represents the set of all right circular cones whose axes coincide with the z-axis and when we differentiate this partially with respect to x what do we get, 2x=2 times z-c*partial derivative of z with respect to x that is p*tan square alpha.

And when we differentiate this with respect to y we get 2y=2 times z-c*q tan square alpha. So eliminating the constants c and alpha what we get, x/y=p/q and this gives us the same partial differential equation as in the previous case. So here we get again the same partial differential equation yp-xq=0.

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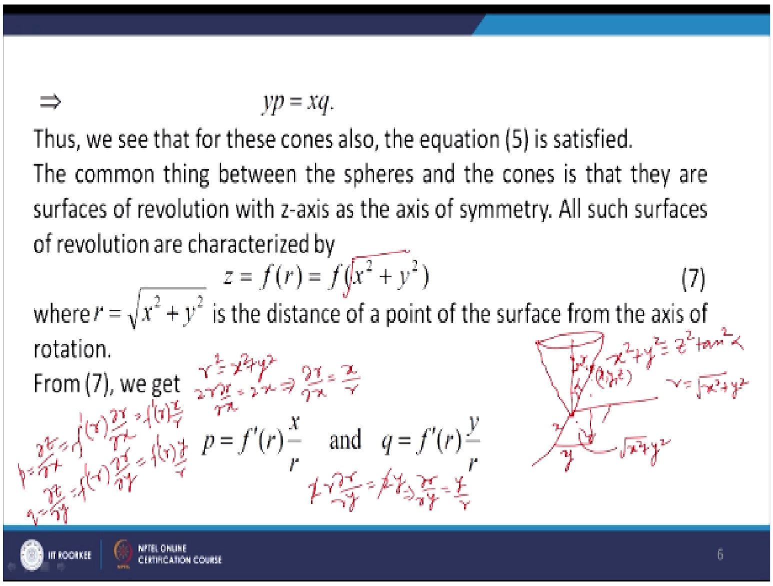

Thus, we see that for these cones also the equation 5 that is yp=xq is satisfied. The common thing between the spheres and the cones is that they are surfaces of revolution with z-axis as the axis of symmetry. They can be generated, this sphere can be generated with z-axis as the axis of symmetry and similarly cone the right circular cone can be generated with z-axis as the axis of symmetry.

Now all such surfaces of revolution are characterized by z=f r which is f of x square+y square where this is f z=f r=f under root x square+y square where r is=under root x square+y square is the distance of a point of the surface from the axis of rotation. Say for example you have this, suppose you take this sphere, you take the right circular cone, here the vertex is at 0, 0 so this is x square+y square=z square tan square alpha.

I have taken c=0 here and this is right circular cone with semi-vertical angle alpha. So here if you take any point on the surface of the cone then its distance from the z-axis is you can consider the orthogonal projection if this point is x, y, z. Then, this is your x and this one is y, so this distance is under root x square+y square and therefore distance of the point x, y, z on the surface of the cone from the z axis this is perpendicular at 90 degree.

So this is also under root x square+y square this one. So r is underfoot x sqare+y square where r is the distance of a point of the surface from the axis of rotation, so z is a function of r and from this equation z=f r you can see that when we differentiate this partially with respect to r we get derivative of z with respect to x, f prime r, z=f r so f prime r*partial derivative of r with respect to x.

And when we differentiate partially with respect to y what do we get f prime r*partial derivative of r with respect to y. Now r square=x square+y square so you can see 2r del r/del x=2x and what do we get, we get the derivative of r with respect to x=x/r okay. So this is f prime r*x/r and this similarly when you differentiate r square=x square+y square with respect to y you get 2r the derivative of r with respect to y=2y and what do we get, derivative of r with respect to y is given by y/r so this is=f prime r *y/r.

So you see this is p, this is q okay so p=f prime r*x/r q=f prime r*y/r and now we can eliminate the arbitrary function f here by dividing this equation p=f prime r*x/r by the equation q=f prime r*y/r and we get p/q=x/y.

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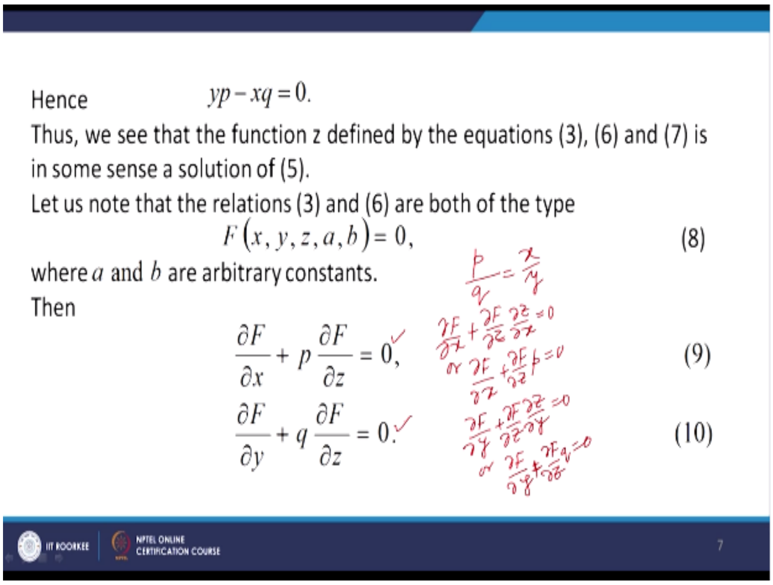

So p/q=x/y which gives yp=xq. Thus, we shall see that the function z defined by the equation 3, 6 and 7 in some sense is a solution of equation 5 this equation okay. Now let us note that the relation 3 and 6, you can see the relation 3 and 6 this relation 6 here we have two arbitrary constants alpha and c and here we have in 3 two arbitrary constants a and c. So they are both of this type F x, y, z, a, b=0 where a and b are two arbitrary constants.

