Ordinary and Partial Differential Equations and Applications

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Lecture - 20
Regular Singular Points-V

Hello friends, welcome to this lecture. In this lecture we continue our decision of regular singular point and if you recall in previous lecture we have discussed the case when the roots of the indicial equations differ by an integer and as an example we have taken the Bessel's equation of order 1/2 and we have seen that the equation is this t square/double dash + ty dash + t square – 1/4 y =0.

So this is a Bessel equation of order 1/2 and we have already checked that here t = 0 is a regular singular point and we have also seen that the roots of the indicial equation is given by r1 = 1/2 and r2 = -1/2 and here we can say that r1-r2 = 1.

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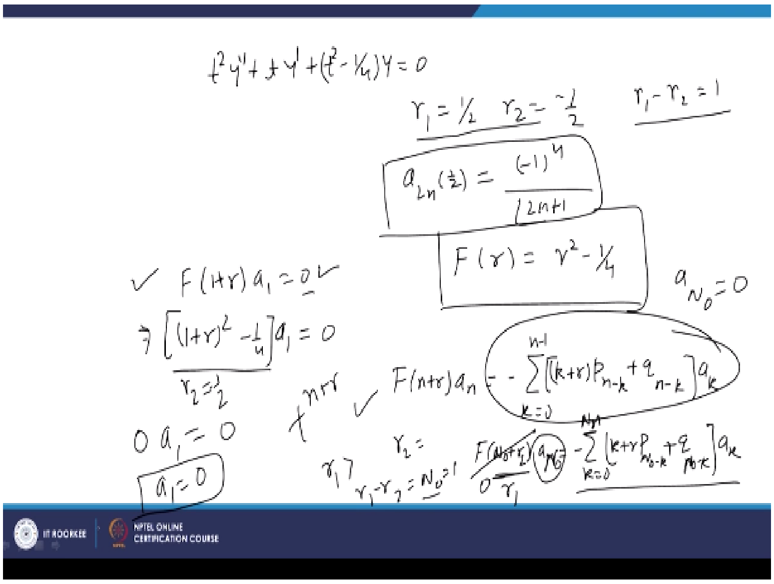

So it means that this case was in the category when the roots of the indicial equation differ by an integer and we also have calculated the coefficient corresponding to r1 = 1/2 that is a2n 1/2 = -1 to the power n/factorial 2n +1 and here please note down that your indicial equation is given as f of r = r square -1/4 by which we have calculated the roots of the indicial equation that is 1/2 and -1/2.

And we have also seen that if you look at the coefficient of t to the power n + r in general theory we have already seen that f of n + r an = -k = 0, 2n-1 k + r Pn – k +qn - k * ak. So here we have seen that if for some let us take r2 as r1 is bigger than r2 and if you take the case corresponding to r2 and that r1 - r2 is differed by some integer.

Let us say n0 then this n = n0 this is what, f of n0 + r2 an0 = - summation k = 02 n0 -1 k + r P n0 – k +q n0 – k and ak. Now here we already know that this n0 + r2 is r1 and r1 is the root of this (()) (03:07) f of r so it means that this is going to be 0. So a n0, 0 * a n0 = this quantity. Now if this quantity is nonzero then we do not have any possibility to find out this an0.

And hence we can say that we do not have any series solution of the form of Frobenius, but if this part is 0, if the righthand part is 0 then we can take any value of this a n0 and in particular we take value of a n0 = 0 and find out the other solution. So in this particular case your n0 is basically 1. So a n0 = 1. So here we look at the corresponding equation for this, the corresponding equation is f of 1 + r a1 = 0.

So here it is quite easy to see that righthand side is already 0 so this righthand side is already 0 and if you look at the values of f of 1 + r that is 1 + r whole square – 1/4 a1 = 0, now if you put r2 as -1/2 then it is what, it is automatically 0, so it is 0 a1 = 0. So it is in this particular problem this righthand side is already so. So this righthand side is already 0 in this particular example.

So here I can take value of a1 as any value in particular I am taking the value of a1 as 0. So it means that here we have the second series solution in the form of Frobenius form.

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(Case 2). $r_2 = -\frac{1}{2}$ and set $a_0 = 1$. Equation (ii) is automatically satisfied, regardless the value of a_1. We will set $a_1 = 0$. The recurrence relation (*iii*) becomes

$$a_n = \frac{-a_{n-2}}{(n-\frac{1}{2})^2 - \frac{1}{4}} = \frac{-a_{n-2}}{n(n-1)}, n \geq 2. \quad (13)$$

Since $a_1 = 0$ so by (13) we have $a_3 = a_5 = a_7 = \cdots = 0$.
The even coefficient are given by

$$a_2 = \frac{-a_0}{2.1} = -\frac{1}{2!}$$
$$a_4 = \frac{-a_2}{4.3} = \frac{1}{4!}$$

and so on.

