Ordinary and Partial Differential Equations and Applications
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Lecture - 03
Existence and Uniqueness of Solutions of a Differential Equations - I

Hello friends. Welcome to this lecture. In previous lecture, we have discussed some basic concept of ordinary differential equation and we will continue our discussion in this lecture also and in this lecture we basically want to discuss the existence and uniqueness theorem for ordinary differential equation. So let us discuss the following.

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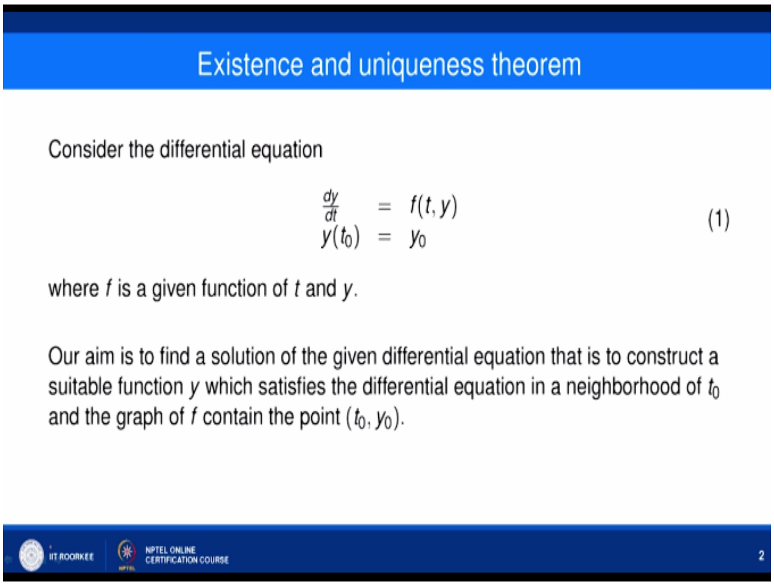

So it means what is the problem that consider the differential equation dy/dt=f t, y with the initial condition y t0=y0 and here f is a given function of t and y. Our aim is to find a solution of the given differential equation that is to construct. It means what, it means we need to find out a suitable function y which satisfy the differential equation dy/dt=f t, y in a neighborhood of t0 and the graph of f contain the point t0 and y0.

So we have discussed certain problem of this dy/dt=f t, y and y t0=y0. Let us discuss some more problems and with the help of these examples we try to discuss more theories. So if you look at this example 1, here we have the linear differential equation.

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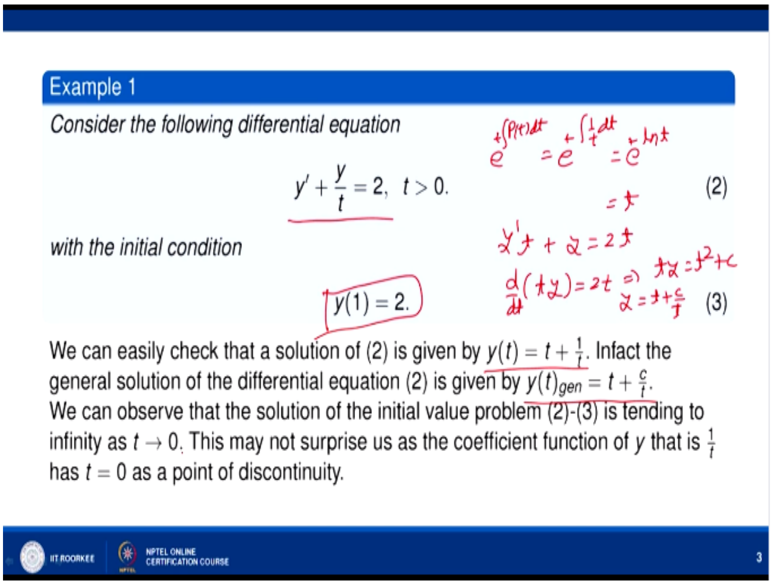

So consider the following differential equation y dash+y/t=2 and here t is>0 and the initial condition along with this differential equation is y 1=2 and we can easily check that a solution of 2 is given by y t=t+1/t. In fact, if you look at this is a linear differential equation y dash+y/t=2 and you can find out the integrating factor here and if you use integrating factor I think integrating factor is e to power –pt dt.

So if you can calculate this then it is coming out to be and pt is what the coefficient of y that is 1/t. So it is going to be what it is e to power –integration of 1/t dt. So it is e to power +lnt, it is not – it is + here integrating factor is this, so it is coming out to be t here. So integrating factor is t here so you just multiply by t here so it is coming out to be y dash t+y=2t and if you simplify this the first term is what d/dt of ty=2t.

And if you simplify this is what you can simply say that it is ty=t square +c. So you can say that y is coming out to be t+c/t. Now this c is a constant which you can find out using the initial condition y 1=2 and you can say that when you put y 1=2 your c is coming out to be 1. So your solution of this problem 2 along with the initial condition 3 is given as yt=t+1/2 and we have already seen that the general solution of this initial value problem is given by t+c/t.

And we can observe that what we want to observe here is that the solution y t=t+1/t is having problem when we take t is tending to 0. What do you mean by problem? that here the solution is tending to infinity as t tending to 0 and that we can say that since the equation has a similarity at t=0 here if you look at the coefficient of y is 1/t and is t tending to 0 then y/1/t is tending to infinity.

So it means that this differential equation itself has say similarity at t=0, so we may consider that of course solution may also have the similar kind of nature. It means that solution will also tend to infinity as t tending to 0. So that is the observation we can observe from this.

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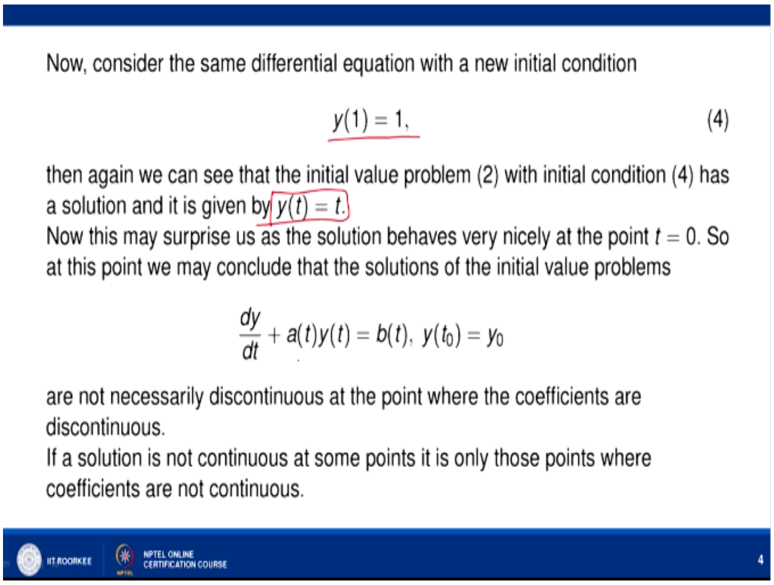

But if you look at the same differential equation but with a new initial condition that is y 1=1. In a previous, we have condition y 1=2 but now we are considering the same differential equation with y 1=1, so if you look at the general solution is given as y t=t+c/t. Now if you put the condition that y1=1 in place of 2 then we can say that c is coming out to be 0. So it means that here the solution is given by y t=t.

