Ordinary and Partial Differential Equation and Applications

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Lecture - 21
Critical Points

Hello friends. Welcome to my lecture on Critical Points of a Linear System.

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Consider a system of the form

$$\frac{dx}{dt} = P(x, y),$$
$$\frac{dy}{dt} = Q(x, y), \qquad (1)$$

where P and Q have continuous first partial derivatives for all (x,y).

Such a system, in which the independent variable t appears only in the differentials dt of the left members and not explicitly in the functions P and Q on the right, is called an ***autonomous system.***

Let us consider a linear system of the form dx/dt=P xy, dy/dt=Q xy where P and Q have continuous first order partial derivatives for all the points x, y, such a system in which the independent variable t appears only in the differentials dt of the left hand side members and not explicitly in the functions P, x, y and Q, x, y is called an autonomous system.

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Definition: Consider the autonomous system (1). A point (x_0, y_0) at which both $P(x_0, y_0) = 0$ and $Q(x_0, y_0) = 0$, is called a ***critical point or equilibrium point or singular point*** of (1).

Let us consider the autonomous system given by the equation 1 dx/dt= P, x, y, dy/dt= Q, x, y. A point x0 y0 at which both P, x0 y0=0 and Q x0 y0=0 is called a critical point or equilibrium point or singular point of the autonomous system. So if the functions P, x, y and Q, x, y (()) (01:30) at a certain point x0 y0 then that x0 y0 will be called as a critical point of the autonomous system.

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Example: Consider the linear autonomous system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x, \qquad (2)$$

or

$$\dot{X} = AX \Rightarrow \dot{X} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X.$$

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

let $X = \begin{pmatrix} x \\ y \end{pmatrix}$

then $\frac{dX}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} X$

$\dot{X} = AX$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Now let us consider a linear autonomous system dx/dt=y dy/dt= -x. This can be expressed as we can write it like this dx/dt dy/dt= 0, 1-1, 0, x, y. So let us take vector x to be having components x, y then the column vector dx dt dy/dt will be= dx/dt and this will be= 0, 1, -1, 0 and x or we can write it as X dot AX where A is the 2/2 matrix 0, 1 -1, 0. So the linear autonomous system can be expressed in this form.

We know that the solution of system X dot= AX where A matrix is given by 0, 1, -1, 0 can be found easily. This we have discussed earlier.

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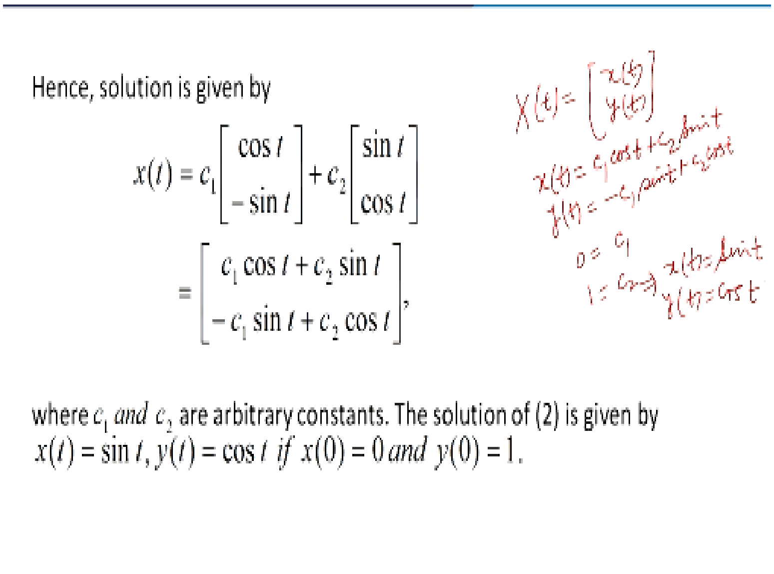

And we know that the system X dot=AX where A is the matrix 0, 1, -1 0 is given by x t= c1* cos t- sin t + c2 times sin t cos t or we can write it as c1 cos t+ c2 sin t and -c1 sin t+ c2 cos t. So here c1 and c2 are arbitrary constants. Now the solution of this linear autonomous system 2 is given by xt= sin t yt= cos t if we assume that x0=0 and y0=1. So that means that if at t=0 if x=0 then what will happen.