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So let us continue our study, so here we take r2 = -1/2 and we set a2 = 1 and we say that equation 2 is automatically satisfied and you can take any value of a1 and we will set a1 = 0. If we take any nonzero value, then this I am leaving it to you that if you take a1 as any nonzero value let us say that a1 = 1 for example or a1 = alpha if you take it and then you do all these calculations.

And at the end you will see that y2, the second solution which we are trying to find out corresponding to r2 = -1/2 will contain a part of y1 so that I am leaving it to you. So here now let us find out the solution corresponding to r2 = -1/2 and here we have seen that here the problem which we are discussing is not occurring because here the righthand side is already 0.

So I can choose any value of a1 and we have taken a1 = 0 and since a1 = 0 and by the recurrence relation we can simply say that all the odd terms are 0. Look at the recurrence relation an + -an-2/n-1/2 whole square -1/4 and we can simplify a square – b square and we can write n-1/2 + 1/2 that is n, n-1/2-1/2 that is n-1. So we can write an as –an -2/n * n-1 that is for r2 = -1/2.

So we can calculate a2 as –a0 upon 2 * 1 and a0 is 1 so it is -1 upon factorial 2, a4 as –a2 upon 4.3, so a2 is we have already calculated this, so a4 is 1 upon factorial 4 and so on.

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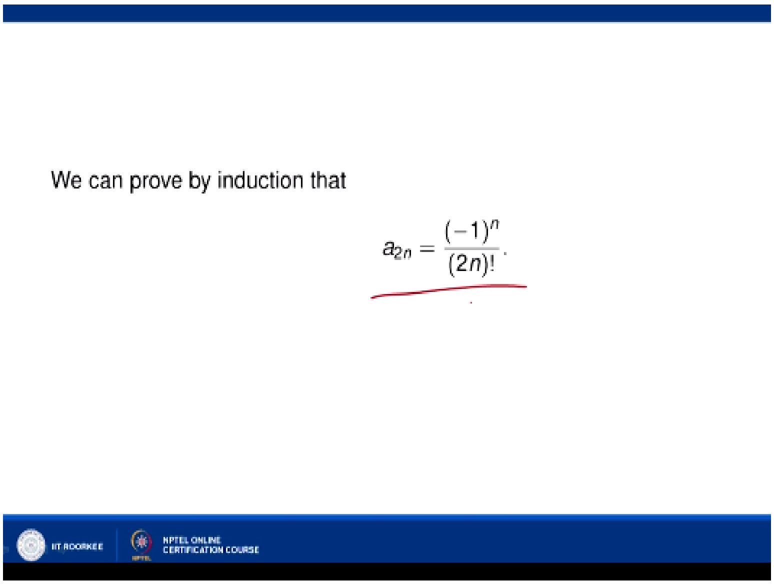

So we can calculate that a2n = -1 to the power n upon 2 to the power factorial 2n. So we can say that by induction we can say that a2n = -1 to the power n/factorial 2n.

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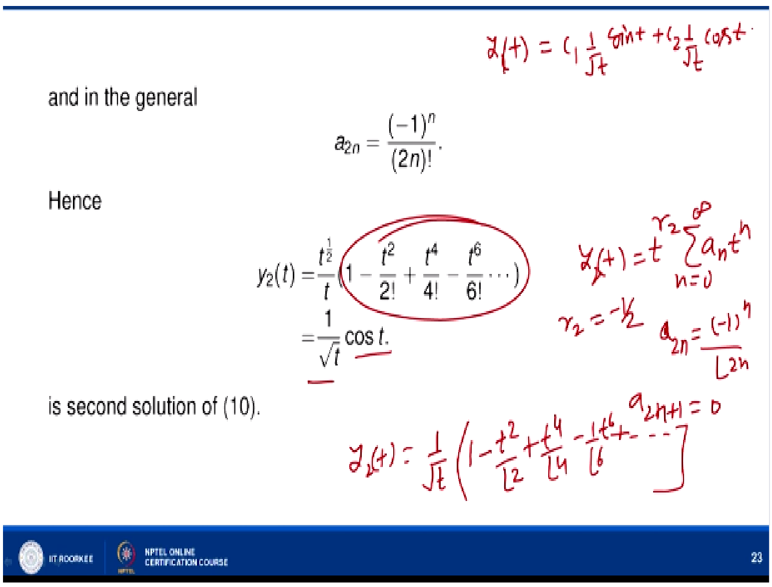

Now corresponding to this we try to find out the solution as y2t = root t, y2t = t to the power say r2 summation an t to the power n, n is from 0 to infinity. So r2 is -1/2 and an we have already calculated a2n as -1 to the power n factorial 2n, a 2n + 1 is simply 0. So we can write it y2t as 1 upon root t because t to the power -1/2 is 1 upon root t and a 0 is 1, so 1-a2, t square upon factorial 2, a4 is what? t to the power 4 upon factorial 4.

A4 is basically 1 upon factorial 4 and A6 is going to be -1 upon factorial 6 t to the power 6 and so on. So if you look at this is 1 upon root t and in the bracket it is this term. So this is what? this is the formula of cos of t. So y2t is written as 1 upon root t cos of t and this is our second solution or Bessel's equation of order 1/2 and we can write down the general solution and we can write down general solution y t as c1, 1 upon root t sin t and c2 1 upon root t cos of t. So this is our general solution of Bessel's equation of order 2.