Now let us see when we differentiate such an equation relation in x, y, z containing two arbitrary constants we are taking them as a and b then let us see what partial differential equation we get from here after we eliminate a and b. So when we differentiate this with respect to x what we get partial derivative of F with respect to x+partial derivative of F with respect to z*partial derivative of z with respect to x=0 or we can say so we get this equation.

Similarly, when we differentiate the equation 8 partially with respect to y we get partial derivative of F with respect to y+partial derivative of F with respect to z*q=0.

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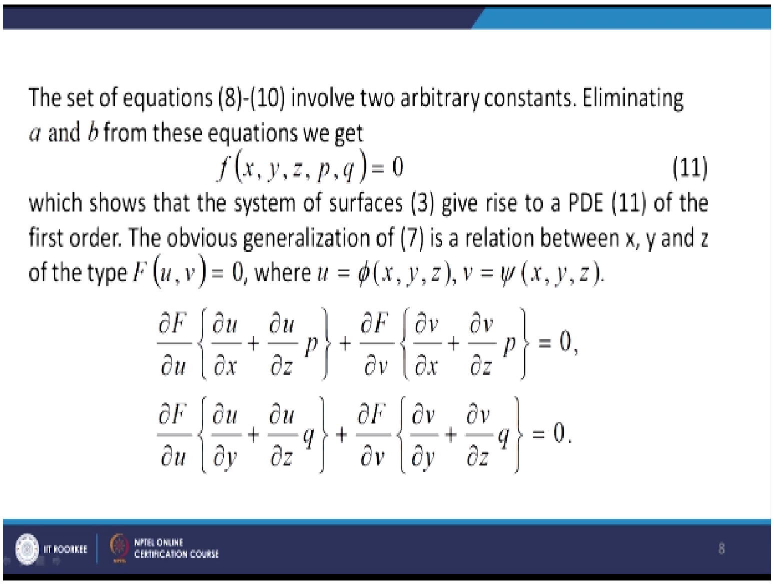

Now you see the equation 8 to 10 involve two arbitrary constants, so eliminating a and b from these 3 equations we get the equation f x, y, z, p, q=0 which shows that the system of surfaces 3 give rise to PDE 11 of the first order. The system of surfaces 3 or you can say 6 okay give rise to a PDE of the type f x, y, z, p, q=0. Now let us look at a generalization of equation 7, this case.

Here we are writing z=function of under root x square+y square and we got the differential equation of first order yp-xq=0, so let us consider a generalization of this case. Here we consider F u, v=0 okay where u and v are functions of x, y, z, u is function of x, y, z say phi x, y, z, v is the function of x, y, z say psi x, y, z.

Now when you differentiate this equation F u, v=0 partially with respect to x what you get, partial derivative of F with respect to u*partial derivative of u with respect to x+partial derivative of u with respect to z*partial derivative of z with respect to x which we are writing as p+partial derivative of F with respect to v*partial derivative of v with respect to x+partial derivative of v with respect to z*partial derivative of z with respect to x which we are writing as p=0.

Similarly, when we differentiate F u, v=0 partially with respect to y we get partial derivative of F with respect to u*partial derivative of u with respect to y+partial derivative of u with respect to z*partial derivative of z with respect to y which we are writing as q+partial derivative of F with respect to v*partial derivative of v with respect to y+partial derivative of v with respect to z* partial derivative of z with respect to y which we are writing as q=0.

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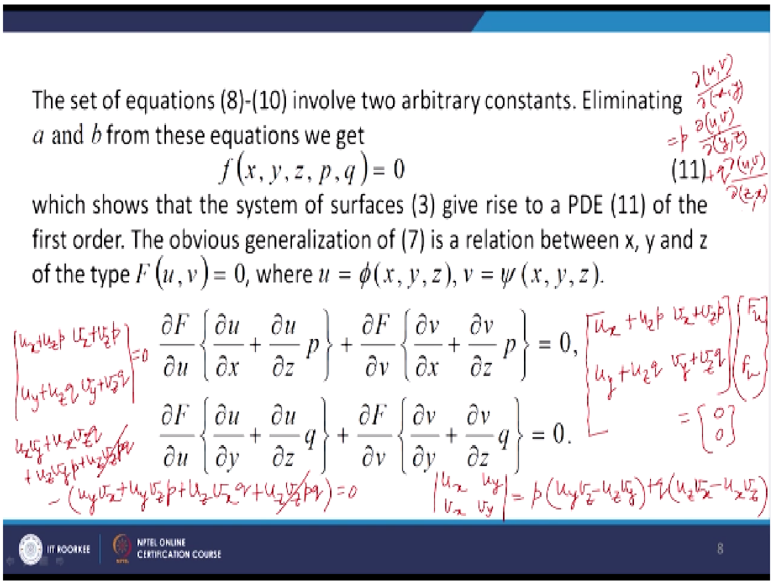

Now what we do, when we eliminate the arbitrary function F from here okay you can see we can write this system of equations as the partial derivative of u with respect to x in short we can write as ux+similarly uz*p then we get uy+uz*q and then second row we can write as vx+vz*p and then vy+vz*q*the column vector F u and F v=0 okay. So we can write these two equations as a matrix equation.

