Ordinary and Partial Differential Equations and Applications
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Lecture - 31
Green's Function and its Applications - II

Hello friends. Welcome to this lecture. In this lecture will continue our study of Green function and with the help of Green function how to solve the nonhomogeneous problem and we have seen one example in previous lecture. So in previous lecture we have discussed how to construct Green function for homogeneous self-adjoint equations and also we have seen how to make linear ordinary differential equation to self-adjoint in a differential equation.

And how to convert nonhomogeneous boundary condition to homogeneous boundary condition. Now let us proceed further and discuss how to use Green function method to solve the nonhomogeneous linear ordinary differential equation.

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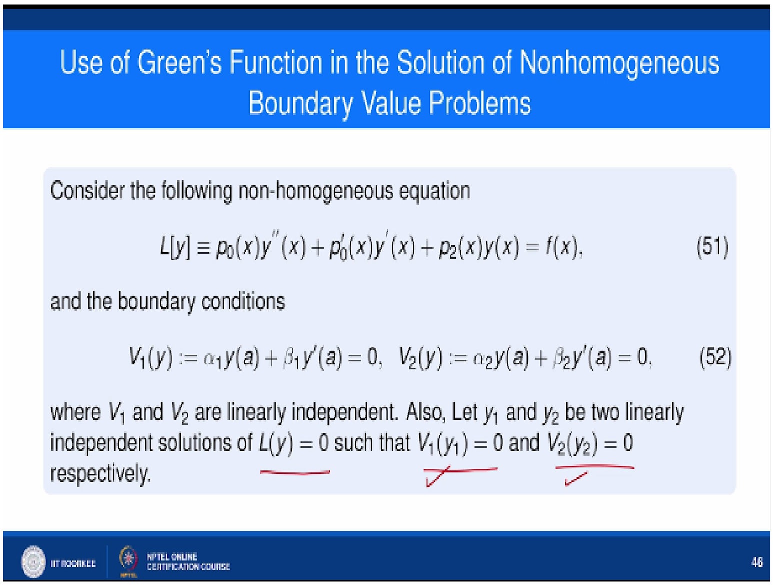

So here this is the theorem. Consider the following nonhomogeneous equation p0 x y double dash x+p0 dash x y dash x+p2 x y x=f of x. Please note down here that the coefficient of y dash is p0 dash x so it is basically a self-adjoint nonhomogeneous linear differential equation. So it is L y=f is the equation given here and the boundary conditions are given as V1 y and V2 y where V1 y is alpha1 y a+beta1 y dash a=0 and V2 y is alpha2 y a+beta2 y dash y dash a=0.

So here this is the condition given at the point x=a and this is the condition given at the point b here. So here V1 and V2 are linearly independent also let y1 and y2 be 2 linearly independent solution of L y=0 such that it satisfy these 2 of the condition respectively. So y1 is the solution which satisfy L y=0 and satisfy the first boundary condition and y2 be another solution which satisfy the second boundary condition.

And here we say that if G x, xi is a Green function of the homogeneous boundary value problem L y=0 with boundary condition Vk y=0 where k is from 1 to 2 then y x is the solution of boundary value problem 51 and 52 that is nonhomogeneous equation with homogeneous boundary condition.

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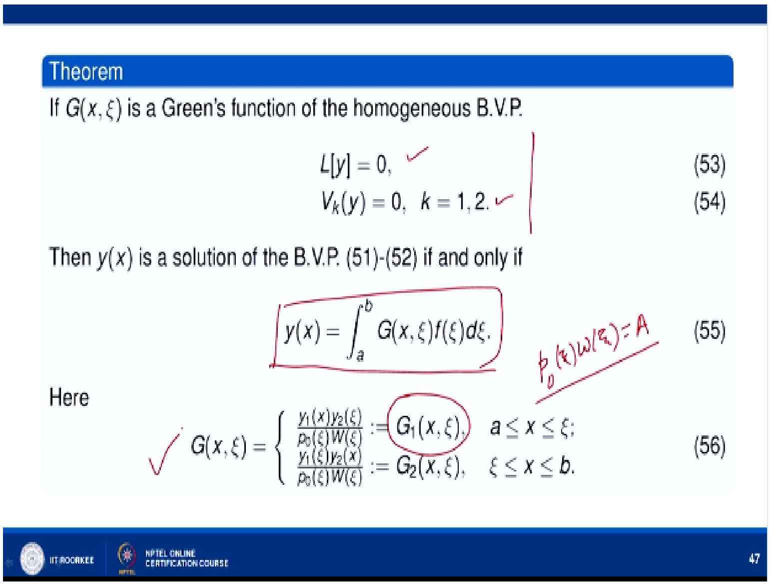

If and only if y x is given by a to b G x, xi f xi dxi where G x, xi is given as y1 x y2 xi/p0 xi W xi which I denote as G1 x, xi and it is given in this domain from x from a to xi and G2 x, xi which is given by y1 xi y2 x/p0 xi W xi and x is from xi to b and please note down here that this p0 W xi is a constant given at a and we have already discussed how to get this constant using Lagrange's identity.

