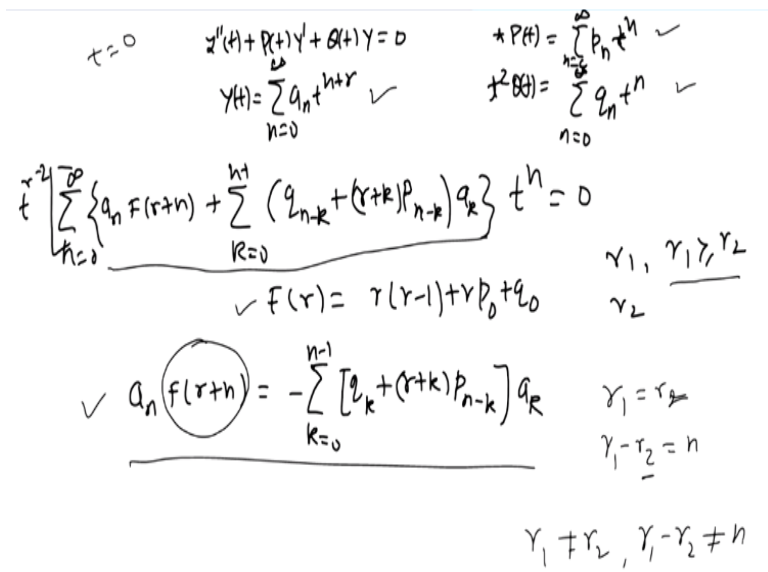
Ordinary and Partial Differential Equations and Applications
Dr. D. N. Pandey
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture - 18
Regular Singular Points - III

Hello friends, welcome to this lecture. In this lecture, we will continue our discussion of finding probability series solution method for linear ordinate differential equation having regular singular point. So what we have seen in previous lecture that we are considering this equation that y double dash t + pty dash - qty=0 here and here we are resuming that t=0 is a regular singular point.

It means that is a singular point. It means that pt and qt is not having a Taylor series expansion, but tpt and t square qt is having a power series solution.

(Refer Slide Time: 01:07)



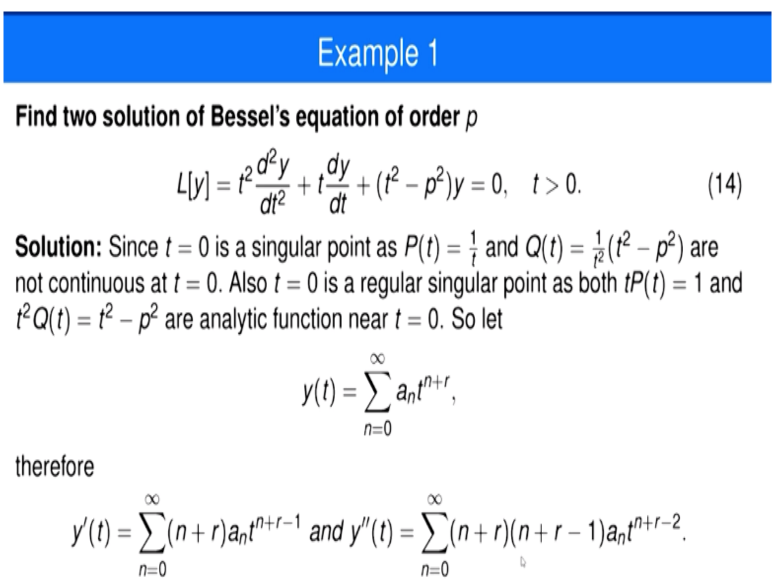
Power series expansion like this, tpt is written as summation, n=0 to infinity, pnt to power n and t square qt is having power series expansion as n=0 to infinity qnt to power n and here we have seen that in this case, we may try our solution like yt=n=0 to infinity and t to power n+r and using this series solution, we have just calculated that if you plug in this value, then we have t to power r-2 * n=0 to infinity and f(r+n)+k=0 to n-1 qn-k+r+kpn-k ak*t to power n.