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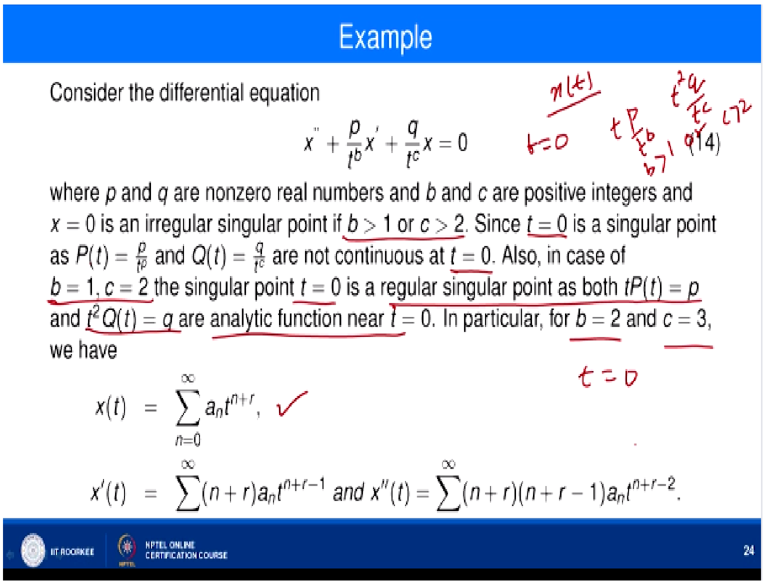

Now so far we have discussed the case of regular singular point and we have seen that in case of regular singular point our possible solution is given in terms of summation an t to the power n + r and by taking this as a solution we are plugging the value of y, y dash, y double dash in the equation and with the help of the equation we try to find out an indicial equation from which we can find out the roots of the indicial equation.

And we try to find out the coefficient with the help of recurrence relation that we have already done. Now question comes that what happen if we have a singular point, but it is not a regular singular point. Then our question is can we apply the same method for which we have applied for regular singular point. So we are going to discuss 2 example which shows that to a certain extent we can do so, but not always.

So it means that that is why our method which we have already discussed is valid are working very good in the sense of regular singular point, but in case of irregular singular point we may not guarantee that our solution work or not. So let us consider the differential equation x double dash + P/t to the power b x dash + q t c x = 0. Here our unknown variable is x and independent variable is t.

So here we are writing our equation like this, xt. So here p and q are nonzero real numbers and b and c are positive integers and x = 0 is an irregular singular point. If b is > 1 or c is > 2. So first of all we can already check that t = 0 is a singular point because the coefficient of x dash and coefficient of x are not continuous at = 0 and then we want to check where it is a regular singular point or not.

But if you take b is > 1 or say c is > 2 then we can check that the coefficient means t * p/t to the power b and t square q upon t to the power c may not be analytic function if b is greater than 1 or c is greater than 2. So either in each of these 2 cases we simply say that t = 0 is not an irregular singular point.

So here let us say that that is already written here that t = 0 is a singular point because pt = p upon t to the power p and qt = q upon t to the power c are not continuous t = 0 and also in the case of b = 1 and c = 2 that is the case which we have already discussed and we say that in case when b = 1 and c = 2 the singular point t = 0 is the regular singular point as both tPt and t square Qt are analytic function.

But in particularly if we take any value for b which is greater than 1 let us say b =2 and let us say c = 3 value and we try to show that in this case we will get 1 Frobenius series solution so for that let us assume that solution is given as xt = n = 0 to infinity an t to the power n + r. So here I am assuming that let us see whether this solution will work for this equation or not. We already know that for b = 2 and c = 3 your t = 0 is not a regular singular point. So let us see whether it will work or not.

For x dash t you calculate and x double dash t you calculate and that is summation n = 0 to infinity n + r, an t to the power n + r -1 and x double dash t is n = 0 to infinity n + r * n + r -1 an t to the power n + r -2. Now plug it back.

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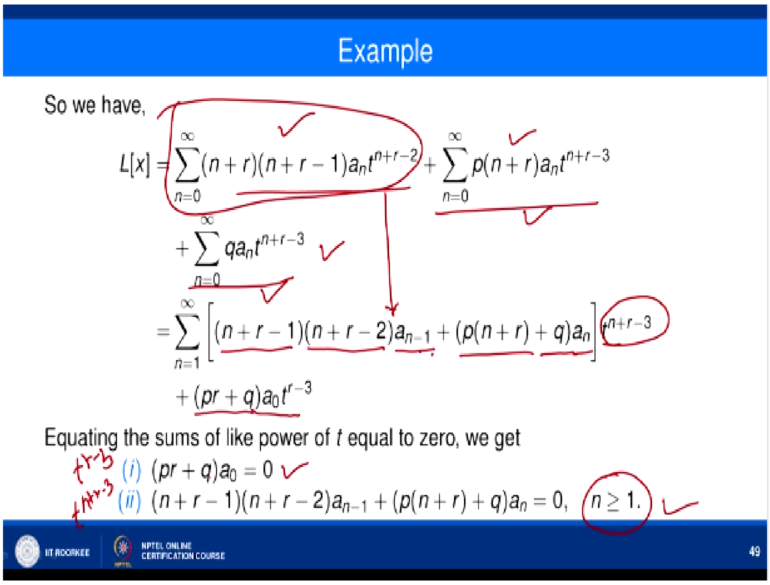