But this create a little bit problem why because if you look at the differential equation, this differential equation has a problem at t=0 means as t tending to 0 the coefficient function of y is going to be unbounded as t tending to 0 but if you look at the solution here that solution y t=t is very nice and we can say that the solution behaves very nicely at the point t=0. So at this point we may say that that in case of linear differential equation that is dy/dt+a t y t=b t along with the initial condition y t0=y0.

The solution may not be necessarily discontinuous at the point where the coefficients are discontinuous. So it means that if a t is discontinuous at a particular point say t=t0 then solution may not be discontinuous at that particular point but if a solution is not continuous at some point it is only those point where coefficients are not continuous. So it means that it may not necessary that solution may have discontinuity at the point where this a t is discontinuous.

But if at all solution has a discontinuity, it has to be along the point where a t is discontinuous, that is the observation we have observed from this current example.

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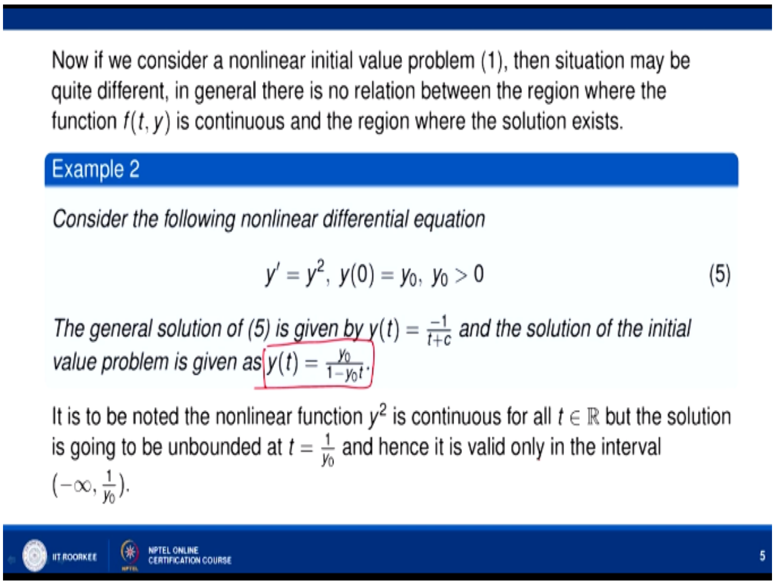

Now we want to consider one more example but this time in place of linear differential equation now we want to consider the nonlinear initial value problem and we can say that if we consider nonlinear initial value problem then situation may be quite different. In general, there is no relation between the region where the function f t, y is continuous and the region where the solution exists right.

So initially in case of linear differential equation we have seen that solution may not be discontinuous at all but if it has a discontinuity it has to be only at those points where the coefficient function is having discontinuity but in case of nonlinear initial value problem this is quite different. Here we may see that there is no relation between the region where the function f t, y is continuous and the region where the solution exists.

For example, consider the following nonlinear differential equation y dash=y square with the condition y0=y0, here y0 we are considering as a positive real number. Now the general solution of 5 this equation number 5 is given by y t=-1/t+c. This you can easily calculate, it is a separable equation and you can solve this separable equation along with this initial condition.

So you can find out the value of c using the initial condition y0=y0 and if you apply that initial condition we can say that the solution is given by y t=y0/1-y0 t and here we can see that the function this y dash=y square, the nonlinear function y square is continuous for all t belonging to R. So here this is a continuous function for all t but the solution is going to be unbounded at t=1/y0.

If you look at here since y0 is positive, so 1-y0 t is going to be 0 at t=1/y0. So it means that the solution is going to be unbounded when t=1/y0. So it means that the solution is valid only in the interval -infinity to 1/y0.

So here we have seen that the solution may not have any relation that where f t, y is continuous and solution may not be continuous in that kind of interval.

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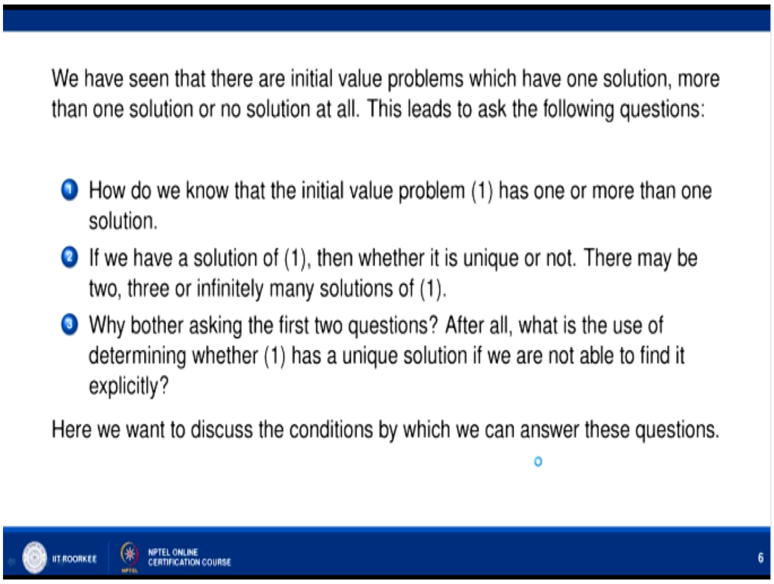

And also we have seen that there are some initial value problems which have one solution, more than one solution and no solution at all. This we have seen in previous lecture. So if you consider these 2 examples along with the initial value problem which we have discussed in previous lectures we may ask the following questions. First that how do we know that the initial value problem has one or more than one solution?