So it means that if Green function is given for this homogeneous boundary value problem with homogeneous boundary condition then the following formula for y of x will solve the nonhomogeneous equation with homogeneous boundary value problem where G x, xi is we have already obtained in previous lecture like this. So that theorem we have given in previous lecture, we have just stated but we have not proved. In this lecture, we wanted to prove the given theorem here.

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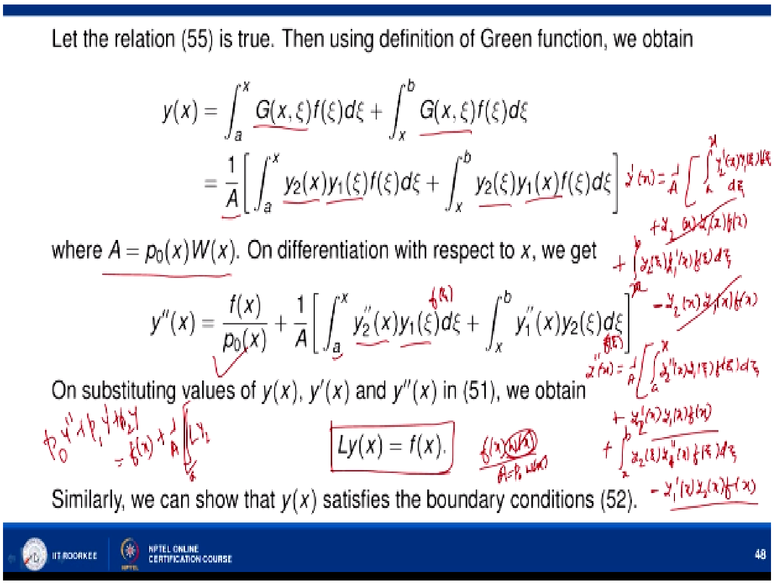

So let the relation 55 is true so 55 is true means y x is given by a to b G x, xi f xi dxi and then we want to show that it is a solution of the nonhomogeneous boundary value problem. So we have to show that if y x is given by this then it satisfy the problem Ly=f of x so for that we have to show that y satisfy this nonhomogeneous equation. So for that let us write down y x=a to x G x, xi f xi dxi+x to b G x, xi f xi dxi.

So here we have just truncated into 2 part depending on xi, xi is<=x and xi>=x. Now in xi<=x we have G x, xi is given by xi<x so xi< means here so here we use this value. G x, xi is y1 xi y2 x/A, so y1 xi y2 x/A since A is constant we can take it out, dxi+x to b now here G x, xi means xi>=x. Now look at xi>x means here so here we are using y1 x y2 xi/A that we have taken out.

So y x is given by this thing now here A is your p0 x W x wronskian of y1, y2 given at point x and we would know that this is constant. So now y x is given here. Now differentiate with respect to x. So when you differentiate with respect to x what you will get, let me write it here so y dash x=1/A as it is. Now here if you differentiate with respect to x so here we can differentiate it like this that a to x and inside it is y2 dash x y1 xi f xi dxi+.

Now since integral involves your x so here we use Leibniz formula in differentiation under the sign of integration and you will get this as here you put xi=x so it is y2 x y1 x f of x and differentiation of x, so it is this so this is + okay + when you integrate this, it is what a to b y2 xi y1 dash x f xi dxi-, now it is in lower so it is – here y2 x y1 x f of x that is all. So if you look at this will cancel out right.

So what you will get y dash x=1/A a to x y2 dash x y1 xi f xi dxi+x to b y2 xi y1 dash xi f xi dxi. Now when you differentiate it again what you will get y double dash x=1/A, here you will get a to x y2 double dash x y1 xi f xi dxi and+. Now I will write it here y1 dash x y1 x f of x+x to b and y2 xi y1 dash x f xi dxi-, here we have y1 dash x y2 x f of x. Now if you look at this quantity this is basically what, it is something it is y1 y2 dash-y1 dash y2*f of x.

And that we can write it, this is what wronskian of y1 y2*f of x so if you write it so it is what, a to x y2 double dash x y1 xi f xi dxi+x to b y1 double dash x y2 xi dxi*f of x. Here if you take it out the term what you will get is f of x wronskian of x here, wronskian of y1 y2 and we already know that this quantity is what, this quantity is basically and divided A so that you can take it out.