So when you plug it back, you have n = 0 to infinity n + r * n + r -1 an t to the power n + r -2 + n = 0 to infinity b * n + r an and here it is what, here we are assuming that b = 2 here then it will give you an t to the power n + r-3 and last term it is n = 0 to infinity q an t to the power n + r-3, so here I am assuming that t or c = 3. So using this we have this term. Now just collect all the term and we write this as.

Look at the term of t to the power r-3 here then here we have this term and this term will give you the coefficient of t to the power r-3 and that is corresponding to n = 0 so that is what pr, so p r + q a0 * t to the power r -3 is given when n = 0, but when you put n = 1 then this will give you the term t to the power n + r - t to the power r-2 so it means that then this term will also come into picture.

So it means that when you start from n = 1 to infinity, then this will give you what, this will give you n + r-1 * n + r-2 an-1, let us say, take the coefficient of t to the power n + r-0. So here t to the power n + r-3 will come when you take n = c-1. So here we write n + r-1 * n + r-2 an-1 + Pn + r q an. So basically what we are doing, we are just first calculating the coefficient of t to the power r -3 that we can take it from this term as well as this term.

Now we want to write a general term. Let us say that general term is t to the power n + r-3, so these 2 term is already given in terms of t to the power n + r-3 so p n + r + q * an that is already coefficient of t to the power n + r-3 and if you look at from this term if you want to calculate the coefficient of t to the power n + r -3 then you simply replace n/n-1.

When you replace n/n-1 in this then you will get here n = 12 infinity n + r-1 * n + r-2 an-1. So from this you are getting this term. So it means that now we have, we simplify like this. Now equating the sums of like power of t = 0 so first let us say t to the power r-3 and we have pr + q * a0 = 0 and then you write t to the power n + r -3 and that is valid when n is >= 1.

So here when you put then we have n + r-1 n + r-2, an-1+ p n + r + q an = 0 here. So from this in this first equation you can get your indicial equation as pr +q = 0.

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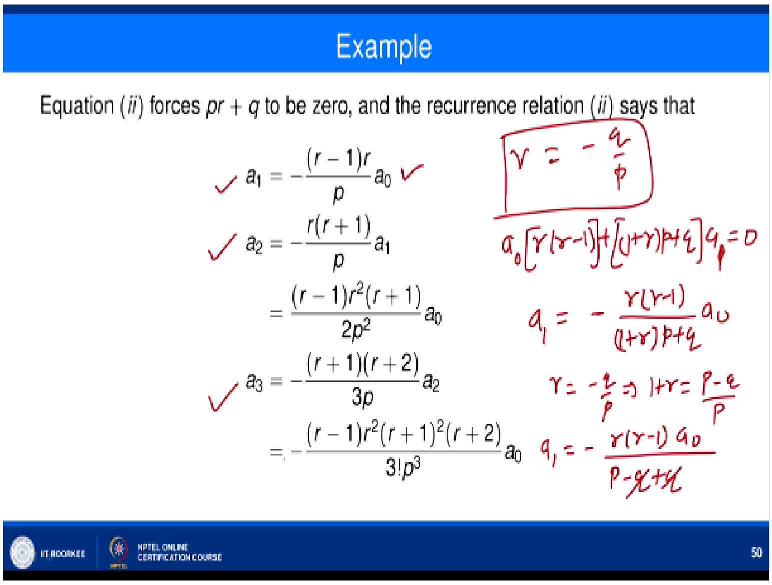

And we can simply say that here since it is a linear equation we will have only one solution that is r = -q/p as your one root of indicial equation. So here please look at here since it is a irregular singular point then only we can say that here we are getting only one root of indicial equation. In fact, here indicial equation is not a quadratic equation, it is a simple linear equation.

So here one root we have obtained as r = -q/p. From this we try to calculate all other coefficients. So a1 is basically what? so here to calculate a1 let us look at here. So here when you put n = 1 then you will get a1 in terms of a0. So use the second and put n = 1. So when you put n = 1 it is what? p1 + r + q a1. So that is what let me write it here.

So here we put n = 1 here then what you will get it is r r-1 a0 + p and 1 + r + q a1 = 0. So you can calculate your a1 like this. So a1 is coming out to be this value. So a1 is going to be what? here this is a0 * r * r -1 + here we have 1+ r P + q a0 = 0. So here you can get this is a1 sorry, so a1 = -r * r - 1 a0/this quantity 1 +r p + q that we have already seen that it is this here.