First of all, we are not sure whether a given initial value problem may have a solution or not but if we have a solution of 1 then whether it is unique or not. It means our second question maybe like this that if we have a solution of 1 then we need to worry about the uniqueness of the solution. It means that it may happen that the same initial value problem may have 2, 3 or infinitely many solutions of 1, so this is our second doubt.

Third doubt is why we are worrying about the existence of solution, after all what is the use of determining whether one has a unique solution if we are not able to find it explicitly. It means that it may happen that a given initial value problem has a solution, it means that we are able to find out the solution of first 2 problems that it has a solution, it may have one solution or more than one solution.

But we are not able to find out the solution in an explicit manner. It may happen that the existence is proved but we are not able to find out the solution in explicit manner but this question 3 may not have much problem because nowadays we have several softwares, very useful softwares are available which can find out the solution not in analytical manner but they can find out the solution in a numerical manner.

It means that we can always find out a solution which is accurate up to 3, 4 decimal place or whatever desired accuracy we want if we have very good software. So I think this third question may be solved if we have a software and we can find out the solution which is accurate up to decimals of few places, maybe 4 places, 8 places or 16 places. So here in this lecture and coming lecture we want to discuss the condition by which we can solve these 3 questions.

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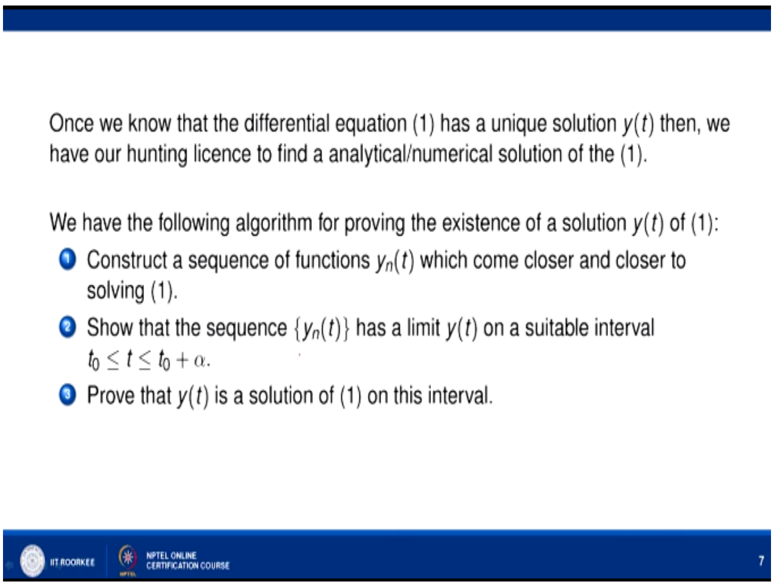

So once we know that we want to understand the importance of the first 2 problems that once we know that the differential equation 1 has a unique solution y t, then we have a license to find out analytical or numerical solution of the 1. So it means that first thing we want to consider that a given differential equation has a solution or not. If it has a solution, we need to worry whether it has a unique solution or not.

If the initial value problem has unique solution, then we try to find out the method to find out that particular solution and it is very advantageous situation when we have a unique solution. So how to find out that unique solution, what we try to do here, we try to approximate the solution of the initial value problem using different iteration schemes. So here we want to find out one iteration scheme which is known as Picard iteration scheme.

And what we are going to do is the following that we have the following algorithm for proving the existence of a solution y t of 1. Construct a sequence of function y and t which come closer and closer to solution of 1. It means that somehow we want to find out a sequence of function which is going to be converse to the solution of the problem 1.

So first we need to construct a sequence and then we want to find out that this constructive sequence has a limit y t in some kind of interval say t0 to t0+alpha that we need to find out in this particular procedure and third very important thing that we need to show that this limit which we have considered as a limit of the sequence. This limit is the solution of 1 in the said interval that is the say outline of the following theorem which we are going to discuss.

So first thing is construct a sequence, second is that we need to show that this sequence has s limit in some interval and third is that limit is actually a solution in that particular interval that we are going to see in this lecture.

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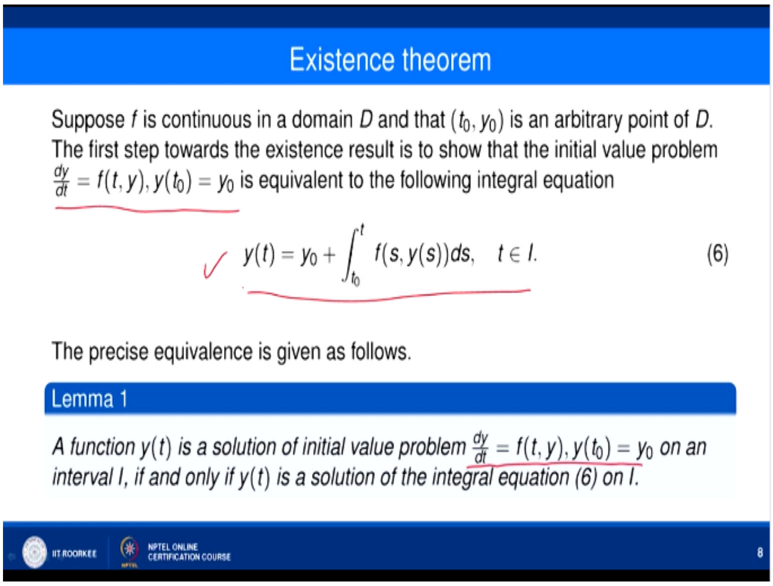

So to consider the existence theorem first we let us assume that suppose f is continuous in a domain D and that t0, y0 is an arbitrary point of t. So first step towards the existences result is to show that the initial value problem dy/dt=f t, y with the condition y t0=y0 is equivalent to the following integral equation y t=y0+t0 to t f s, ys ds where t is belonging to some interval.

And to show the equivalence between this differential equation and the integral equation, we consider the following lemma which says that a function y t is the solution of initial value problem dy/dt=f t, y with y t0=y0 on an interval I if and only if y t is a solution of the integral equation 6 on the interval I. So first we need to show the equivalence and then we proceed with existence theorem while showing the solution of this integral equation.