Xt= c1 cos t+ c2 sin t so this is Xt= xt, yt. So xt= c1 cos t+ c2 sin t and yt= -c1 sin t+ c2 cos t. So if we put the conditions that at t=0 x=0 then we will have 0=c1 cos t cos 0 will be=1 sin 0 is 0 so c1=0 and yo=1 so 1=-c1 *0+c2. So we have c2=1 and therefore what we will get if you put xt=0 y0=1 then xt will be= sin t and y t will be= cos t. So if we put the initial condition that is at t=0 x is 0 at t=0 y=1 then the solution of the autonomous system 2 is given by xt= sin t yt= cos t.

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This solution defines a path C_1 in the xy plane. If we take $x(0) = -1$ and $y(0) = 0$, then we get $x(t) = \sin(t - \frac{\pi}{2}), y(t) = \cos(t - \frac{\pi}{2})$.
This solution is different from the earlier solution but it also defines the same path C_1. Thus the ordered pairs of functions $(\sin t, \cos t)$ and $(\sin(t - \frac{\pi}{2}), \cos(t - \frac{\pi}{2}))$ are two different solutions of the system (2), which are different parameterizations of the same path C_1.
By eliminating t we get that, path C_1 is the circle with centre at (0,0) and radius 1.

$x(t) = c_1 \cos t + c_2 \sin t$
$y(t) = -c_1 \sin t + c_2 \cos t$
$x(0) = -1$
$y(0) = 0$

Now this solution xt= sin t and yt= cos t defines a path let us call that path as C1 in the xy plane if we take x0=-1 so we have xt= c1 cos t+ c2 sin t is xt and yt=-c1 sin t+ c2 cos t will be general solutions. So xt=c1 cos t+c2 sin t yt=-c1 sin t+c2 cos t, if you take at t=0 x0=-1 and y0=0. We will put these conditions and then from here we shall get xt=sin t-pi/2 and here we will get yt= cos t-pi/2 if you put t=0 here then sin –pi/2 is -1 so we get x0=-1 and when you t=0 here we get cos –pi/2 which is 0 so we get y0=0.

Now we can see this solution is different from the earlier solution, but it also defines the same path c1. Thus the ordered pairs of function sin t cos t and sin t-pi/2 cos t- pi/2 are 2 different solutions of the system 2 which are different parameterization of the same path C1. When we eliminate t we get that see if we have xt= sin t yt=cos t so if you eliminate t you can get x square+ y square=1.

And here also you get x square +y square=1. So when we eliminate t we get that the path c1 is a circle with centre at 0, 0 and radius 1. Now eliminating t is the system 2 gives us see we have these equations dx/dt=y dy/dt=-x. So dy/dx here will be given by –x/y. So dy/dx=-x/y and this gives the slope of the tangent to the path of 2 which passes to the point x, y. So this is your 2 dx/dt=y dy/dt=-x so dy/dx=-x/y.

And therefore this gives us the slope of the tangent to the path of 2 at the point x, y.

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Eliminating t, the system (2) $\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$,

which gives the slope of the tangent to the path of (2) passing through the point (x,y), provided $(x, y) \neq (0,0)$.

$$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow x\,dx + y\,dy = 0$$

$$\Rightarrow x^2 + y^2 = c^2,$$

which is a one parameter family of paths in the xy-phase plane.

And c1 describes the path of the system 2 which gives the slope of the tangent to the path of 2 passing through the point x, y provided x, y is not the origin. Now dy/dx=-x/y gives you x dx+y dy=0. When we integrate these equations we get x square+ y square= C square where C is a parameter. So it is a one parameter family of paths in the x, y plane and we see that it describes a circle with center at 0, 0 in radius C.