Here f(r) is defined like this, f(r) is a quad equation in terms of r, which is given as r * r-1 +rp not +q not, where p not and q not is coefficient here. So using this expression, we have this following term and this is the coefficient of t to power n. So here equating the coefficient of t to power n to 0, we have this expression that tnf(r+n)=-k=0 to n-1, qk+r+kpn-k*ak, so here using this formula, we can find out the coefficient of n depending on r and the previous coefficient a not 2, an-1.

So here we have considered several cases that here there are possibilities that he r1 and r2 are roots f(r), states that r1>/= r2, so here one possibility is that when r1=r2 that we cannot use this formula to find out the second solution and another possibility where we are not able to find out the second solution is that case when r1-r2 is some kind of integer. In that case also we are not able to utilize this formula because this f(r+n) is going to be 0 when we are considering the lower value that is r2 here.

So the case when we can use this formula to obtain 2 Frobenius series solution method is the case when r1 and r2 are not equal and r1-r2 is also not equal to any kind of integer and in this case, we can find out 2 Frobenius series solution method and let us find out the example where we can have this possible case.

(Refer Slide Time: 03:44)



So here we are considering this case. So we are considering following Bessel's equation of order p, so here we need to find out 2 solutions of Bessel's equation of order p where Bessel equation is defined at t square d2y/dt square+t/dt+t square-p square y=0 where t>0. So first thing we have to look at that t=0 is a singular point. So if you look at the coefficient of highest order derivative present here, that is d2y/dt square, if you look at the coefficient is t square.

Coefficient of t square means that at t=0, this t square=0. So it is a singular point of this equation and how to check that it is a regular singular point. For that, you divide by t square and look at the coefficient of dy/dt, so it means that when you divide by t square, it is reduced to d2y/dt square+1/t dy/dt+t square-p square/t square y=0. So here it is your pt and it is your qt. So tpt is what? tpt=1 here. So we say that it is analytic.

Similarly, you can look at t square qt, that is t square qt=t square-p square here. So here this is also having a series expansion or we can say that tpt and t square qt both are analytic and in fact here this series expansion have only one term that is p not=1 and here we have 2 terms in expansion of t square qt and here q not=-p square. So here we can say that since t=0 at regular singular point as pt=1/t and qt=1/t square t square-p square are not continuous at t=0.

So this is another way to check that a given point is a singular point or not. So here we have to check that pt and qt are not continuous function at t=0. So in this way also, we can check that t=0 is a singular point. Now t=0 is a regular singular point as both tpt and t square qt are analytical function. So here tpt is 1 and t square qt is t square-p square and both are analytical function or we can say that both are having Taylor series expansion here, t=0.

So it means that it satisfies the condition, which we have discussed earlier that t=0 is a singular point and t=0 is a regular singular point. So here assuming that yt=n=0 to infinity ant to power n+r is a solution of this and then we try to find out the coefficients an and the possible values of r, for which it acts as a solution here. So again do it in a same way, you calculate y dash t, y dash t=n=0 to infinity n+r ant to power n+r-1 and calculate y double dash t, that is n=0 to infinity (n+r) (n+r-1) ant to power n+r-2.

(Refer Slide Time: 07:01)

So we have,

$$L[y] = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n t^{n+r}$$
$$+ \sum_{n=0}^{\infty} a_n t^{n+r+2} - p^2 \sum_{n=0}^{\infty} a_n t^{n+r}.$$
$$= \sum_{n=0}^{\infty}[(n+r)(n+r-1)+(n+r)-p^2]a_n t^{n+r} + \sum_{n=2}^{\infty} a_{n-2} t^{n+r}$$

So we can write ly as n=0 to infinity (n+r) (n+r-1) ant to power (n+r)+n=0 to infinity (n+r) ant to power (n+r) going to pty dash and here it is qty that is submission n=0 to infinity ant to power (n+r)-(n+r+2)-p square n=0 to infinity ant to power (n+r)=0 here. So here we can write, this is as n=0 to infinity, we just collect the term of t to power (n+r) and t to power (n+r+2). So here we write (n+r) (n+r-1)+(n+r). So here this term + this term + this term.