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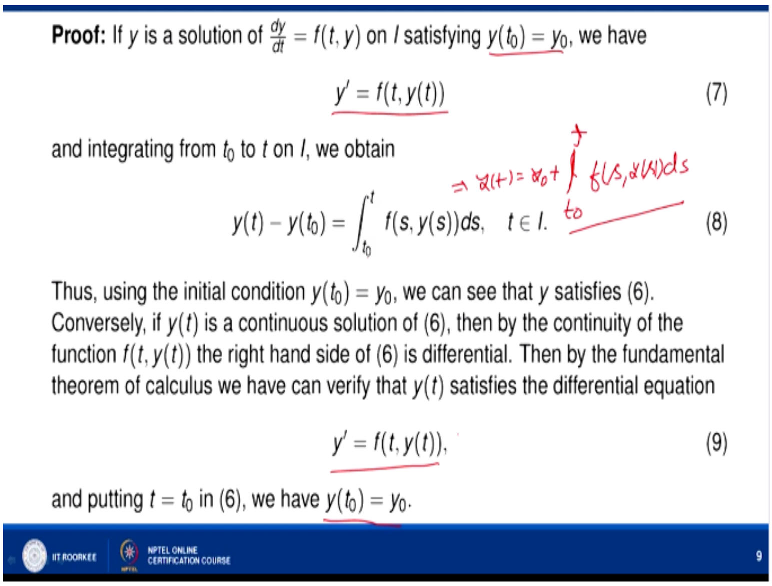

So let us prove the existence equivalence of these 2 differential equation and integral equation. So if y is the solution of dy/dt= f t, y on I satisfying y t0=y0 we have y dash=f t y t. Here y dash denotes the symbol dy/dt and we integrate from t0 to t on this and we have what y t-y t0=t0 to t f s, y s ds when t belonging to I. Now here we already have this condition that y t0=y0.

So it means that this I can write as y t=so this we can write as y t=y0+this integral t0 to t f of s, y s ds and which is nothing but equation number 6. So it means that we have shown that if y is the solution of this differential equation initial value problem then y is the solution of this integral equation. Now we want to show the converse part, conversely if y t is a continuous solution of 6 then by the continuity of the function f t, y t the right hand side means this side is differentiable.

So differentiable then by the fundamental theorem of calculus we can verify that y t satisfy the differential equation. So we simply differentiate this and when we differentiate this, this will give you y dash t. This is a constant so this will be 0 so y dash t=derivative of this using (()) (17:28) theorem we can consider that it is nothing but y dash t=f t, y t and if you look at this integral equation and if you put t=t0 then we will get y t0=y0 because this term is going to be 0.

So it means that from the integral equation we can say that y t0=y0 and y dash=f t, y t. So it means that we have seen the equivalence of the initial value problem and this integral equation.

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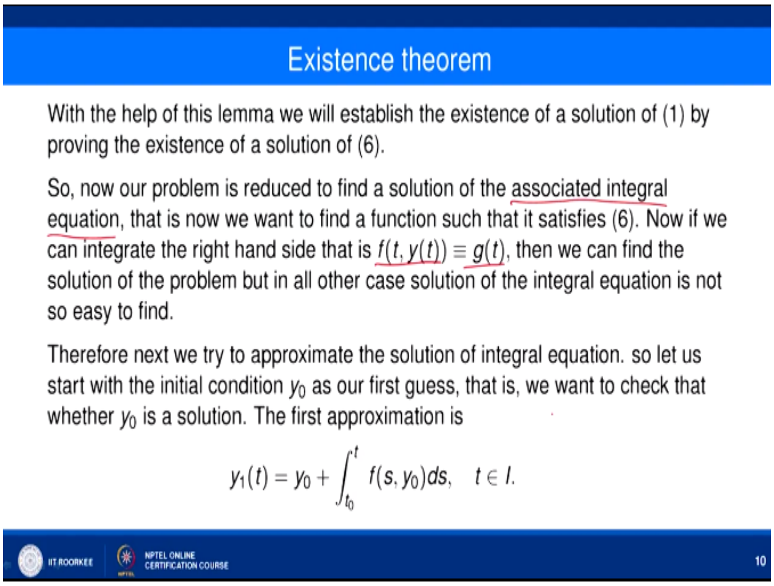

So once we have equivalence we want to discuss the existence theorem. So with the help of this lemma we will establish the existence of a solution of 1 by proving the existence of a solution of 6. So it means that our working procedure is to find out the solution of the integral equation rather than solution of the differential equation. Now these 2 are equivalent, so we can show that once we have the existence solution of integral equation then we have a solution of differential equation.

So now our problem is to reduce to find a solution of the associated integral equation that is our problem now and that is now we want to find out a function such that it satisfies 6. Now here it may happen that we can integrate right hand side that is f t, y t is some function of t which we can integrate or you can simply say that if this part if you look at this if this is integrable f s we can find out this integral in a precise manner.

Then we can find out the solution y t in a precise manner in explicit manner but problem occurs when we do not have a function like this which can be integrable. So it means that therefore next you try to approximate the solution of integral equation because we are not able to find out the solution in an explicit form. So let us start with the initial condition y0 as our first case.

So once we want to find out the approximate solution the first guess is the condition is y t0=y0 because we already know that at point t=t0 it satisfy the initial value problem. So our first approximate is this y t is y0.

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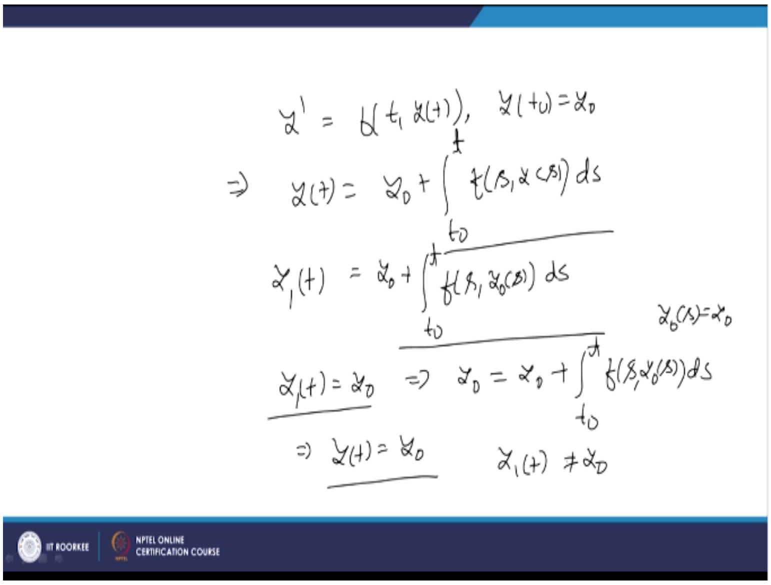

So here if you look at what is our problem, here if you consider this we have this problem, y dash=f t, y t and here y t0=y0. So we have seen that it is equivalent to the following integral equation that is this t0 to t f of s y of s ds so that we have already seen. Now it all dependent on this integral. If we are able to find out this integral in explicit manner, we have a solution but if we are not able to find out this integral then we are in trouble that we are not having a solution.