So look at this figure we have here this is path C1 now you can see here that the directions on the circles are all in the clockwise direction. Now let us see how we get the direction to be the clockwise direction. See we have C1 xt= sin t yt= cost t in fact 0=0 y0=1. So when you take this we have here x0=0 so at t=0 x is 0 and y is 1. This means we have this point 0, 1 and x, t sin t, yt is cos t so when t increases from say 0 to pi/2 sin t increases while cos t decreases that means x increases while y decreases and both of them are positive.

So we move in this direction and when t becomes pi/2 we get xt=1 and cos 2=0 so we reach here and then when t increases from pi/2 to the pi the value of sin t starts decreasing from 1 to 0 while y becomes from 0 to -1. So this is 0-1 point and then again we can move and come here. So when t increases further from 3 pi/2 to 2 pi we come here and this is your -1, 0 point. So you can see here when we take t=0 y at t=1 and xt= sin t and y t= cos t gives us the solution path as x square + y square=1.

And the direction on the curve the circle is in the clockwise direction.

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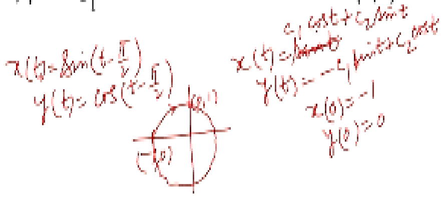
This solution defines a path C_1 in the xy plane. If we take $x(0) = -1$ and $y(0) = 0$, then we get $x(t) = \sin(t - \frac{\pi}{2}), y(t) = \cos(t - \frac{\pi}{2})$.

This solution is different from the earlier solution but it also defines the same path C_1. Thus the ordered pairs of functions $(\sin t, \cos t)$ and $\left(\sin(t - \frac{\pi}{2}), \cos(t - \frac{\pi}{2})\right)$ are two different solutions of the system (2), which are different parameterizations of the same path C_1.

By eliminating t we get that, path C_1 is the circle with centre at (0,0) and radius 1.



Similarly, if you take the other solution xt=sin t-pi/2 and yt= cost t-pi/2 the other solution. Then it is another parameterization of the same path C1. Here what we have at t=0 we have -1. So x=-1 so we have here -1 and then at t=0 y=0 so we have this point and then when t increases from 0 and goes up to pi/2. We get sin 0 sin 0 is 0 so x pi/2 is 0 and when t is pi/2 here we get 1.

So this is we are moving in this direction and reach the point 0, 1 and then we have this clockwise direction. So we are moving along the circle in the clockwise direction. So you can see here the x square+ y square = c square describes the one parameter family of circles in the x y phase plane and the direction along the circles is in the clockwise direction.

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The autonomous system (1) could be interpreted as a system defining a velocity vector field V, where

$$V(x, y) = [P(x, y), Q(x, y)]$$

The x-component of the velocity vector V is $P(x, y)$ while $Q(x, y)$ is the y-component of V. The vector V is the velocity vector of any point R, which describes a path of (1) defined parametrically by a solution $x = f(t), y = g(t)$. At a critical point, both components of this velocity vector are zero. Hence at a critical point, the point R is at rest.

$\frac{dx}{dt} = P(x,y)$

$\frac{dy}{dt} = Q(x,y)$

$V(x,y) = (P(x,y), Q(x,y))$

Now the autonomous system 1 could be interpreted as the autonomous system dx/dt= P xy and dy/dt= Q xy. This autonomous system could be interpreted as a system defining here velocity vector field where V xy= P xy having components P xy Q xy. The x component of the velocity vector V is P xy while the Qxy is the y component of V. The velocity vector v is at any point R which describes the path of one defined parametrically by a solution x=ft y=gt

So you take any path of the system one which is defined vt the solution x= ft y=gt R is a point on that. So then V is the velocity vector say for example you take this so you can take any point R on this where X=ft y=gt. Then this is the velocity vector. So V is the velocity vector and at a critical point both components of this velocity vector are 0 because we know that at a critical point Pxy is 0 and Q x, y is 0.