Now A is basically p0 x W x so if you use the value of A that is p0*W x so this will cancel out, we have f of x*p0 x+this quantity 1 to a, a to x y2 double dash x y1 xi f xi dxi+x to b y1 double dash x y2 xi f xi dxi. Now when you substitute the values of y double dash, y dash and y and calculate this quantity that p0 y double dash+p1 y dash+p2 y, you will see that it is what, it is inside if you multiply here then we have only f of x+1/A and if you look at inside you will get what.

Inside in integral L of y2 a to x let me write it here in separate thing.

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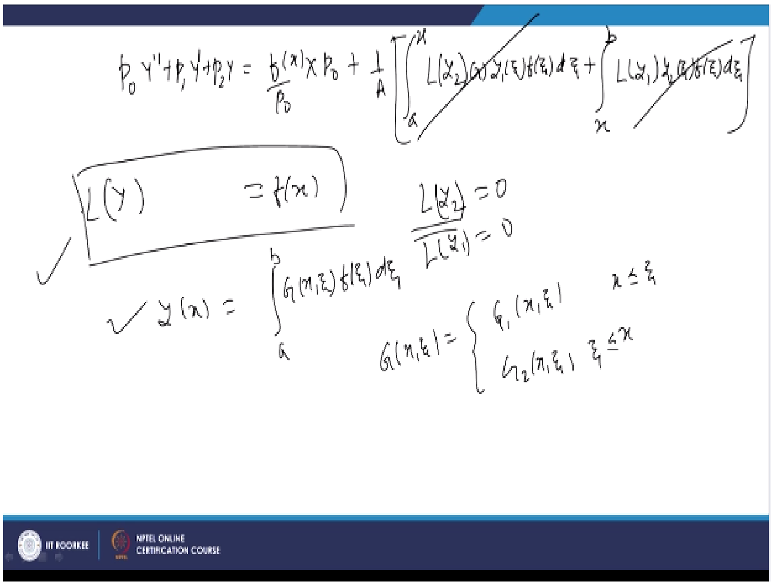

So here we want to calculate this p0 y double dash+p1 y dash+p2 y and it is f of x/p0*p0+1/A and here let me collect all these things and it is a to x when you collect you get L of y2 of x y1 xi f of xi dxi+ and x to b here then you will get L of y1 y2 xi f of xi dxi and so on. So that is what we will obtain here okay. So if you collect from end term here. So now we already know that this y2 is a solution of L of y2 satisfying the boundary condition that is not required in fact L of y2 is=0 similarly L of y1 is=0.

So using this this is simply gone so what is left is that f of x, so it means that L of y is given as f of x where y is given as a to b G x, xi f of xi dxi. So we have shown that if y x is given by a to b G x, xi f xi dxi where G x, xi is defined as G1 x, xi, G2 x, xi, x<xi, x>xi. Then your L y=f of x that y x will solve the nonhomogeneous boundary value problem. So that is the content that is one way.

Now we want to show that it also satisfy the boundary conditions, so boundary conditions are what, that is Vk y=0 so here just note down that look at this particular problem.

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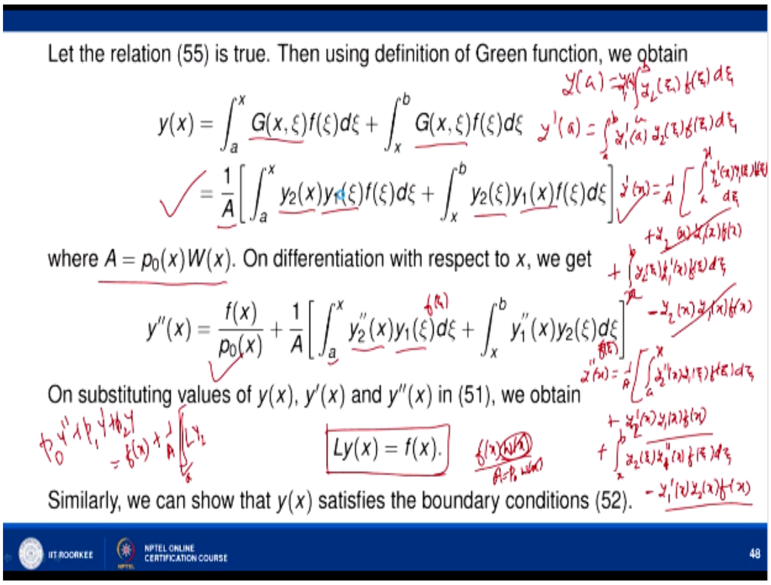

So here if you look at y of a will be what, so if you look at y of a will get this a to b and here you will get what y1 a, so y1 a you will get, y1 a and this is y2 xi f xi dxi and similarly your y dash a will be what, y dash a will be simply a to b and y1 dash a you can take it out, y1 dash a y2 xi f of xi dxi. Now y a you can write it from this equation and y1 dash a you can get it from this. Similarly, you can get y b and y dash b from these set of equations.