Now let us use that value of r as –q/p, so using r = -q/p, then 1 + r is going to be p-q upon p. So using this we can write this as a1 as –r * r-1 a0/so it is p-q + q. So we can cancel out, so we can write a1 as –r-1r/p * a0. So a1 is calculated and similarly you can calculate your a2 and so on. So we can calculate all the coefficient using your recurrence relation 2 here.

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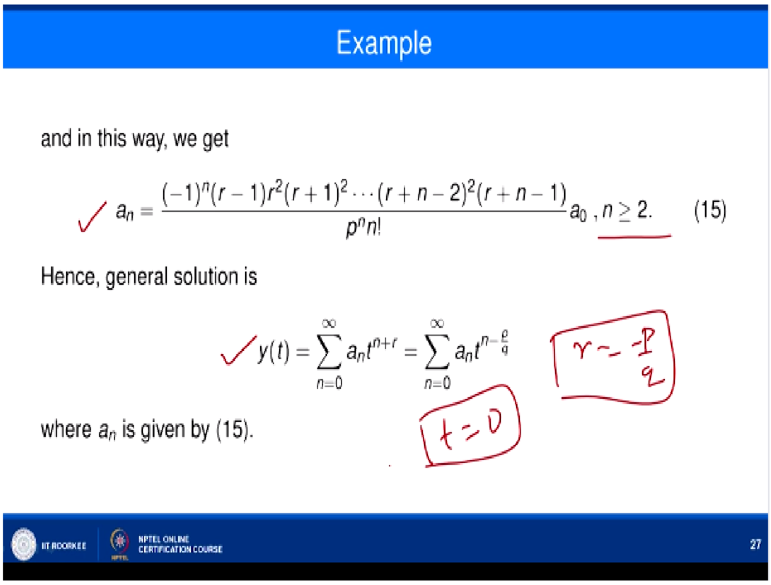

And we can say that our solution is calculated like this, an = -1 to the power n r-1 r square r + 1 and so on and p to the power n factorial n that is n is >= 1 here. So we can write down our solution as yt as summation n = 0 to infinity an t to the power n + r and r is given as -p/q. So even if here t = 0 is your singular point not only singular point it is irregular singular point but still we are able to find out series solution in fact Frobenius series solution in case of irregular singular point.

So here since t = 0 is a weak irregular singular point then we can find out one solution of this form. We are not able to find out the second solution of the form of Frobenius series solution. So how to find out the second series solution for that you can use your variation of parameter method or any given method. So here in this case we have seen that one solution we can obtain.

Now let us, now I am leaving it to you that here I am assuming that your b = 2 and c = 3. Now I simply say that if you take b = 3 and c = 4 in that case you will not have any solution of the form of Frobenius series that I am leaving it to you, that check that when case b = 3 and c = 4 we do not have any solution of the form of Frobenius series. Of course it will have a solution but not of the form of Frobenius series solution.

So here we simply say that here t = 0 is a weak irregular singular point of this differential equation. Now let us consider one more example and example is this.

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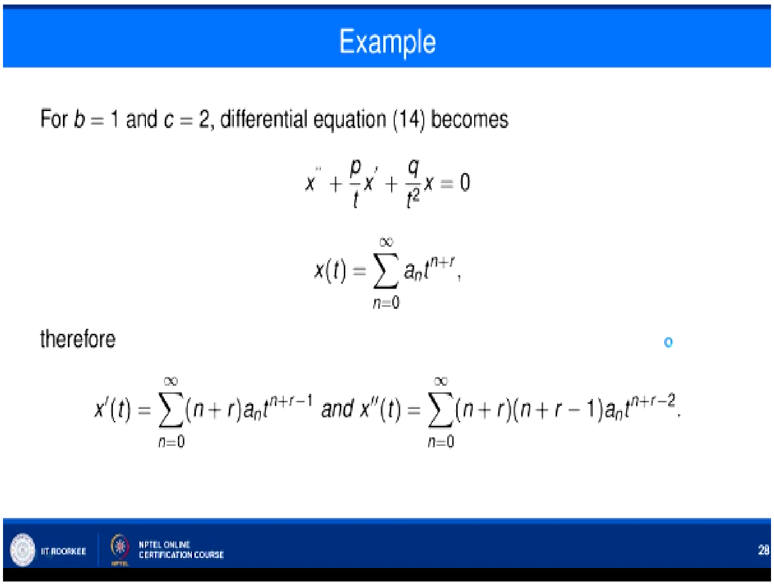

We already know that for b = 1 and c = your differential equation is reduced to this and that is your x double dash + p/t x dash + q/t square x = 0. If you look at this is nothing but your Cauchy-Euler equation right. This is a simple Cauchy-Euler equation and we can simply say that here also if you want to apply the solution xt = n from 0 to infinity an t to the power n + r.