That means both the components of these velocity vector is 0 and therefore, we say that the point R is at rest. V describes the velocity vector of the point R the x component and y component that is P xy and Qxy both are zero at a critical point and so we can say that the velocity of vector at the critical point is 0 so the critical point is at rest. Now so critical point is also called an equilibrium point.

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In particular, let us consider the special case

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = F(x, y) \qquad (3)$$

which arises from a dynamical system described by

$$\frac{d^2x}{dt^2} = F\left(x, \frac{dx}{dt}\right) \qquad (4)$$

At a critical point $P(x, y) = y = 0, \text{ and } Q(x, y) = F(x, y) = 0$ which implies

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 0.$$

So in particular let us consider the special case dx/dt = y dy.dt= Fxy which arises from a dynamical system d square x y dt square= Fxy you can take dx/ dt= y here and then dy/dt will be= F xy and we will get this linear system. We get this autonomous system dx/dt = y dy/dt= Fxy from this dynamical system. Now at a critical point we know that P xy and Q,xy are 0. So P xy=y0 and Qxy= Fxy=0.

Now Y=0 means at the critical point dx/dt=0 and Fxy=0 means dy/dt=0. So at a critical point dx/dt=0 and dy/dt=0 and further since d square x/dt square= dy/dt you can see from here d square dx/dt=y. So d square x/dt square= dy/dt. Now at a critical point P xy and Q xy are both zeros and therefore y=0 and F xy=0. So we get dx/dt=0 and dy/dt=0 and further from the equation dx/dt=y follows that.

D square x/dt square=dy/dt, but dy/dt is 0 so d square x/dt square=0.

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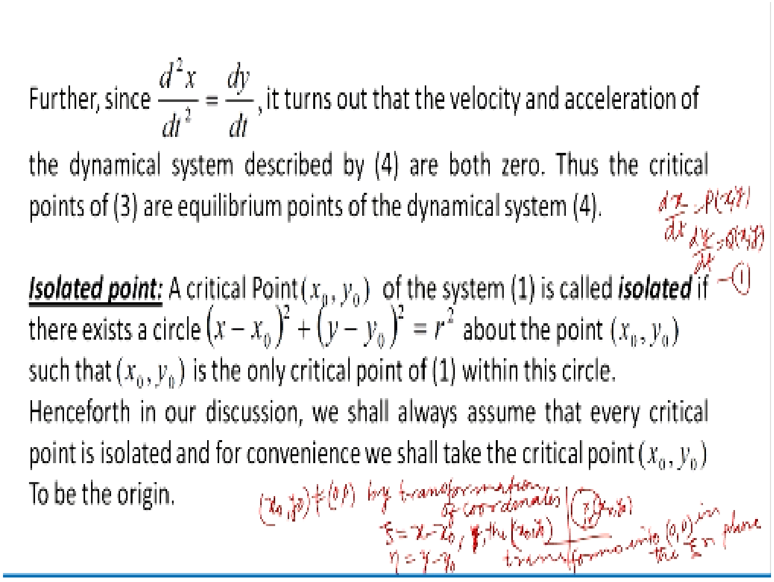

And therefore we can say that the velocity and acceleration of the dynamical system described by the equation 4 are both zero. And thus the critical points of the system 3 are equilibrium points of the dynamical system. Now let us defined an isolated point the critical point x0 y0 of the system 1 that is dx/dt= P xy dy/dt=Q xy this is system 1. So a critical point x0 y0 of the system 1 is called isolated.

If we can find the circle which center at the point x0 y0 of some positive radius R such that x0 y0 is the only critical point within this circle. So you can say this is your point x0 y0 we will call it an isolated critical point if we can find the circle which center at x0 y0 of radius say R such that inside the circle there is no other critical point. Now henceforth in our discussion let us assume that every critical point is isolated.