So what we try to show that if we try to approximate the solution in the following manner, suppose we say that here we say that if we replace t0 to t f of s, now in place of y s let us say that this y0 s is an approximation of the solution y s. So we want to find out the approximation of y t. So first guess is let us say that in place of y if we put y0 s so just calculate this and we call this as y1 t and we call this as a first approximation.

So calculate this quantity y0+t0 to t f s y0 s ds. Now if we calculate this quantity and if we say that y1 t is coming out to be y0, this implies that y0=y0+t0 to t f of s y0 s ds. Here this y0 s is nothing but y0. So it means that what we have shown here that if y1 t is y0 it means that y0 satisfy this integral equation. This implies that your y t is nothing but y0 and y0 is a solution if it happened.

But if it not happened it means if y1 t is not y0 then we have to move further, so that we are going to discuss here. So here we say that the first approximation is given as this y1 t=y0+t0 to t f s y0 ds where t belongs to I.

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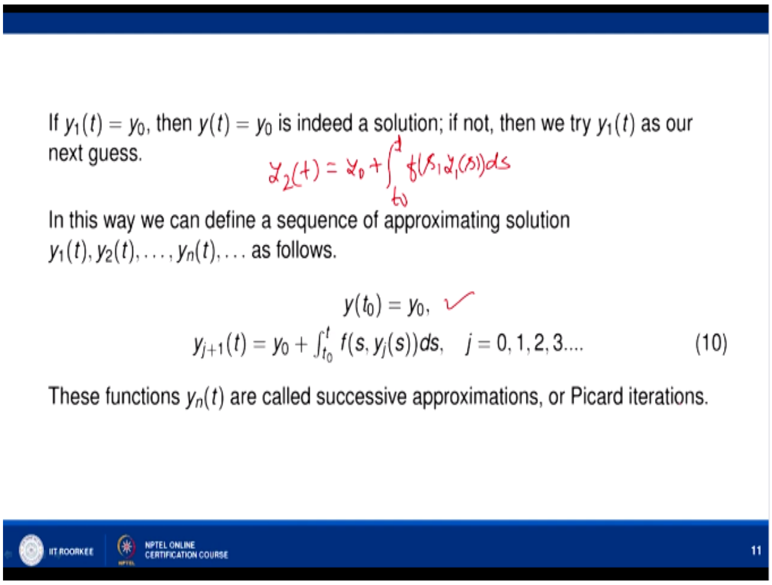

And if y1 t is y0 then we have shown that y t=y0 is indeed a solution. If not then we try y1 t as our next guess. So by taking this y1 t we define y2 t as y0+t0 to t f of s y1 s d of s and then we try to see that if y2 t is coming out to be y1 t we stop and we say that y1 t is our solution. If we say that y2 t is not y0 then we move to next guess and in this way we can define a sequence of approximation solution y1 t, y2 t, yn t as follows.

Here y t0=y0 and yj+1t=y0+t0 to t f s yj s ds where j is from 0, 1, 2, 3 and so on and these function y and t are called successive approximation or Picard iteration. So the first process is done means we are able to construct the sequence which we trying to prove that this sequence will converge to the solution of the integral equation.

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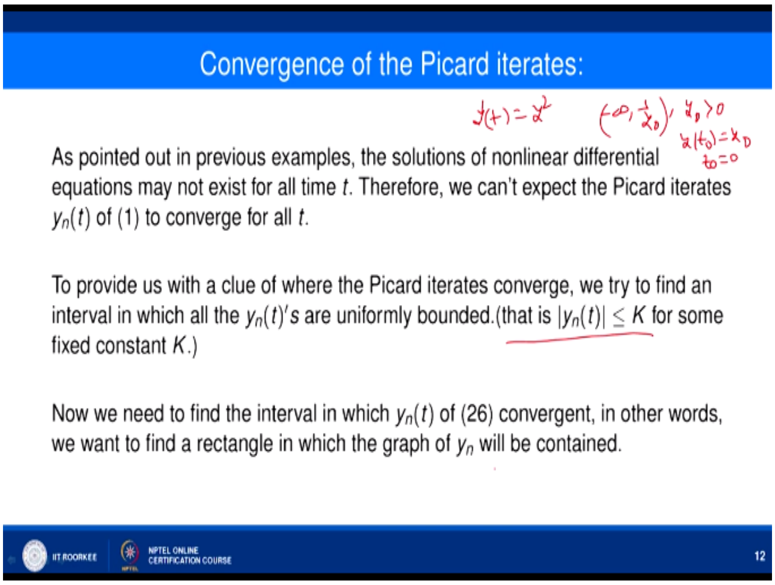

So as pointed out in previous example that the solution of nonlinear differential equation may not exist for all time t. Therefore, we cannot expect the Picard iteration yn t of 1 to converge for all t because we have already seen this example if you remember this example that y dash t=y square right and we have seen that the solution exist only for this interval –infinity to 1/y0, here y0 is positive and y t0 is defined as y0, in fact here t0 is=0.

So we have seen that in this particular example we have seen the solution may not exist for all time t. So it means that whatever we are going to prove we will prove that it will not converge for all time t. So it means that to provide us with a clue of where the Picard iterative is converge we try to find out an interval in which all the yn t's are uniformly bounded. It means that we need to find out the interval such that your yn t is bounded by some say constant K where K is some fixed constant K.

So now we need to find the interval in which yn t of the previous equation convergent. In other words, we want to find out a rectangle in which the graph of yn will be contained. So that we are going to do and that is very much related to the nonlinear function f.