And then this equation will have a non-trivial solution provided determinant of the coefficient matrix is 0. So we have determinant of this coefficient matrix=0 that will give you ux+uzp uy+uzq then vx+vz*p vy+vz*q=0. So for a non-trivial solution of this matrix equation we must have the determinant of the coefficient matrix=0 then value of the determinant what you get, uxvy then uxvz*q then uzvy*p+uzvz*pq and the –we have uyvx and then we have uyvzp and we have uzvx q and we have uzvz pq.

So you can see uzvz pq will cancel with this uzvz pq okay, uxvy-uyvx we can write as ux uy vx vy becomes ux vy-uy vx can be written as ux vx uy vy and then we have here ux vz let us take the coefficient of p, let me write it on the right side okay, the coefficient of p will be equal to what we have uyvz-uzvy and the coefficient of q will be what we will have uzvx-uxvz.

So this is what we have and we can write it in the form of the Jacobian. So this is nothing but I can write it as in the form of Jacobian delta u, v/delta x, y=p times here we will have y, z+ q times z, x. So this is what we have.

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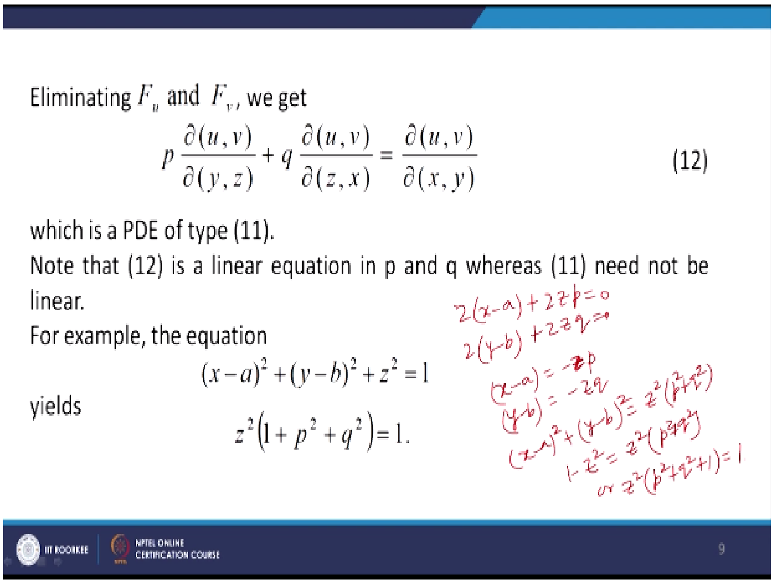

So we have p*delta u, v/delta y, z then we have q times delta u, v/delta z, x=delta u, v/delta x, y which is a PDE of the type 11. You can see this is the PDE of this type f x, y, z, p, q=0 we have p and q of the first degree and then their coefficients are functions of x, y, z. So this is the PDE of the first order and it is of the type f x, y, z, p, q=0. Now this is the linear equation in p, you can see the p and q occur in the first degree whereas in the equation 11 the p and q need not be of the first degree.

So this equation is a special case of the equation 11. Now for example let us consider this equation x-a whole square+y-b whole square+z square=1 you can see when we eliminate the constants a and b here, it will result into this partial differential equation of first order but it is nonlinear. So you can see we can also differentiate it partially with respect to x we get 2 times x-a+2z*p=0 okay.

And when we differentiate it partially with respect to y what we get 2 times y-b+2zq=0 so what we get x-a=-zp and y-b similarly=-zq okay. Now we want to eliminate a and b so let us square so then x-a whole square+y-b whole square will give you z square*p square+q square. Now by the given relation x-a whole square+y-b whole square=1-z square. So 1-z square=z square times p square+q square.

And therefore we can write it as z square times p square+q square+1=1 and it is not a linear equation in p and q okay.