And for convenience we shall take the critical point x0 y0 to be the origin. Suppose x0 y0 is not the origin then by transformation of coordinates xi = x-x0 eta=y- y0 the point x0 y0 is transformed into the point x0 y0 is transformed into 0,0 into xi eta plan. So without any loss of generality we can assume that the point x0 y0 is the origin.

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Definition: Let 'C' be a path of system (1) and let $x = f(t), y = g(t)$, be a solution of (1), which represents 'C' parametrically. Let (0,0) be a critical point of (1). We say that path 'C' approaches the critical point (0,0) as $t \to \infty$ if

$$\lim_{t \to \infty} f(t) = 0, \lim_{t \to \infty} g(t) = 0. \qquad (5)$$

Thus, a point R tracing out 'C' according to equations $x = f(t), y = g(t)$ approaches to the point (0,0) as $t \to \infty$ independent of the solution used to represent 'C'.

In other words, if 'C' approaches (0,0) as $t \to +\infty$, then (5) is true for all solutions $x = f(t), y = g(t)$ representing 'C'.

Now let us assume C be a path of the system 1 and that x=ft y=gt be a solution of 1 which describes C parametrically which represents C parametrically and let us take 0,0 to be a critical point of 1. As we have already said this point 0,0 is an isolated critical point. Now we say that the path C approaches the critical point 0,0 as t goes to infinity if limit t goes to infinity of t=0 and limit t goes to infinity gt=0.

Thus a point R tracing out C according to equations x=ft y=gt approaches to the point 0, 0 as t goes to infinity independent of the solution use to represent C. In other words, if C approaches 0,0 as t goes to +infinity then 5 is true for all solutions x=ft y=gt representing C. So it is independent of the solution.

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Similarly, a path C_1 approaches the critical point (0,0) as $t \to -\infty$ if

$$\lim_{t \to -\infty} f_1(t) = 0, \ \lim_{t \to -\infty} g_1(t) = 0,$$

where $x = f_1(t), y = g_1(t)$ is a solution defining the path C_1.

Definition: Let 'C' be a path of system (1), which approaches the critical point (0,0) of (1) as $t \to +\infty$ and let $x = f(t), y = g(t)$ be a solution of (1), which represents 'C' parametrically. Then we say that 'C' enters the critical point (0,0) as $t \to +\infty$ if $\lim_{t \to +\infty} \frac{g(t)}{f(t)}$ exists or if $\lim_{t \to +\infty} \frac{g(t)}{f(t)}$ becomes $+\infty$ or $-\infty$.

Similarly, a path C1 approaches the critical point 0.0 as t goes to –infinity if limit t tends to –infinity (()) (21:34) =0 and limit t tends to –infinity (()) (21:36) where x=f1t y=g1t is a solution defining the path C1. Now another definition let C be a path of system 1 which approaches the critical point 0, 0 of the system 1 as t goes to + infinity and let x=ft y=gt be a solution of 1 which represents C parametrically then we say that c enters the critical point 0, 0 as t goes to infinity.

If the limit of the functions gt/ft exist as t goes to + infinity are the limit of gt/ft becomes +infinity and –infinity as t goes to +infinity. So either the limit gt/ft as t goes to +infinity is finite or it is +infinity or – infinity then we shall say that the path C enters the critical point 0, 0 as t goes to infinity.

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Since $\frac{g(t)}{f(t)}$ gives the slope of the line joining critical point (0,0) and a point $R(f(t), g(t))$ on 'C'. Hence "by a path 'C' enters the critical point (0,0)", we mean that the line joining (0,0) and a point R tracing out 'C' approaches a definite limiting direction as $t \to +\infty$.

Types of critical points:

The isolated critical point (0,0) of (1) is called a ***center*** if there exists a neighbourhood of (0,0), which contains a countably infinite number of closed paths $P_n (n=1,2,.....)$, each of which contains (0,0) in its interior and which are such that the diameters of the paths approach 0 as $n \to \infty$ (but (0,0) is not approached by any path either as $t \to +\infty$ or $t \to -\infty$).