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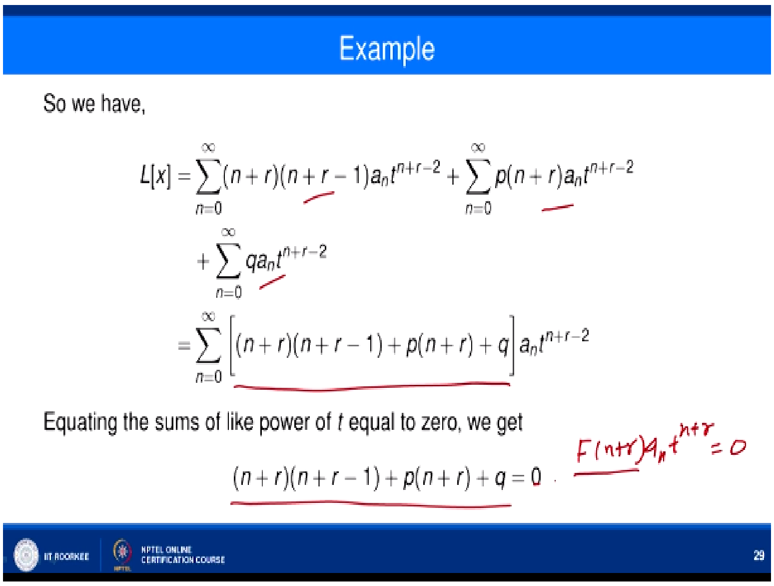

And similarly calculating x dash t, x double dash t we can simply say that lx = what n = 0 to infinity n + r * n + r -1 an t to the power n + r-2 + n = 0 to infinity pn + r an t to the power n + r –2 + q = 0 to infinity q an t power n + r-2 and if you calculate coefficient will be what, n = 0 to infinity n + r n + r-1 +p n + r +q * an t to the power n + r -2 so here we say that here when you start from n = 0 your all terms will come into picture.

So equating the sums of like power of t = 0 will get this term, so this is your general term, so here we write f of n + r an t to the power n + r = 0, so here Fn n + r is given by n + r * n + r -1 + p n + r + q = 0 here.

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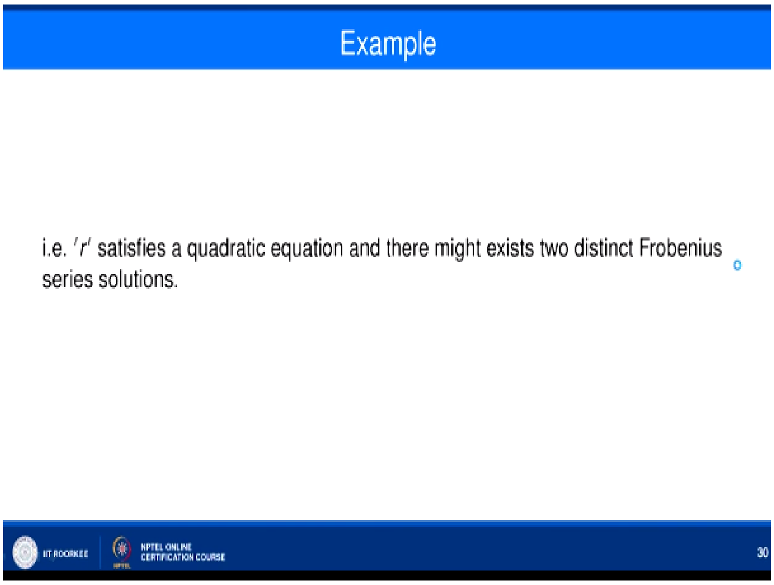

And we can simply say that here this indicial equation when you put n = 0 your indicial equation is what, this indicial equation is a quadrating indicial equation and this will give you 2 roots of indicial equation and we with the help of 2 roots we can find out 2 solutions. One solution is given surely in terms of Frobenius series solution and the other you can get depending on the cases.

But here if you look at this is a simple Cauchy-Euler equation and we can simply say that if you put n = 0 here then it is what r * r -1 + p r + q = 0 and it is the indicial equation which you have obtained for Cauchy-Euler equation. So we simply say that Cauchy-Euler equation is the simplest case of irregular singular point and we can use the method to find out the solution for the Cauchy-Euler equation.

And we have already seen that t to the power r1 and t to the power r2 by suitably defining r1 and r2 we can get other solution like this.