And then you can verify, I am leaving it to you that it also satisfy the boundary condition Vk y=0. So I am not doing it because it is going to be a little bit lengthier.

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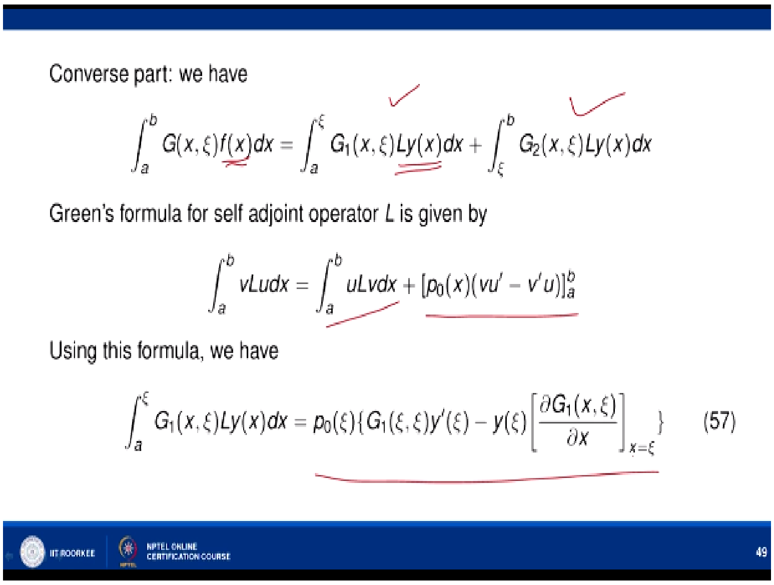

So now let us look at the converse part. The converse part means that once we have G x, xi then basically it will also satisfy the relation that a to b G x, xi f xi is basically y of x. So let us start with this part and you show that this is equal to your y of x. So for that you just calculate this a to b G x, xi f of x d of x. Now here we already know that f of x is given by L of y of x, so let me write it G x, xi f x now in place of f x I am writing Ly of x.

So now truncate this into 2 parts, a to xi G1 x, xi Ly x dx+xi to b G2 x, xi Ly x dx. Now we already know that Green's formula for self-adjoint operator is this a to b vLudx=a to b uLvdx+p0 x vu dash- v dash u from a to b. So here using this formula we have a to xi G1 x, xi Ly x dx I mean you look at this part this is going to be y x L of G1 and L of G1 is basically constant multiple of y1 and y1 is a solution of L.

So that part will be 0 and what is left is this point. Here a is this and b is=xi and when you simplify you will get this formula p0 xi G1 xi, xi y dash xi-y xi dou G1 x, xi/dou x at the point x=xi.

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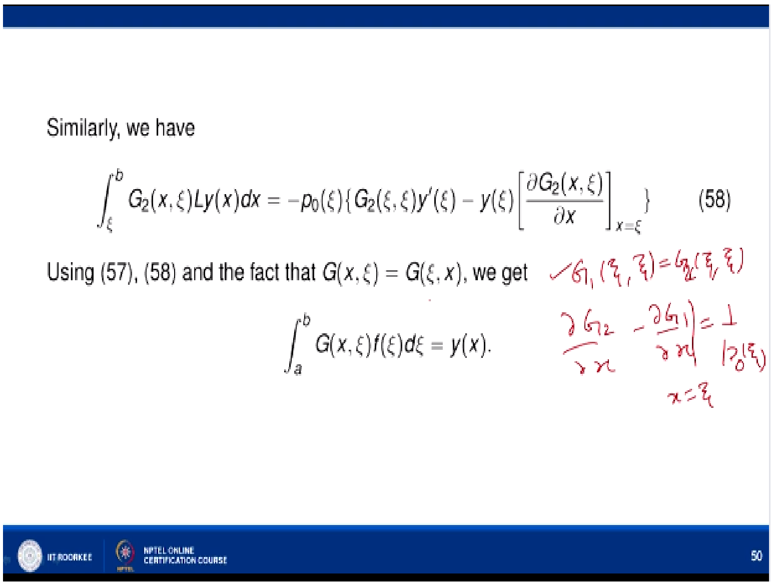

Using this similar formula, we can get this part xi to b G2 x, xi Ly x dx=then again using the same formula here, here you will get y and L of G2 and G2 is a multiple of y2 so we can write it here that part is gone to be 0 and using this boundary term here we can simplify our expression as –p0 xi G2 xi, xi y dash xi-y xi dou G2 x, xi/dou x at x=xi. So here we have xi to be G2 x, xi Ly x dx=-p0 xi G2 xi, xi y dash xi-y xi dou G2/dou x at x=xi here.