Since gt/ft gives the slope of the line joining critical point 0,0 and point R with coordinate ft on C let us see how we get that. Suppose this is your 0, 0 critical point and this is your path describing c R is any point here ft gt. Then the slope of the line of **or as** will be gt/ft. Since gt/ft gets the slope of the line joining the critical point 0, 0 and point R which coordinate ft gt on c hence by a path c enters the critical point 0, 0.

We mean that the line joining 0, 0 and point R tracing out C approaches a definite limiting direction as t goes to infinity. So when a line joining 0.0 and a point R which is stressing out the curve C approaches at definite limiting direction as T goes to infinity we shall say that the path c enters the critical point 0.0. Now let us discuss the various types of critical points. The isolated critical point 0, 0 of 1 is called a center.

If we can find a neighborhood of 0,0 which contains a countably infinite number of closed paths each of which contains 0,0 units interior and which are such that the diameters of the paths approach 0 as n goes to infinity, but 0,0 is not approached by any path either as t goes to + infinity t goes to –infinity.

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By a neighbourhood of (0,0), we mean the set of all points (x,y) lying within some fixed (positive) distance 'd' of (0,0).

Since an uncountably infinite set always contains a countably infinite subset, the definition of center does not exclude the possibility of an uncountably infinite set of closed paths each of which contains (0,0) in its interior.

Now by a neighborhood 0,0 we mean that it is a set of all points x, y lying within some fixed distance d of 0,0. Since an uncountably infinite set always contains a countably infinite subset the definition of center does not exclude the possibility of an uncountably infinite set of closed path each of which contains 0,0 units interior. Now let us look at this figure you can see here these are closed path.

Each path is containing 0,0 inside at its center and they are approaching 0,0, but you can see the diameters of the path approaches 0 as n goes to infinity, but 0,0 is not approached by any path as gt to + infinity or t goes to infinity and each path is containing 0,0 in its interior. Here you can see each path is containing 0, 0 in its interior and the diameters of the path are going to 0, but no path is approaching to 0.

So 0, 0 is not approached by any path either t goes to + infinity or t goes to – infinity so here 0, 0 is a node.

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Saddle point:

The isolated critical point (0,0) is called a ***saddle point*** if there exists a neighbourhood of (0,0) such that the following conditions hold:

i. There exist two paths which approach and enter (0,0) from a pair of opposite directions as $t \to +\infty$ and there exists two paths which approach and enter (0,0) from a different pair of opposite direction as $t \to -\infty$.

ii. In each of the four domains between any of the four directions in (i) there are infinitely many paths which are arbitrarily close to (0,0) but which do not approach (0,0) either as $t \to +\infty$ or as $t \to -\infty$.

Now let us discuss Saddle points. The isolated critical point 0,0 is called a Saddle point if we can find in neighborhood of 0,0 such that the following conditions holds. There exists 2 paths which approach an enter 0, 0 from a pair of opposite directions as t goes to + infinity and there exist 2 path which approach and enter 0, 0 from a different pair of opposite directions.

As t goes to -infinity. In each of the 4 domains between any of the 4 directions in 1 there are infinitely many paths which are arbitrarily close to 0,0, but which do not approach 0,0 either as t goes to + infinity R or as t goes to – infinity.

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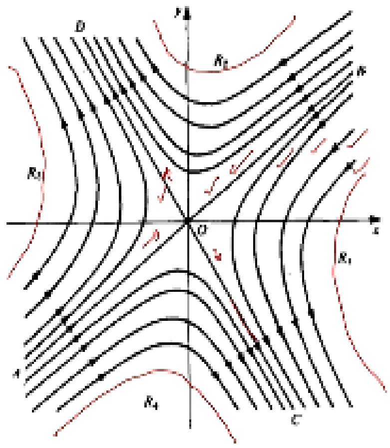

Fig.3

Now you can see here these are 2 paths which approach and enter 0, 0 and these another 2 paths this one. This one and this one they are in opposite directions. Here you can see the direction is this here this direction, this direction. So there exist 2 paths which approach and enter 0, 0 from a pair of opposite directions as t goes to + infinity like this as t goes to + infinity like this and there are 2 paths which approach and enter 0,0 in the opposite directions as t goes to – infinity.