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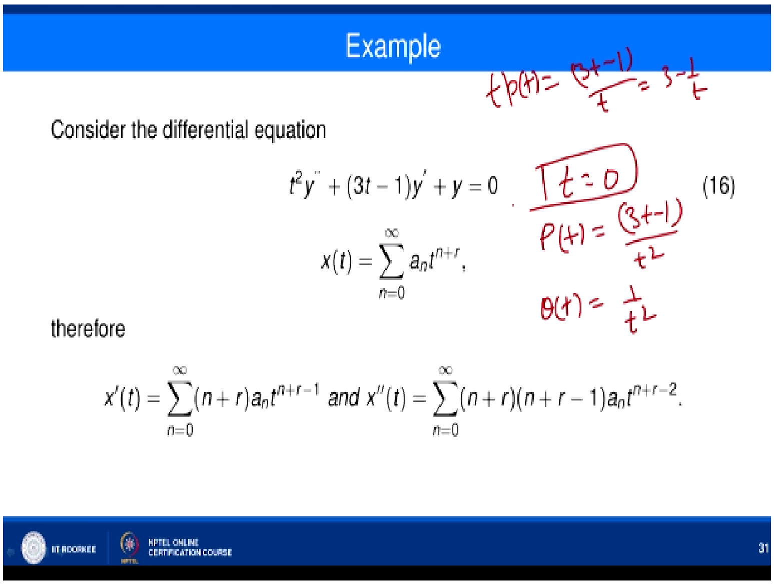

Now look at one more example here also if you look at example is what t square/double dash + 3t-1 y dash + y = 0. If you look at here also t = 0 is a singular point, so because here pt is what? pt is your 3t-1/t square and qt is what? qt is 1 upon t square. So we simply say that pt and qt are not continuous. So t = 0 is a singular point. Now just look at whether it is a regular singular point or not.

So look at tPt, so tPt is coming out to be 3t - 1/t or you can say 3-1/t and you say that this is not analytic function. So it means that here this tPt feels to be analytical so it means that t = 0 is the case of irregular singular point. Right so if it is a irregular singular point and still if you want to find out that whether the method which is given for regular singular point will work for this or not.

Let us assume our solution yt as summation n = 0 to infinity an t to the power n + r and just calculate y dash y double dash and so on and put it back to equation what you will see.

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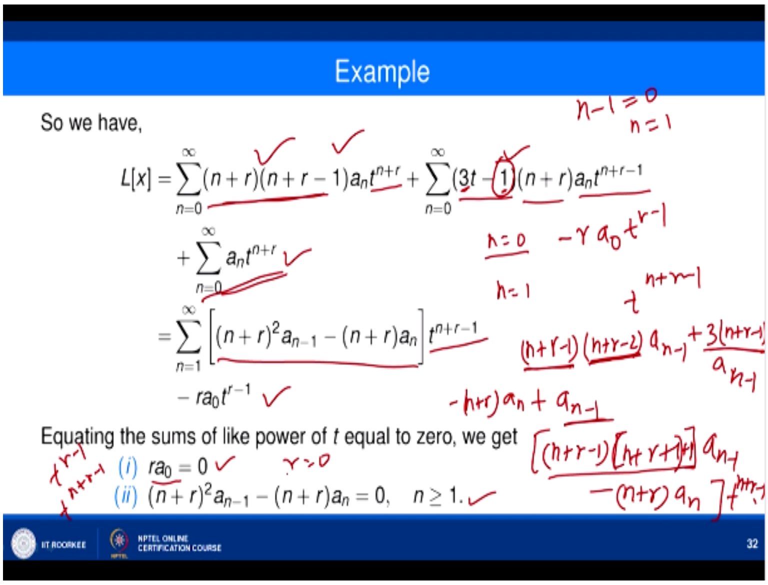

It is n = 0 to infinity n + r n + r -1 an t to the power n + r + summation n = 0 to infinity 3t-1 n + r an t to the power n + r -1 + summation n = 0 to infinity an t to the power n + r. Now here if you simplify you will get what? so here let us calculate the coefficient of t to the power n + r. So here this term and this term will conduit in t to the power n + r and this will give a term of like if you multiple 3t into this then it will also contribute in t to the power n + r and if you look at this term will simply say the term of t to the power n + r - 1.

So your term when you put n = 0 then this will give you, let me write it, n = 0 here then corresponding to this you will get – r a0 t to the power r-1, which I am writing here. So this is the term which we obtain by putting n = 0 in the second term and then when you put n = 1 and onward then all other term will also contribute and we say that in general if we look at the coefficient of t to the power n + r-1 then from first term we get t to the power n + r -1 if we replace n by n-1.

Then it is what n, let me write it here, so we just want to calculate t to the power n + r-1, so first term will give you what? n + r -1 * n + r -2, I am replacing n by n-1 * an-1 fine, from this I can write 3 and it is what n + r-1 and again a1, I am writing because of this, because this will not give term like this. So here we will get like this. This is a1+, if we multiply - here then what do you will get.

It is an-1 from last you will get an-1 and if you just calculate here it is an-1 and if you calculate this is what you write this as what? n + r-1 you take it out then it is what n + r-2 +3. So we will get n + r + 1 then there is one more term left out because of this that is here I have to write one more term that is here –n + r an, so that I am writing here, so we will get n + r-1 + n + r + 1 an-1 -n+r an.

And these are the coefficients of t to the power n + r-1, which I am writing here, so if you simplify it is what? so here we are calculating the coefficient of t to the power n + r-1 so one term we are getting from this by replacing n by n-1 that is n + r-1 * n + r-2 *an-1 here we are getting term from this then here 3n + r-1 en -1 so writing 3 n + r-1 an-1 and 1 we are getting from here when we replace n by n-1.