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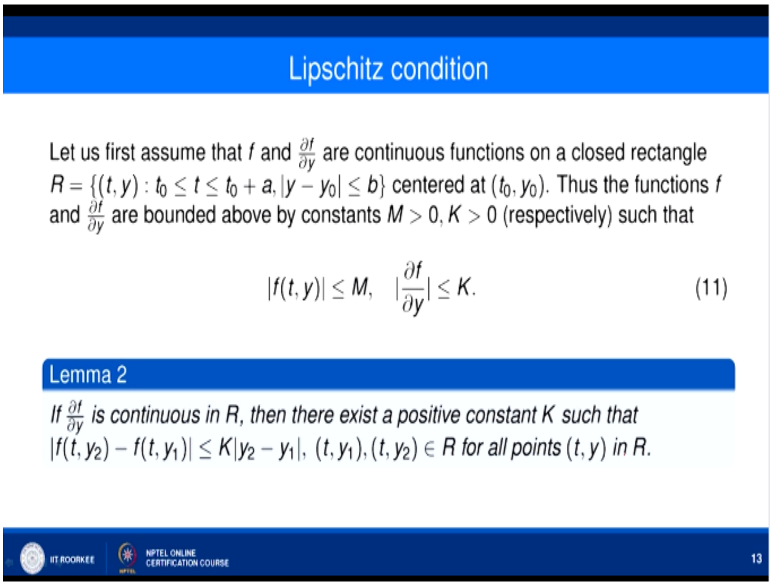

So let us first assume that f and dou f/dou y are continuous function on a closed rectangle where rectangle R is defined like this. It is set of all t and y where t is lying in this interval t0 to t0+a and y-y0 is<=b where a and b are some real constant and this rectangle is centered at t0 and y0 and it means that the function f which we have assumed as a continuous function and dou f/dou y which we also assume that it is continuous function though this function f and dou f/dou y are bounded above by constants M>0.

So we have assumed that there exist a constant M says that modulus of f t, y is<=M where t, y belongs to this R and dou f/dou y is also bounded by another constant K for t and y belonging to this rectangle R. So we say that if dou f/dou y is continuous in R then there exist a positive constant K says that modulus of f t, y2-f t, y1 is<=K y2-y1 where t, y1 and t, y2 belongs to R for all points t1, y1 and t2, y2, t, y1 and t, y2 belongs to R.

So we show that dou f/dou y is continuous in R is equivalent to the condition of this.

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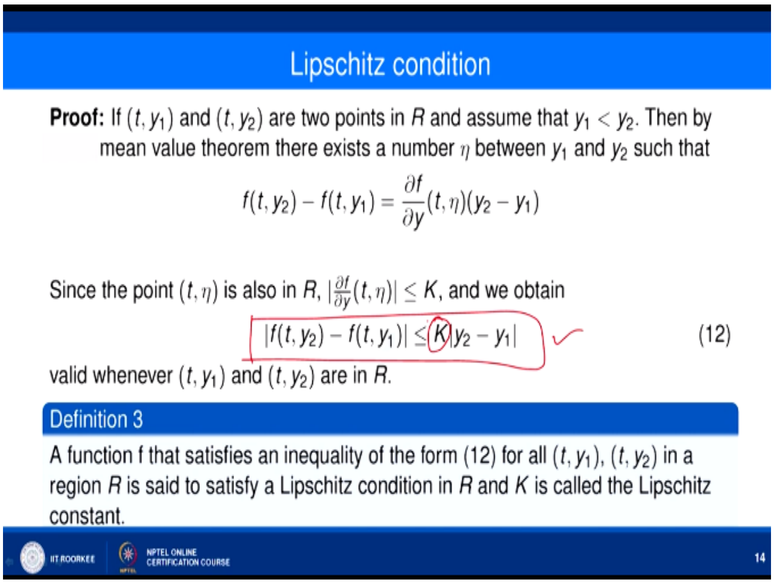

And this condition has a very important name that we are going to discuss in next slide. So first let us prove this lemma. So here we say that f t, y1 and t, y2 are 2 points in R and assume that this y1 is<y2 then by Rolle's mean value theorem we can say that there exist a number eta between y1 and y2 such that f t, y2-f t, y1=dou f/dou y t, eta y2-y1 and we can say that since the point t, eta is also in R then we can say that dou f/dou y at this point t and eta is bounded by this K that we have already assumed here in this thing.

Here we assumed that modulus of dou f/dou y is<=K so it is true for all t, y in this rectangle R. So in particular it is also true for this t and eta, so dou f/dou y at the point t, eta is also bounded by K. So here from this we can say that modulus of f t, y2-f t, y1 is<=K y2-y1 and it is valid whenever t y1 and t y2 are in R in this rectangle and this condition has a name and we try to give this inequality as a following thing.

A function f that satisfy an inequality of the form 12 for all t y1, t y2 in the region R is said to satisfy a Lipschitz condition in R and K is called the Lipschitz constant. So it means that if f satisfies the condition given in 12 then we say that f satisfy the Lipschitz condition and the constant appearing here is called the Lipschitz constant.

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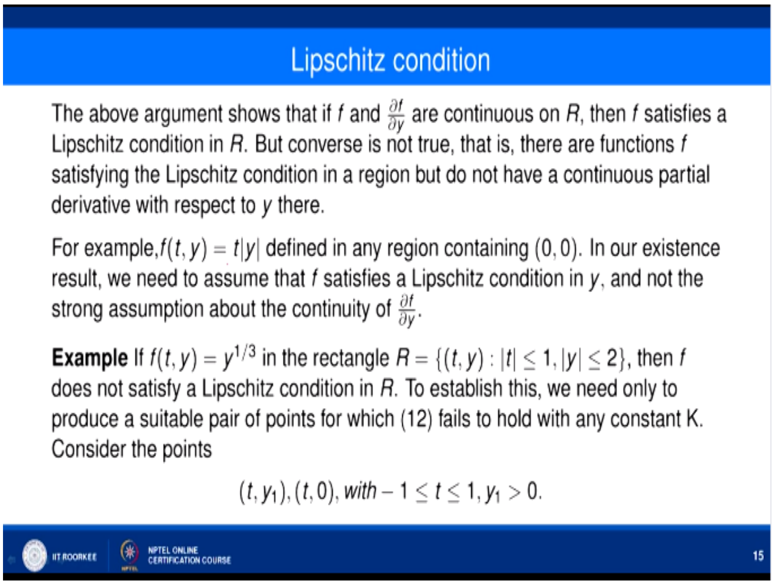

So the above argument shows that if f and dou f/dou y are continuous on R then f satisfy a Lipschitz condition in R but converse may not be true. In fact, there are some function f which satisfy the Lipschitz condition in a region but do not have a continuous partial derivative with respect to y there. For example, if you consider this function f t, y= t modulus of y, you define in a region containing the point 0, 0.

So in our existence result we need to assume that f satisfy a Lipschitz condition in y and not the strong assumption about the continuity of dou f/dou y. So we are discussing this Lipschitz condition because in existence and uniqueness theorem we are going to assume that f satisfy a Lipschitz condition that is why we are discussing the Lipschitz condition.