So when we calculate, it is (n+r)*(n+r-1)+(n+r)-p square ant to power (n+r), what is left is this that n=0 to infinity ant to power n+r+1+2 that is retained here. So ly is given as this. So once we calculate ly, then we can say that y is a solution of this provided that this quantity =0. So we need to find out that this expression has to equate corresponding term of t to power n to 0 here. So first look at the coefficient of t to power 0=0.

If you look at the coefficient of t to power 0=0, her we will get the lowest degree term is t to power r here. So t to power r, when you put n=0 here, so when you put n=0 here, we have r*r-1+r-p square*a0

(Refer Slide Time: 08:56)

Equating the sums of like power of t equal to zero, we get

$$(i)\ F(r)a_0 = [r(r-1)+r-p^2]a_0 = (r^2-p^2)a_0 = 0$$

$$(ii)\ F(1+r)a_1 = [(1+r)r+(1+r)-p^2]a_1 = [(1+r)^2-p^2]a_1 = 0$$

$$(iii)\ F(n+r)a_n = [(n+r)^2-p^2]a_n = -a_{n-2},\quad n \geq 2.$$

So that we are writing f(r)a0=0 that is r*r-1+r-p square a0=0. So if you simplify, this is what r*r-1 that is r square –r+r-p square. So it is written as r square-p square a0=0, so that is corresponding to equating the coefficient of t to power 0. So equating the coefficient of t to power 0, we have r square-p square a0=0. Similarly look at the coefficient of t to power 1 and this we will get from this.

So here when you put n=1, then we will get the coefficient of t to power r+1 that is r+1*r+1+r-p square, which we call as f(r+1). So here f(r+1) is given as a1=(1+r)*r+(1+r)-p square*a1. You will not any contribution from this term. Because here n is starting from 2 here.

(Refer Slide Time: 10:20)

So we have,

$$L[y] = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n t^{n+r}$$

$$+\sum_{n=0}^{\infty} a_n t^{n+r+2} - p^2\sum_{n=0}^{\infty} a_n t^{n+r} = 0$$

$$= \sum_{n=0}^{\infty}[(n+r)(n+r-1)+(n+r)-p^2]a_n t^{n+r} + \sum_{n=2}^{\infty} a_{n-2} t^{n+r} = 0$$

So here if you look at here we are looking at this r, this is r+1 and here t to power r+n, so if you look at the coefficient of t to power r, we are getting f(r)*a0. Coefficient of t to power 1+r is given as f(1+r)*a1. Similarly look at the coefficient of t to power (r+n) coefficient of t to power (n+r) is this quantity, which is nothing but f(n+r) an. Here we have this summation an-2, so this is your coefficient of t to power n+r. That is what we are writing here.

So here we are writing f(n+r) an=(n+r) square –p square an=-an-2 and here we are writing n>=2 here. So here we are getting these 3. This is corresponding to t to power r, this is corresponding to t to power r+1, this is corresponding to t to power (r+n), so first equation will give you the possible values of r, for which we will get a solution here. So from equation 1, which is known as indicial equation, so here this will give the value of r.

(Refer Slide Time: 11:25)

Equation (i) is the indicial equation, and it implies that $r_1 = p$, $r_2 = -p$

(Case 1). $r_1 = p$. Equation (ii) forces a_1 to be zero, and the recurrence relation (iii) says that

$$a_n = \frac{-a_{n-2}}{(n+r)^2 - p^2}, n \geq 2. \qquad (15)$$

Since $a_1 = 0$ so by (33) we have $a_3 = a_5 = a_7 = \cdots = 0$.