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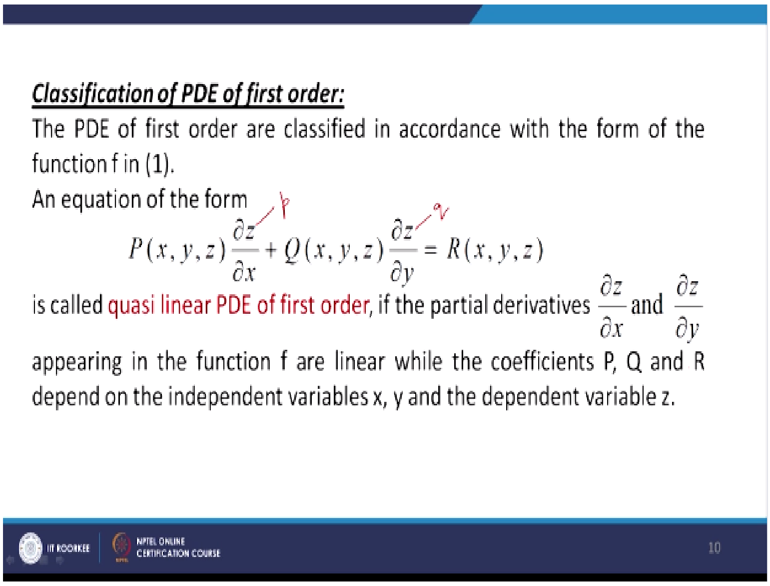

Now classification of PDE of first order, so let us look at the first order partial differential equation and discuss their classification. The first order partial differential equations are classified in accordance with the form of the function f in the equation 1 okay. The classification of the first order PDE will depend on the form of this function f x, y, z, p, q=0 okay. So if this first order PDE is of this form P x, y, z*this is p okay Q x, y, z*q=R x, y, z.

Then, such a partial differential equation of first order is called as the quasi-linear PDE of first order. Here the coefficient of this small p and small q can be functions of x, y, z and the right hand side R x, y, z okay, which is free from P and Q can also be a function of x, y, z. So if the coefficients of p and q which occur in first degree and separately okay their coefficients are functions of x, y, z.

And then the right hand side R x, y, z is also function of x, y, z then this partial differential equation is known as quasi-linear partial differential equation of first order where p and q are linear and the functions P, Q, R depend on the independent variables x, y and the dependent variable z.

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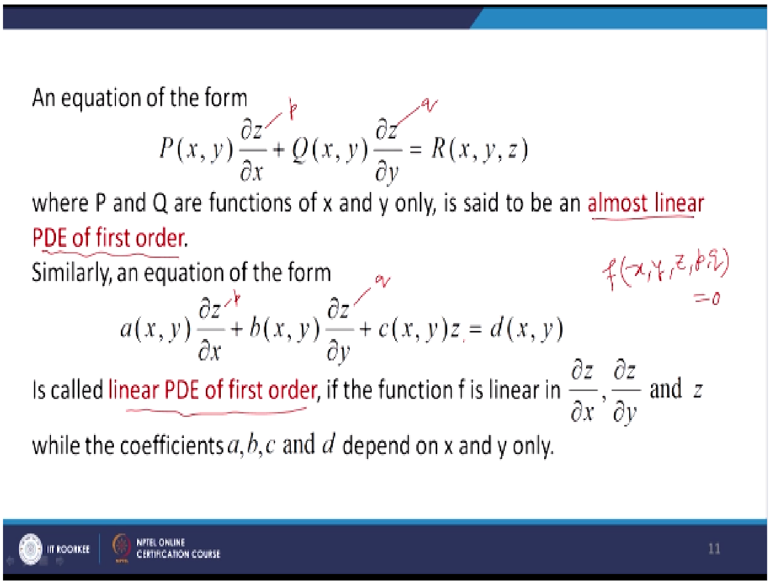

Now this is p, this is q, if the coefficient of p and the coefficient of q are independent of z they only depend on the independent variables x and y so the coefficient of p is the function of x, y only, the coefficient of q is the function of x, y only while this term which is free from p and q is a function of x, y and z then such an equation is called almost linear partial differential equation of first order.

Now if we have an equation of this type where the coefficient of p is a function of x, y, the coefficient of q is the function of x, y and z okay z which occurs in first degree okay. If the coefficient of z is the function of x, y=a function of x and y, the term which is free from p, q and z is a function of x, y only then such a partial differential equation is called as a linear partial differential equation of first order.

So here the function f which occurs in f x, y, z, p, q=0, if this function f is linear in p and q, p, q and z while the coefficients a, b, c, d depend on x and y only, then we shall call it as a linear PDE of first order.

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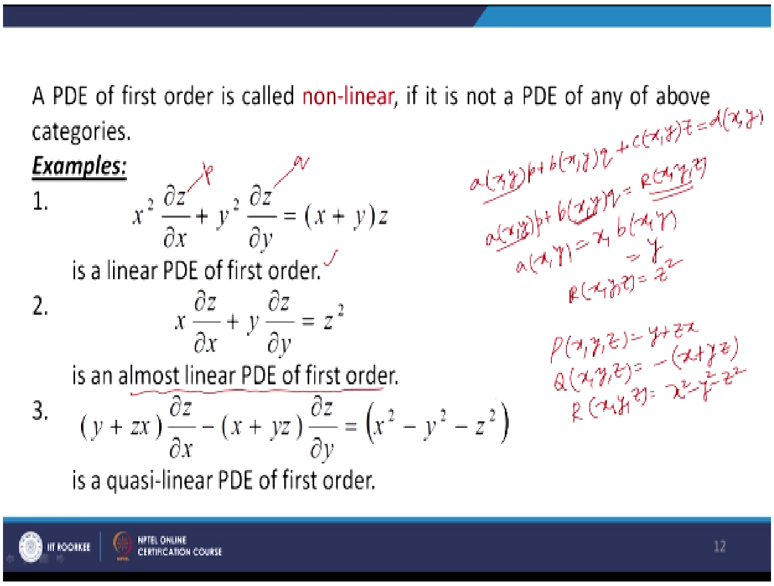

A PDE of first order is called non-linear if it is not a PDE of any of the above types okay, so like for example you can see now let us look at these examples, x square this is p, this is q okay and then we have x+y*z. This PDE is of the type a x, y*p b x, y*q+c x, y*z=d x, y. Here a x, y=x square, b x, y=y square and c x, y=-x+y while d x, y=0, so it is the linear PDE of first order.