Here we are writing an-1, and the term corresponding to t to the power n + r -1 we are getting form this then we are getting –n + r an and when you simplify n + r-1 * n + r-2 + 3n + r-1 + 1 and we write it like this. So n + r-1 * n + r+1 + 1*an-1 -n+r a * t to the power n + r-1 when you simplify this you will get what? n+r whole square -1+1. So it is n+r whole square an-1, -n+r an t to the power n + r-1 when we start from n=1 to infinity.

And then because you are replacing n by n-1, so n-1 = 0 means n starting from 1 onward. So it means that here we have the lowest degree term is t to the power r-1 so equating the lowest degree term t to the power r-1 is 0. So we will get r a0 = 0 and then you write n to the power r-1 so equating the coefficient of t to the power n + r-1 we are getting this recurrence solution that is n + r whole square an-1 –n + r an = 0.

That is for n >= 0. So if you look at the first equation a0 cannot be 0 so r has to be 0. So this will again give you a linear indicial equation and here you will get only one root of indicial equation that is r = 0 so here we have r = 0.

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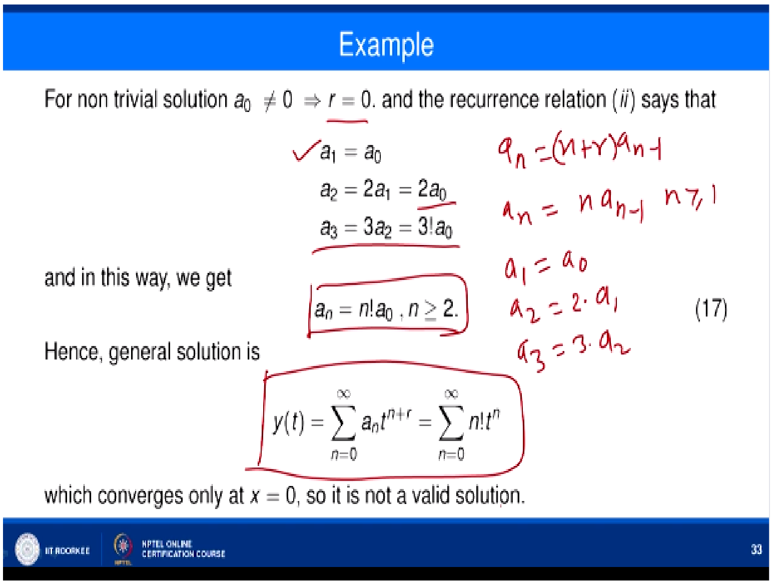

Now we with the help of this root of this indicial equation we try to find out other an. So you can simply say that an = what? an = n + r an-1 so here you have an as n + r you can cancel out because n is positive r is 0, so you can cancel out n + r, so an is given as n + r an -1 and r = 0, so an = n an-1. So here if you start since n is >= 1, so let us start with n = 1 itself so a1 = a0 that we have listed here and using this a2 is what.

A2 = n=2 so 2 * a1 so it is 2 * a1 that is 2 * a0. Similarly, a3 is what 3 * a2, a2 is 2 * a2. So we can write it that a3 is given as factorial 3 a0 and we can use mathematical induction to show that an is given as factorial n a0 and n is >= 2. So hence general solution we can write it as yt = n = 0 to infinity an t to the power n + r that is n = 0 to infinity factorial n t to the power n.

So here theoretically we can say that it looks that we got a solution and solution is given by n = 0 to infinity factorial n t to the power n, but if you have already seen the theory of power series that this power series has 0 radius of convergence that you can calculate using your tools that you can find out that radius of convergence is coming out to be 0. So it means that this power series is actually not a power series because it will never converge.

It will converge only the point at t = 0. So it means that this is not a valid solution r t = 0, so it means that here we have seen that though it looks that our solution workout very fine and we are getting a solution but actually this series solution is having radius of convergence as 0. It means that this is not a series solution of this equation and here we can say that here t = 0 is a irregular singular point.

But we are not getting a single solution of the form of Frobenius series but in previous example though $t = 0$ is your irregular singular point, but still we are able to find out one series solution the form of Frobenius form. So here we say that $t = 0$ is a strong irregular singular point. So that we want to convey that when point is not irregular singular point that we may not guarantee that a series solution exist or not.

Because in first case we have seen that series solution exist and it is given by form which we have already discussed but here in the second case your $t = 0$ is a irregular singular point, but here your solution you can say that it looks like that it is given as a Frobenius series form but actually it has no radius of convergence so we simply say that it has, it is not Frobenius series solution.

So it means that in case of irregular singular point our method of assuming the solution as Frobenius series form may not work at all. So here we want to give a message that when we have a regular singular point then we can find out a best solution using Frobenius series solution but if it is not a regular singular point then we have several other methods to find out solution but this Frobenius series solution may not work.

So with this I conclude the study of Frobenius series solution method of linear second order differential equation. Thank you for listening us. Thank you.