So using these estimates and the continuity property that G1 xi, xi=G2 xi, xi and dou G2/dou x-dou G1/dou x at x=xi=1/p0 xi. We can write that a to b G x, xi f xi dxi=y of x. Here also we have utilized the symmetry property that G x, xi=G xi, x. Let us just simplify this procedure like this.

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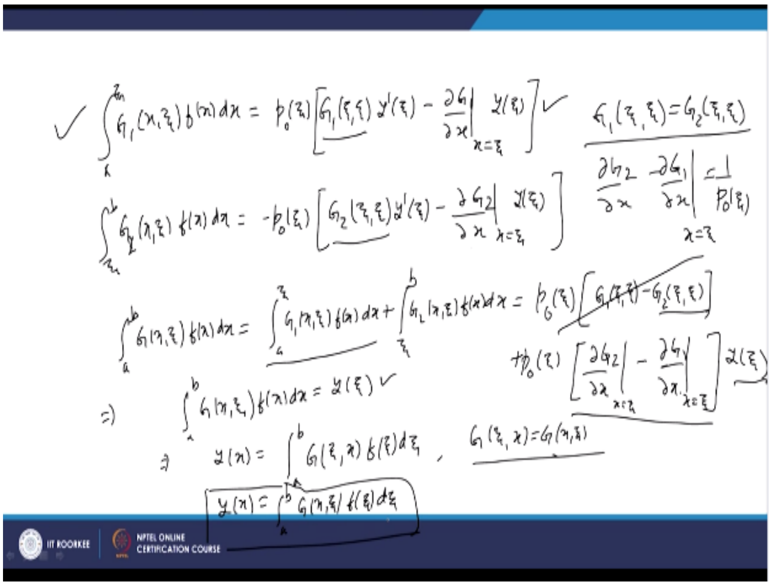

We have calculated a to xi G1 x, xi f of x dx=p0 xi G1 xi, xi y dash xi-dou G1/dou x at x=xi y xi. Similarly, we have calculated xi to b G2 x, xi f x dx=-p0 xi G2 xi, xi y dash xi –dou G2/dou x at x=xi y xi. Now we use this condition so let us see, a to b G x, xi f x dx=a to xi G1 x, xi f x dx+xi to b G2 x, xi f x dx=now use this formula. Let me collect the coefficient of p0 xi.

So p0 xi let us first use this G1 xi, xi-G2 xi, xi+, + if you multiply here then p0 xi=dou G2/dou x at x=xi-here dou G1/dou x at x=xi*y of xi here. Now because of continuity this part is 0 and because of jump discontinuity this part is going to be 1/p0 xi, so 1/p0 xi and p0 xi is here so that will cancel out and we have only y xi. So it means that a to b G x, xi f of x dx=y of xi here.

So now just interchange xi and x and we have y of x=a to b G of xi, x f of xi dxi but here we have noted down that G of xi, x=G of x, xi because in self-adjoint problems your Green function is symmetric Green function so in place of G xi, x we can write G x, xi. So we can write y of x=a to be G x, xi f of xi dxi that is what we wanted to prove that a to b G x, xi f xi dxi=y of x when G x, xi is the given Green function.

So that is what we have proved here that a to b G x, xi f xi dxi=y of x. So we have shown that if G x, xi is a Green function of the homogeneous boundary value problem then y x is the solution of nonhomogeneous boundary value problem with homogeneous boundary condition if and only if y x is written as a to b G x, xi f xi dxi where G x, xi is the Green function given here which we have already obtained for the homogeneous boundary value problem with homogenous boundary condition.

So once we have this result proven then again let us consider one example based on this.