Here xp+yq=z square, this PDE is of the type a x, y*p+b x, y*q=R x, y, z. Here the coefficient of p depends on x and y, the coefficient of q also depends on x and y while this term depends on x, y, z. So a x, y here is x; b x, y=y and R x, y, z=z square. So this is an almost linear PDE of first order okay. Now let us look at this, here the coefficient of p is P x, y, z=y+zx; coefficient of q which is Q x, y, z=-x+yz and R x, y, z=x square-y square-z square. So this is a quasi-linear partial differential equation.

The coefficients of P and Q depend on x, y, z and this term okay R x, y, z also depends on x, y, z. So it is a quasi-linear PDE of first order.

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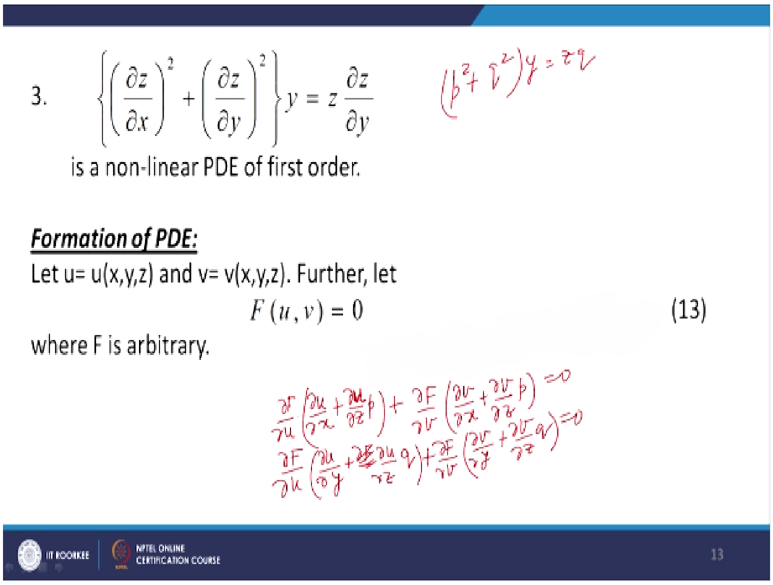

And this equation you can see, this is the partial differential equation of first order but here p and q are not of first degree, so this is p square+q square*y=z*q. It is not a linear PDE of first order, non-linear PDE of first order. Now let us see how the partial differential equations are formed. So let u be a function of x, y, z; v be a function of x, y, z and further assume that F u, v=0 that is there is the relation between u and v which is given by the function F.

So F u, v=0 and F is arbitrary. Then, differentiating this equation with respect to x what we shall have u, v, r functions of x, y, z so we will have when we differentiate with respect to x partial derivative of F with respect to u*derivative of u with respect to x+derivative of u with respect to z*derivative of z with respect to x+derivative of f with respect to v*derivative of v with respect to x+derivative of v with respect to z*derivative of z with respect to x which is equal to p=0.

So similarly when we differentiate this with respect to y we get yeah so when we differentiate F with respect to ux we get partial derivative of F with respect to u then we have to differentiate u with respect to x, so derivative of u with respect to x+derivative of u with respect to z*q*p.

And then we differentiate F with respect to v so partial derivative of F with respect to v*partial derivative of v with respect to x+partial derivative of v with respect to z*p=0. Similarly, when we differentiate this equation with respect to y we get F u*uy+uz*q+F v*vy+vz*q=0.

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Then, differentiating (13) partially with respect to x and y, we get

$$\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right\}+\frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right\}=0 \qquad (14)$$

and

$$\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right\}+\frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right\}=0. \qquad (15)$$

Eliminating $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ from (14) and (15), we obtain

$$\begin{bmatrix} u_x+u_z p & v_x+v_z p \\ u_y+u_z q & v_y+v_z q \end{bmatrix}\begin{bmatrix} F_u \\ F_v \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} u_x+u_z p & v_x+v_z p \\ u_y+u_z q & v_y+v_z q \end{vmatrix}=0$$

So we get these two equations and when we eliminate F u derivatives of the arbitrary function F with respect u and v as we had discussed earlier okay, this will give you the following. This determinant=0 and you can write it in the form of a matrix equation where the coefficient matrix will have this element a11 then a12, a21, a21*the column vector F u and F v=0 okay. We can write this as a matrix equation like this.

So ux+uz*p and then we have uy+uz*q and then vx+vz*p and vy+vz*q*the partial derivatives Fu Fv=0. So we will get this. Now this matrix equation will have a non-trivial solution provided the determinant of the coefficient matrix is 0. So we will have ux+uz*p, uy+uz*q and then vx+vz*p vy+vz*q=0.