Now you can see these 4 directions divide this (()) (27:43) 4 regions R1, R2, R3 and R4. So there are 4 regions and in each of the 4 domains between any of the 4 directions there are infinitely many path you can see these are infinity many path like this. So there is infinity many paths which are arbitrarily close to 0,0 but which we do not approach 0,0 either as t goes to infinity or as t goes to –infinity. So here 0, 0 is a Saddle point.

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Spiral point: The isolated critical point (0,0) is called a ***Spiral point*** or (focal point) if there exists a neighbourhood of (0,0) such that every path 'P' in this neighbourhood has the following properties:

i. P is defined for all $t > t_0$ (or for all $t < t_0$) for some real number t_0;

ii. P approaches (0,0) as $t \to +\infty$ (or as $t \to -\infty$); and

iii. P approaches (0,0) in a spiral like manner, winding around (0,0) an infinite number of times as $t \to +\infty$ (or as $t \to -\infty$).

Now let us discuss Spiral point the isolated critical point 0, 0 is called a Spiral point or focal point. If we can find In neighborhood of 0,0 such that every path P in this never has the following properties. P is defined for all t>t0 and for all<t0 for some real number t0. P approaches 0,0 as t goes to + infinity or as t goes to – infinity. P approaches 0,0 in a spiral like manner, winding around 0,0 an infinite number of times as t goes to +infinity R as t goes to –infinity. So you can see here.

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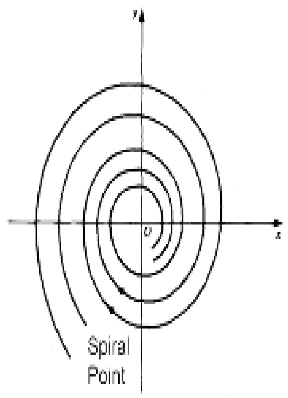

Fig.4

This 0, 0 is a spiral point here because the paths are winding around the point 0, 0. The P is defined every path p you can take any path p is defined for all t>t0 or (()) (29:34) and it is approaching 0, 0 as t goes to +infinity or it is approaching 0, 0. It is approaching 0,0 as t goes to + infinity or -infinity and it is approaching 0,0 in a spiral like manner like this. So as t goes to +infinity or –infinity.

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Node: The isolated critical point (0,0) is called a node if there exists a neighbourhood of (0,0) such that every path 'P' in this neighbourhood has the following properties:

i. P is defined for all $t > t_0$ (or for all $t < t_0$) for some real number t_0;
ii. P approaches (0,0) as $t \to +\infty$ (or as $t \to -\infty$); and
iii. P enters (0,0) as $t \to +\infty$ (or as $t \to -\infty$).

Now we come to Node. The isolated critical point 0,0 is called Node if there exist a neighborhood of 0,0 such that every path P in this neighborhood has the following properties. P is defined for all t>t0 or for all t<t0 for some real number t0. P approaches 0,0 as t goes to + infinity or as t goes to –infinity. And p enters 0, 0 as t goes to +infinity or as t goes to –infinity.

So here p approaches 0,0 in a spiral like manner and it winds around 0,0 and infinite number of times. Here P approaches 0,0 or as t goes to +infinity or t goes to –infinity, but it does not wind around 0.00. So in the case of Node, you can see the path we have p is defined for all>0 for some real number t0 p approaches 0, 0 as t goes to infinity or –infinity and p enters 0, 0 as t goes to + infinity or –infinity.

So you can see this path this one this path A, B and then you can see this path are approaching to origin and we will see that the slope of this path at origin is nothing, but the slope of the line A, B. So this is the characteristic of Node the paths are approaching 0, 0 as t goes to +infinity and – infinity and when we discuss later on in detail we shall see that these paths approach to the origin with the slope of the line A, B. With that I come to the end of this lecture. Thank you very much.