So here we can simply say that since f is not the partial derivative of f does not exist we simply say that dou f/dou y is not continuous in a region where 0, 0 is contained but we can easily see that this satisfy the Lipschitz condition, you can see like this.

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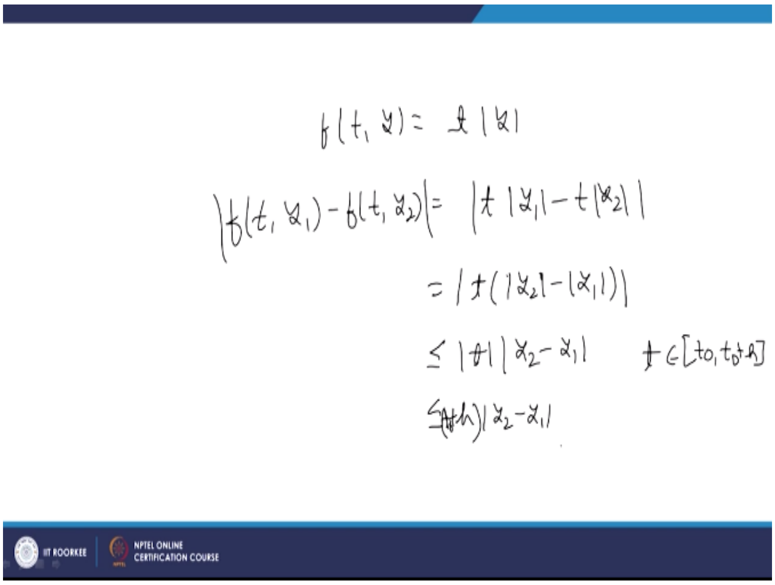

Here we can simply say that f t, y is=t modulus of y so here we can say that f t, y1- f t, y2 it is t modulus of y1-t modulus of y2=modulus of t y2-y1 and we already know that it is what it is here we can say modulus of y2-y1. Now this t belongs to some interval so we can say that if t belongs to t0 to t0+h we can say that t is bounded by h so we can write h/2-y1. So we can say that f satisfy the Lipschitz condition like this.

So since t belongs to t0 to t0+h, we can say that this t is bounded by t0+h. So here we can say that f is satisfying the Lipschitz condition like this.

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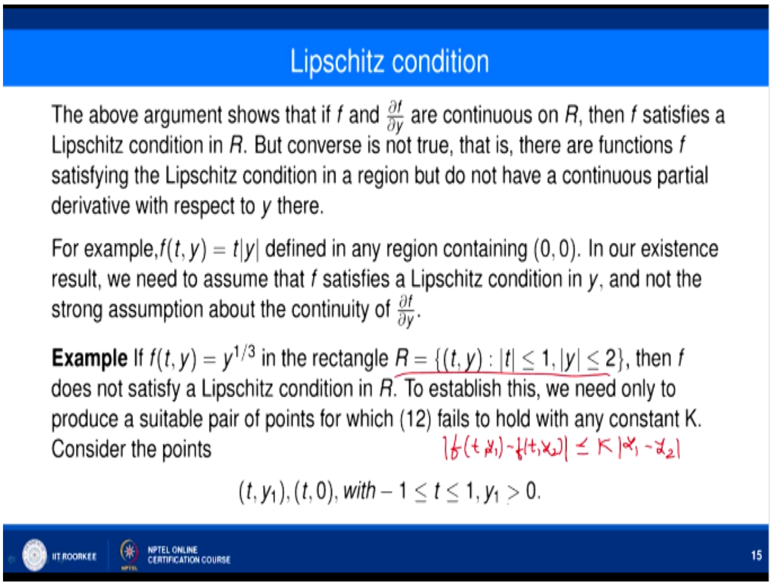

But we have already seen that since it has not partial derivative we say that though f is satisfying the Lipschitz condition but here partial derivative is not continuous and we can say that this existence that continuity of partial derivative is a stronger condition than the Lipschitz condition.

Now we consider one more example where we say that if f t, y=y to power 1/3 in a rectangle R which is defined like this is a set of all t, y such that modulus of t is<=1, modulus of y is<=2 then we can say that f does not satisfy Lipschitz condition in R. To establish this, we need only to produce a suitable pair of points for which this equation number inequality 12 fails.

So here the purpose of this example is that it may happen a given function may not satisfy the Lipschitz condition in some region, some rectangle but it may satisfy the Lipschitz condition in some other rectangle. So to show that first of all that it does not satisfy the Lipschitz condition in this rectangle, we need to show that this condition is not to the condition this f t, y1-f t, y2<=K times y1-y2 so this will not hold for any constant K.

So to show that let us consider 2 points say t, y1 and t, 0 where t is lying between -1 to 1 and y1 is some constant which is bigger than 0.

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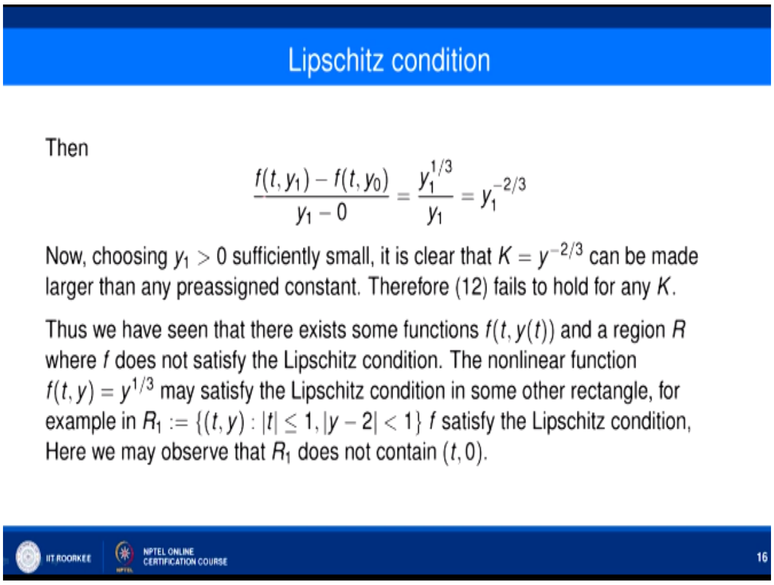

And if you look at this quantity f t, y1-f t, y0/y1-0, here y0 is simply 0 so f t, y1 is nothing but y1 to power 1/3 and f t, y0 is simply 0 and divided by y1. So we can have this as y1 to power -2/3.