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$$\begin{vmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x} & \frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x} \\ \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y} & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y} \end{vmatrix}=0$$

or $$p\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}\right)+q\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial v}{\partial z}\frac{\partial u}{\partial x}\right)=\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right)$$

or $$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)}=\frac{\partial(u,v)}{\partial(x,y)}$$

or $$Pp+Qq=R$$

which is a first order PDE linear in p and q. It is called Lagrange's PDE of first order.

So when you put this determinant=0 we arrive at this equation p times this q times this expression=this expression and which is nothing but in the form of Jacobian we can write it as p*del u,v/del y, z+q del u, v/del z, x=del u, v/del x, y and if we denote it by P, this by Q and this by R then we have Pp+Qq=R which is a first order partial differential equation which is linear in p and q because p and q occur in the first degree. It is called as Lagrange's PDE of first order.

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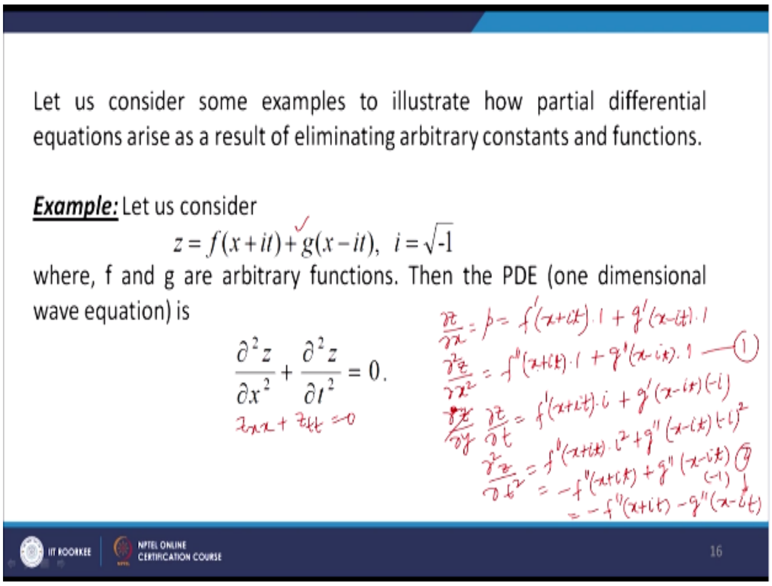

Now let us consider some examples to illustrate how partial differential equations arise as a result of eliminating arbitrary constants and functions. For example, let us consider z=f x+it+g x-it where x and t are independent variables, z is a dependent variable, f and g are arbitrary functions, i is iota that is square root -1. So we have to form the partial differential equation by eliminating the arbitrary functions f and g here.

So let us differentiate it partially with respect to x, so p will be=f dash x+it we can take x+it as u okay, so f dash u*partial derivative of u with respect to x which is 1+g dash x-it, x-it we can write as v*partial derivative of v with respect to x so we have 1 and then this is derivative of z with respect to x. Now we can take one more derivative with respect to x, so second derivative with respect to x will be=this is f prime u, u is x+it okay.

So f double prime x+it*1 and here will get g double prime x-it*1 because v is x-it when we differentiate v with respect to x we get 1. Similarly, if you differentiate it with respect to y. Differentiate this equation with respect to t okay what we get, f prime x+it*i okay and then g prime x-it*-i. When we differentiate it once more with respect to t what we get, f double prime x+it*i square and here g double prime x-it*-i whole square.

So this will be –f double prime x+it here+g double prime x-it. Now let us call it equation 1 and this as equation 2 okay. This is i square, this is also -1 so we get – f double prime x+it-g double prime x-it. So when we add equation 1 and 2, this is 2 okay 1 and 2, the right hand side becomes 0, this will cancel with this, this will cancel with this and will get zxx+ztt=0. So this is a second order partial differential equation we get.

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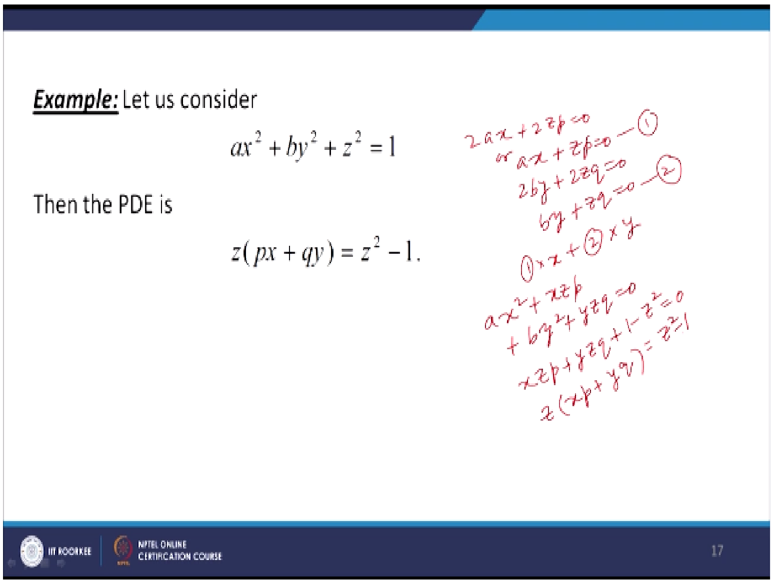

Now let us consider the equation ax square+by square+z square=1. Then, we differentiate it with respect to x, we get 2ax+2zp=0 or we can say ax+zp=0 and then we differentiate with respect to y we get 2by+2zq=0 or we get by+zq=0. So we have ax square+by square=1-z square. So what we do, we multiply this equation 1 and 2 by xy, so 1*x and 2*y and let us add them, what we get, ax square+xz*p+by square+yz*q=0.