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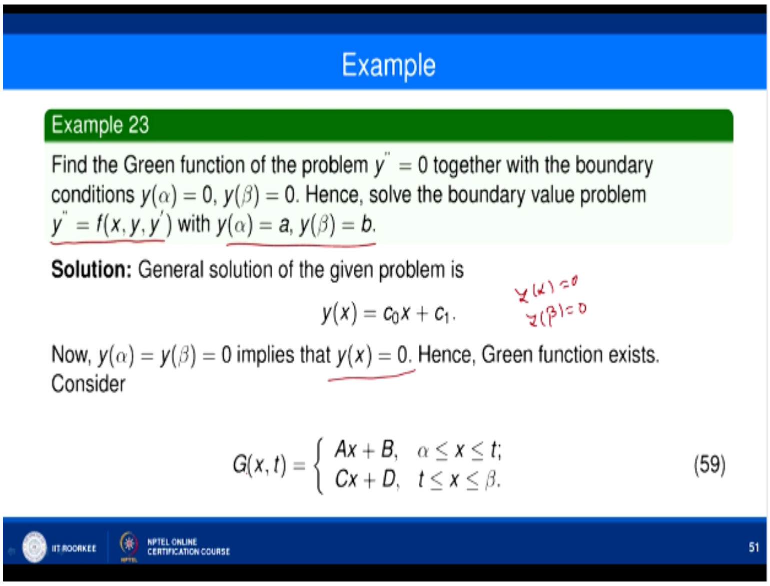

And here find the Green function of the problem y double dash=0 together with the boundary condition y alpha=0, y beta=0 and hence solve the boundary value problem y double dash=f of x, y, y dash with y of alpha=a and y beta=b. So please look at here. Here we have this nonhomogeneous problem and nonhomogeneous boundary condition as well. So first thing is let us find out the Green function for the homogeneous problem.

So y double dash=0 means y x=c0x+c1. Now if you use y alpha=0 and y beta=0 you can say that we have only trivial solution. So only trivial solution exist means your Green function exist and let us write down the Green function as G x, t=Ax+B where x is lying between alpha to t and Cx+D where x lying between t to beta. Now we want to show that G x, t satisfy the boundary condition at alpha and beta.

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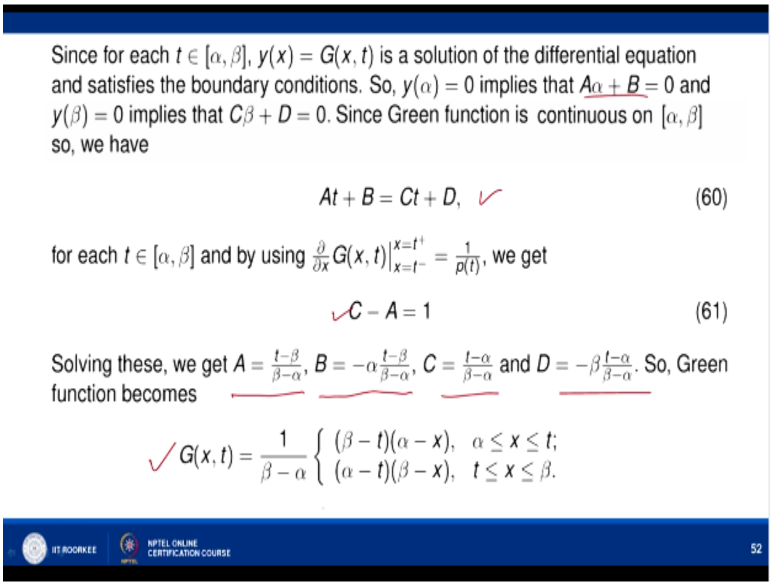

So using this since this G x, t satisfy the boundary condition so it means that this Ax+B will be 0 at x=alpha, so that will give you that A alpha+B=0 and y beta=0 implies that C beta+D=0. So this devises to since Green function is continuous we have one more condition that At+B=Ct+D and using the jump condition we can write dou G/dou x from x, t+- t-=1/p t. Now here p t is what here p t is basically 1 here.

So using this we can write that C-A=1, so C-A=1 and At+B=Ct+D and some more condition is given so using this we can get our solution A, B, C, and D. So once you have A, B, C, D then we can write down G x, t as 1/beta-alpha beta-t alpha-x when x is lying between alpha to t and alpha-t beta-x when x is lying between t to beta and if you remember the proposed problem is self-adjoint problem.

So you can say that G x, t is also looking at symmetry problem. So here we have utilized all the 4 conditions that G x, t satisfy the boundary condition as well as G x, t is continuous and G x, t has jump discontinuity at x=t so using this we have G x, t.

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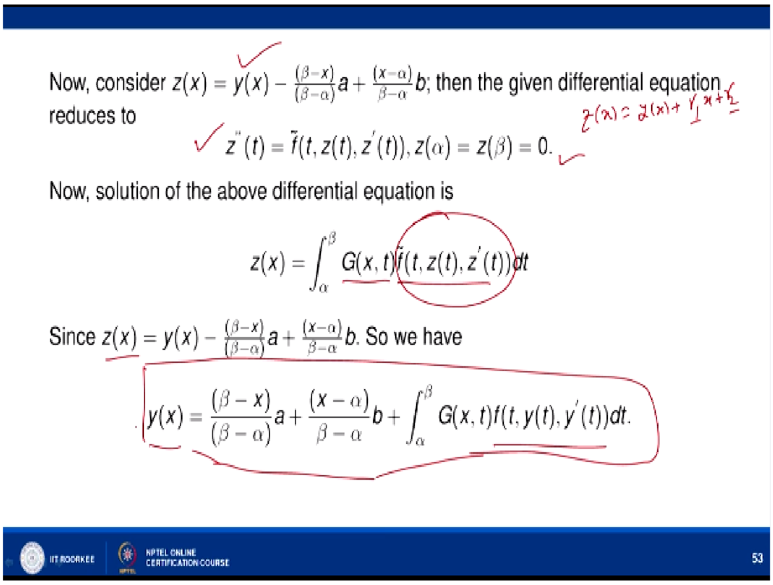