So if you look at the values of r, will be r1=p and r2=-p square. So it is r square-p square=0, so it is r+(-p), so r1 we are getting the p and r2=-p provided that p is >0. This we are assuming. So here we are resuming that p>0 and 2p is !=1. It means that difference is not an integer here. So here we are assuming that p is positive and 2p is !=1 here. So here, once we have the values of r1 and r2, then our next aim is to find out the coefficients.

So let us find out the case that r1=p. first we look at the r1=p and find out the coefficient and then look at the case r2=-p, and then again we find out the coefficient. Let us look at r1=p, then when r1=p, then the first equation will give you that a0 is nonzero. So we cannot calculate a0 value here, but when r=p, we can easily calculate using this formula, find out a1. So from here, we can find out the value of a1 here.

So from this, if you put r as p, then it is (1+p) square-p square*a1=0. Now since this term is non zero, so a1 has to be 0. So here we can say that this equation 2 forces a1 to be 0 and if you look at the recurrence relation that is this, this will give you the values of an in terms of an-2. So you can write an as –an-2/(n+r) square-p square, this expression is valid for n>=2 and in this case, your r=p. So an is given as –an-2/p+ r square-p square.

We can observe here since a0 is non zero and a1=0, so using this expression we can write that a3, which is given in terms of a1, it has to be 0, similarly all the odd coefficient has to be 0 here. So we can say that since a1=0, all odd coefficients are 0 because of this recurrence relation. So we need to find only a2 in terms of a0. So using a2=-n=2, so –a0/2+r, r is p here, (2+p) square-p square that we are writing here.

(Refer Slide Time: 14:02)

The even coefficient are given by

$$a_2 = \frac{-a_0}{2^2(1+p)} a_0$$

$$a_4 = \frac{-a_2}{2^2.2(2+p)} a_2 = \frac{1}{2^4 2!(1+p)} a_0$$

and so on.

That 2 square (1+p)a0, we are taking 2 out, so let me write it here. an is what a(2n), I am writing, that is a-a(2n-2)/(2n+t) square-p square. We can simplify this further an2n-2 and here we can write 2n+p-p, so that is 2n*2n+p+p and this is valid for n=1, 2 and so on. So here I am writing that in place of n, if we write 2n, then we can consider this. If you simplify this further then what is a2n, a2n is going to be –a(2n-2) and 2 we can take it out, 2 square n*n+p, you will get.

So that is what we are using here. So a2 is –a0/2 square(1+p)*a0. Similarly you can calculate a4 using a2, so a4 is -a2 upon 2 square 2(2+p) and if you put the value of a2, then it is 1 upon 2 to power 4 factorial 2(1+p)*a0 and so on. In this way, we can find out all the even coefficient.

(Refer Slide Time: 15:23)

We can prove by induction that

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n}(n)!(1+p)(2+p)\cdots(n+p)}.$$

Hence

$$y_1(t) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}n!(1+p)(2+p)\cdots(n+p)} y^{2n+p}.$$

Choose $a_0 = \frac{1}{2^p\Gamma(p+1)}$, then

$$J_p(y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+p+1)} (\frac{y}{2})^{2n+p} \qquad (16)$$

is one solution of (14) known as Bessel's function of order p.

So once we have even coefficient, we can write that our expression is like this, a2n=(-1) to power n 2 to power 2n factorial n(1+p)(2+p) up to (n+p). So once we have a2n, then we can write down number solution as Y1(t) a0, here a0 is missing, a0 summation n=0 to infinity (-1) to power n, we are writing this expression that is 2to power 2n factorial n(1+p)(2+p) up to (n+p) and Y to power 2n+r, and r is p here, so it is Y1(t) is given as this.