Or we can say this is ax square+by square is 1-z square, so we get xz*p+yz*q+1-z square=0 or we can say this is z times xp+yq=z square-1. So this is the PDE that we get when we eliminate the arbitrary constants a and b here. Now let us look at this f x+y+z, x square+y square+z square=0.

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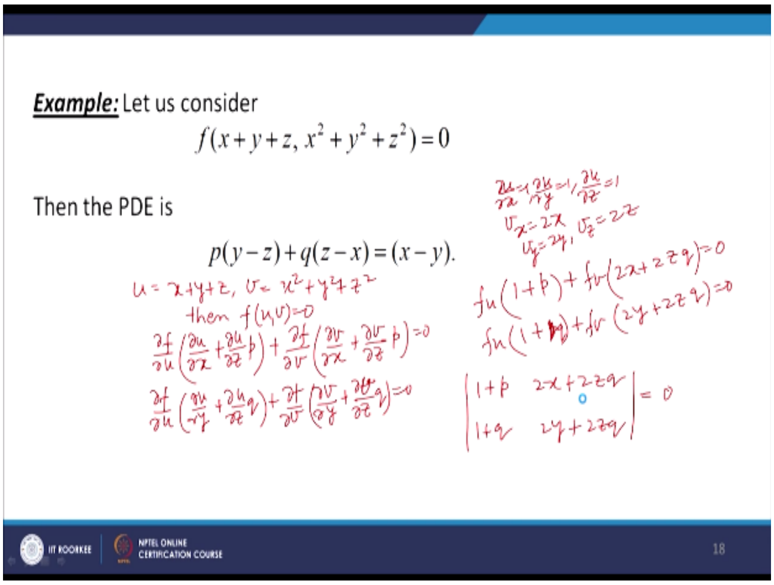

We have an arbitrary function f here, so we have to eliminate this arbitrary function f. So let us take u=x+y+z, v=x square+y square+z square. Then f u, v=0, so let us differentiate this equation f u, v=0 with respect to x what we get, partial derivative of f with respect to u*partial derivative of u with respect to x+partial derivative of u with respect to z*partial derivative of z with respect x+partial derivative of f with respect to v*partial derivative of v with respect to x+partial derivative of v with respect to z*p=0.

Similarly, when we differentiate with respect to y we get partial derivative of v with respect to z*q=0. Now u is x square+y+z so ux=1, uy=1, uz=1 okay and vx partial derivative of v with respect x is 2x, vy is 2y, vz is 2z. So when we put the values here what we get we have fu*ux, ux is 1+uz, uz is 1 so we get 1+p+fv*vx vx=2x, vx+vz vz=2z*p=0 and then fu times uy=1, uz=1 so 1+q and then fv*by=2y+2z*q=0.

Now we can eliminate fu fv from here so 1+p, 1+q and 2x+2 zq, here we have 2y+2zq=0. So when you expand this determinant you arrive at this equation p times y-z+q times z-x=x-y from here.

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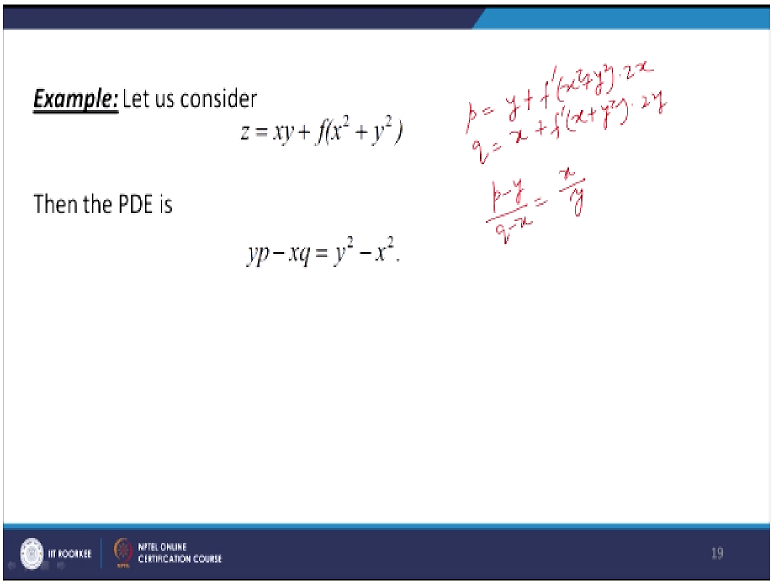

Now when we have z=xy+f x square+y square. When we differentiate partially with respect to x we get p=y+f prime x square+y square, so we are differentiating f with respect to x square+y square*2x and then q=x when we differentiate with respect to y then we have f prime x square+y square*2y. So we can write p-y/q-x=2x/2y or x/y. So we can divide and f prime, x square, y square will cancel and will arrive at this equation.