So once we have G x, t now we want to solve the nonhomogeneous problem but before solving the nonhomogeneous problem we have to convert the nonhomogeneous boundary condition to a homogeneous boundary condition and for that we consider a linear change which we have already discussed z x=y x-beta-x/beta-alpha*a+x-alpha/beta-alpha*b. Again I request that you do it, z x=y of x+some gamma1 x+gamma2 and you need to find out what is gamma1 and gamma2.

So when you fix it you will get this result here. Then our problem is now reduced to z double dash t=f tilde t, z t, z dash t where z alpha=0 and z beta=0 here. So now we can write down our solution of this nonhomogeneous problem is now z of x=alpha to beta G x, t f tilde t, z t, z dash t dt. Now we already know that y x and z x is connected by this formula, so using this we can write down y x= z x+this.

So when you write beta-x beta-alpha a and so on+alpha 2b z x, t f tilde t z, t, z dash t when you rewrite in terms of x then it is coming out to be f t, y t y dash t dt. So your solution is now given by this particular formula that y of x=beta-x/beta-alpha*a+x-alpha/beta-alpha*b+alpha to beta G x, t f t, y t y dash t dt. Now here everything depends on this function f. If this function f is independent of y t and y dash t then will have a close solution.

It means that we can find out the explicit formula of y of x but if this function f is not independent of y t or y dash t then this is basically an integral equation. So it all depends on the form of f here. So let us take a particular form of this f and see one more example here.

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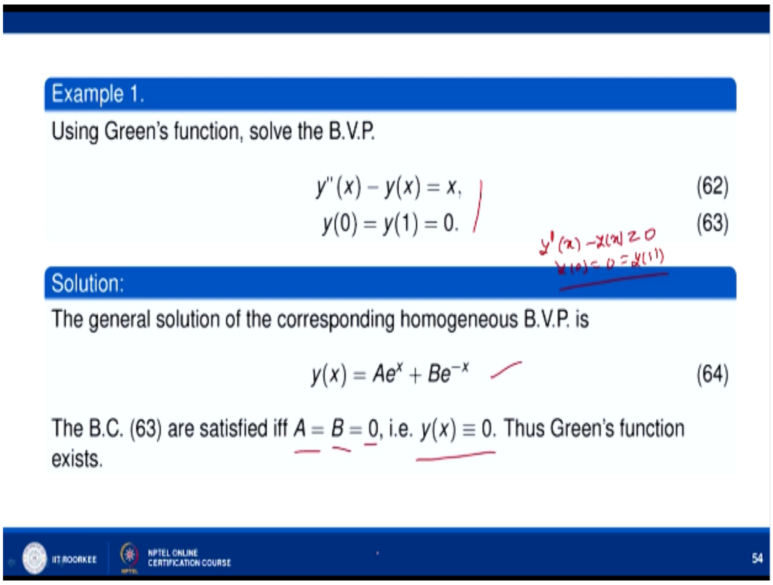

So using Green function solve the boundary value problem y double dash x-y of x=x, y 0=y1=0. So this time we are considering the homogenous boundary condition and nonhomogeneous problem. So first find out the Green function for the homogeneous equation with homogeneous boundary condition. So find out y double dash x-y of x=0 y of 0=0=y of 1 and we try to find out the solution.

Here we can say that general solution of the corresponding homogeneous boundary problem is y of x=Ae to power x+Be to power –x and when you apply the boundary condition we can say that A and B both are coming out to be 0 so it means that we have only a trivial solution of the homogeneous boundary value problem and hence your Green function exist.