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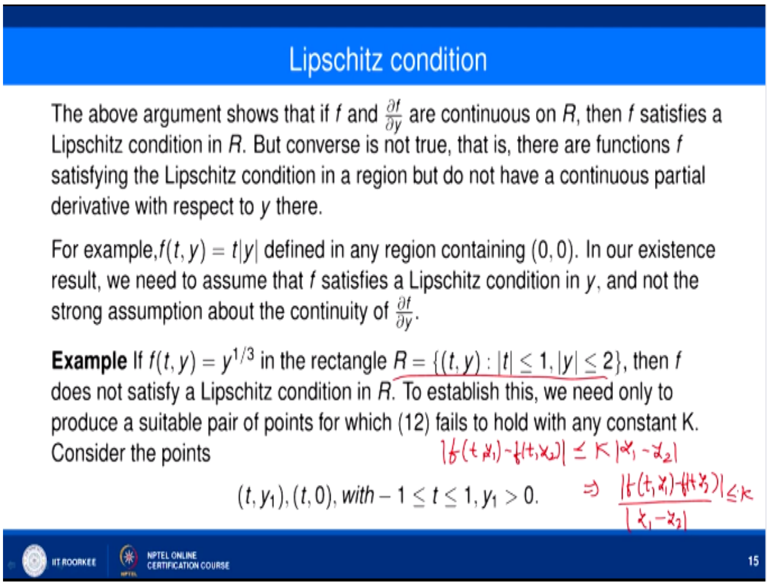

To show that such K does not exist means we have to show that the f t, y1- f t, y2/y1-y2 we can say that this condition is equivalent to this condition. So if we can show that this is an unbounded thing then such a K does not exist. So that is what we are going to see it here. So we can say that choosing y1>0 sufficiently small. So here we can choose this y1 sufficiently small and we can show that this y to power-2/3 can be made larger than any preassigned constant.

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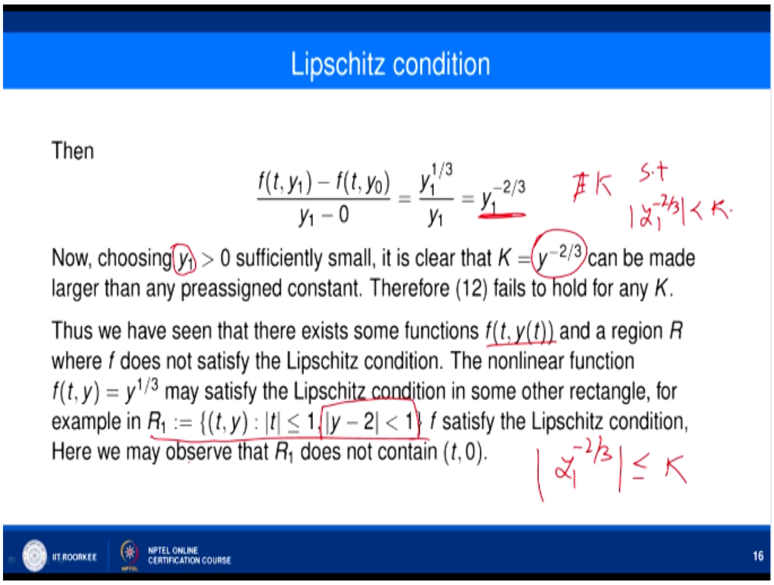

So it means that this quantity y1 to power -2/3 can be made unbounded. So it means that there exist no K such that modulus of y1 to power -2/3 is bounded by any constant K. So it means that if you produce any constant we can take y1 small enough such that it is going to be bigger than that constant. So it means that what we have seen here that this f t, y which is given as y to power 1/3 is not satisfying the Lipschitz condition in the rectangle this.

But we can also see that first of all what we have seen is this that there exist some function f t, y t in a region R where f does not satisfy the Lipschitz condition. The nonlinear function f t, y=y to power 1/3 may satisfy the Lipschitz condition in some other rectangle. For example, if we define a rectangle like this R1=set of all t, y such that modulus t is<=1 and modulus of y-2 is<1.

Look at here, here the quantity y1 to power -2/3 is going to be bounded by some constant K. So here we have seen that in one rectangle it does not satisfy the Lipschitz condition but in some other rectangle it may satisfy the Lipschitz condition. So it means that we are going to use in coming existence and uniqueness theorem. So here we have seen that that this function f t, y=y to power 1/3 is not satisfying the Lipschitz condition in rectangle R which we have defined as this.

Here we have seen that this nonlinear function f t, y=y to power 1/3 does not satisfy the Lipschitz condition in this rectangle R but we have seen that this function may satisfy the Lipschitz condition in some other rectangle for example in this rectangle.

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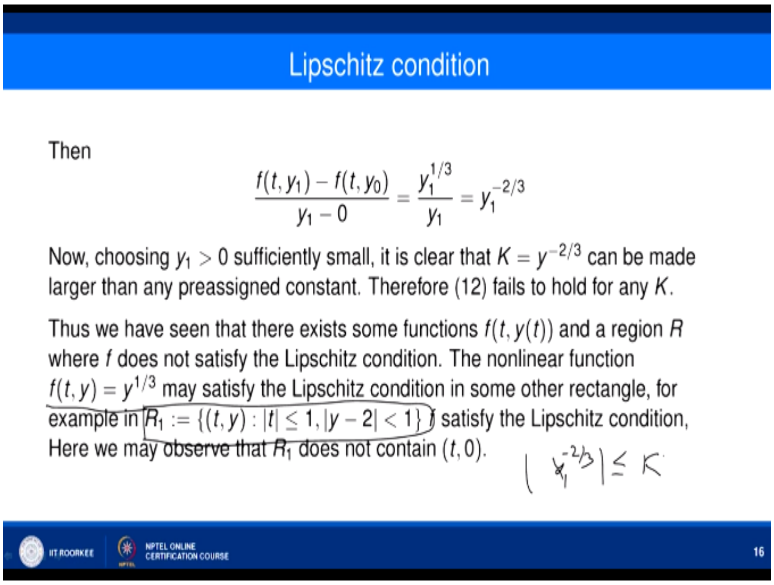

And here we can say that here this quantity y1 to power -2/3 is actually bounded by some constant K. We can always find out the constant K such that dou f/dou y is bounded by a constant K and the existence that modulus of dou f/dou y is bounded implies that f satisfy the Lipschitz condition. So here we stop our lecture. In next lecture, we will continue our discussion after this. So thank you for listening us. Thank you.