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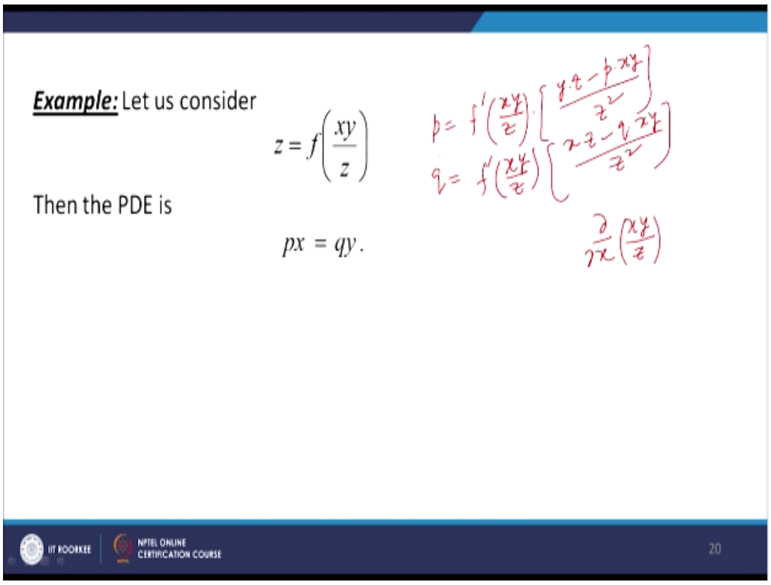

Now similarly z=f xy/z, when we differentiate it with respect to x we get p=f prime xy/z*derivative of this with respect to x. So derivative of this with respect to x means what, we have to differentiate xy/z with respect to x. So we will have derivative of the numerator will be y*z-derivative of z with respect to x that is p*xy/z square. Similarly, when we differentiate with respect to y we get this derivative of numerator with respect y.

So we get x*z-q*xy/z square. Now we can divide to eliminate f prime xy/z and will arrive at px=qy.

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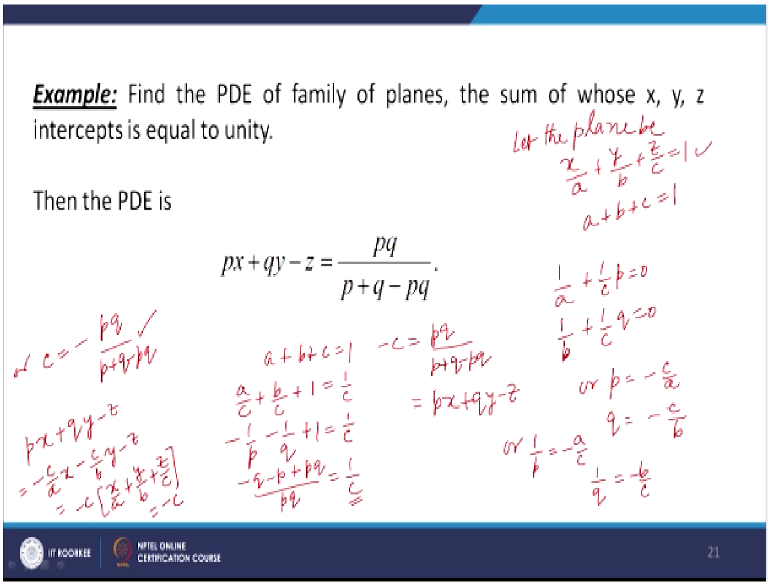

And if we have the PDE of family of planes that the sum of whose x, y, z intercepts is=unity then we can write let the plane be x/a+y/b+z/c=1. Now this plane makes intercepts a, b and c on the three coordinate axis. We are given that the sum of whose x, y, z intercepts is=unity, so a+b+c=1 this is given to us, we have to arrive at this PDE, so let us differentiate this with respect to x we get 1/a.

Now this derivative of y with respect to x is 0, so 1/c, derivative of z with respect to x we get p=0. When we differentiate with respect to y we get 1/b+1/c*q=0. So we get the relations, we get p=-c/a and q=-c/here we get this is b so q=-c/b. So p is –c/a, q is –c/b okay, now we want to find this relation here, so a+b+c=1 gives you when we divide by let us say here we have –c or we can say 1/p=-a/c and 1/q=-b/c okay.

So a+b+c=1 we can divide this equation by c so a/c+b/c+1=1/c and what we get here a/c=-1/p, b/c=-1/q+1=1/c. Now we have to eliminate 1/c and to eliminate 1/c we take the help of what we will do, here we have pq LCM –q-p and then we get pq=1/c. So we get pq upon or we can say c=-pq/p+q-pq. Now let us find px+qy-z. So this is px+qy, p=-c/a*x, q is –c/b*y-z we have.

Now this is –c times x/a+y/b+z/c and we are given that x/a+y/b+z/c=1, so this is –c. So px+qy-z=-c. So what we get we have –c=pq/p+q-pq from here and from here –c=px+qy-z. So we get this partial differential equation. You can see it is nonlinear. Now with that I would like to conclude my lecture. Thank you very much for your attention.