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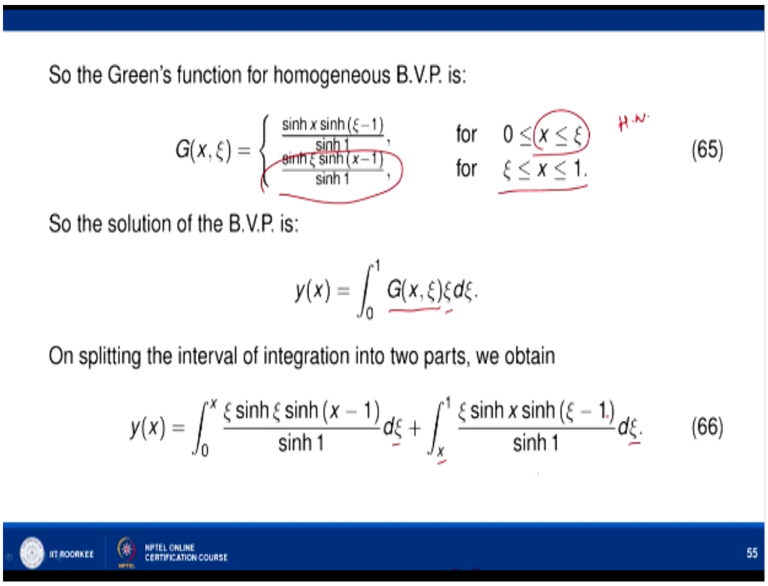

And Green function we can calculate in this form. I am leaving it to you that you find out that G x, xi=sin hyperbolic x sin hyperbolic xi-1/sin hyperbolic 1 when x is lying between 0 to xi and sin hyperbolic xi sin hyperbolic x-1/sin hyperbolic 1 when x is lying between xi to 1. This I am leaving it to homework and you can see that this is a symmetric Green function and the solution of the boundary value problem is given by y x= 0 to 1 G x, xi and f xi is basically x here, f xi is xi here.

So f x=x so f xi=xi, so 0 to 1 G x, xi xi dxi. Now G x, xi is known to you and xi is also given, so you can write down this integration into 2 parts, 0 to x and x to 1. So 0 to x, xi sin hyperbolic xi, here when you look at xi is<x so xi<x means this part. So in first integration you use xi is the function sin hyperbolic xi sin hyperbolic x-1/sin hyperbolic 1 d xi and in this xi is>x so you use this part. So xi sin hyperbolic x sin hyperbolic xi-1/sin hyperbolic 1 dxi and you solve it.

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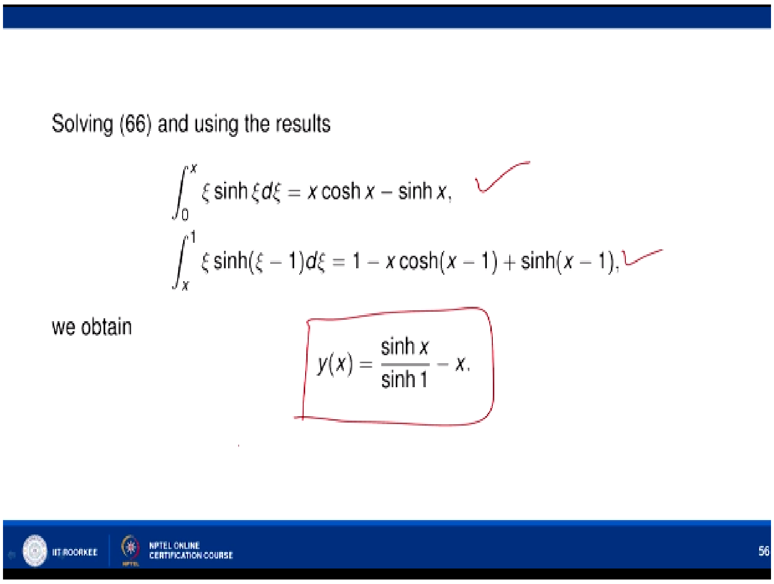

And you can use this following result that 0 to x xi sin hyperbolic xi dxi=x cos hyperbolic x-sin hyperbolic x and x to 1 xi sin hyperbolic xi-1 d xi=1-x cos hyperbolic x-1+sin hyperbolic x-1. So this integration you have to solve. I think it is not very difficult that is why I am using simple result here and we can write down our solution as this, y of x=sin hyperbolic x/sin hyperbolic 1-x.

So here we have solved the nonhomogeneous boundary value problem with homogeneous boundary value problem with the help of Green function for homogeneous boundary value problem. So this is one very important use of Green function and since here your right hand side is independent of y and y dash, so we are getting the exact solution means we are getting the closed form of this solution not an integral equation.

So with this I conclude our lecture. So in this lecture, we have seen that under what condition our Green function exist and with the help of Green function which we have obtained for homogeneous equation with homogeneous boundary value problem, with the help of this we can solve a nonhomogeneous equation with nonhomogeneous boundary value problem. So that is all for this lecture, will continue in next lecture. Thank you very much for listening. Thank you.