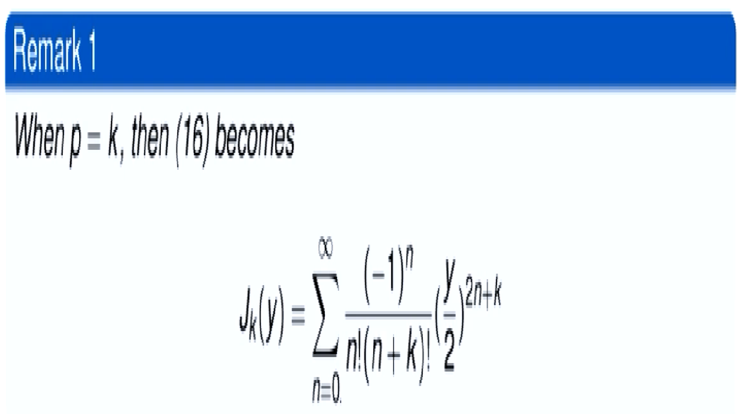
For simplicity, if we take a0 as 1 up on 2 to power p up on factorial p+1, we will see why we are choosing this value. We will discuss later on, so when we choose this value a0=1 up on 2 to power p gamma p+1, then this expression will take a simpler form, that is here if we take a0 as this thing it means that here it is (-1) to power n/2 to power 2n factorial n(1+p) up to (n+p)*a0 value is 2 to power p and factorial p you can write it here and y to power (2n+p) and summation n=0 to infinity.

So we can simplify this as (-1) to power n as it is factorial n as it is, and this we can write is gamma (n+p+1) here and here this Y to power (2n+p) is here and here also we have 2 to power (2n+p) so we can write y/2 power(2n+p), and we using this special value of a0, this will represent a series expansion which call as Jp(Y), and this is known as Bessel's function of order p of first kind.

So here using this value of a0 has this we have expression for Jp(Y) given as this n=0 to infinity (-1) to power n/factorial n gamma (n+p+1)(Y/2) to power (2n+p). So this is the expression for Bessel's function of order p, and this value a0, which we have taken is we can obtain using orthogonal property of Bessel's function. So when we study this Bessel's function of order p, there you can understand how to find out this a0 value.

Since it is valid for any a0, so in particular if you take a0 as this value, then this will take a simpler form and which we are writing as Bessel's function of order p. Now look at the other solution corresponding to r2=(-p).

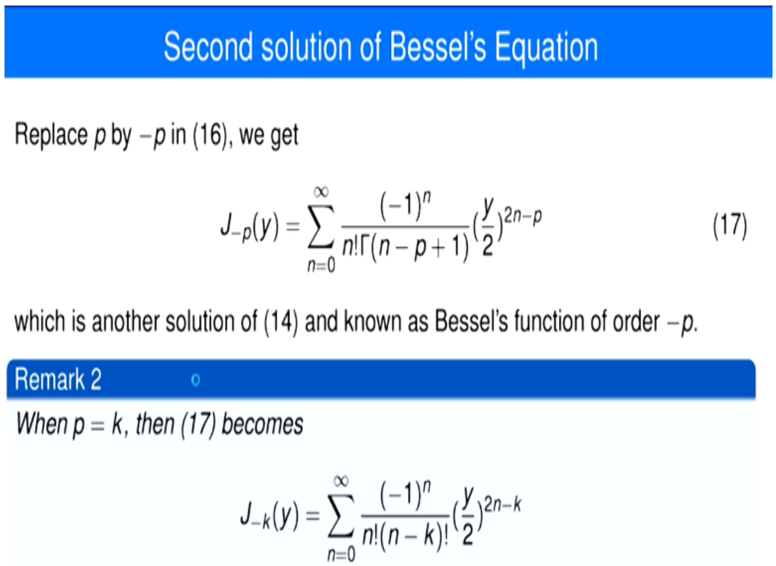
(Refer Slide Time: 18:16)



So we will get the other solution, but before that let us summarise this that when p=k, then 16 becomes Jk(Y) as summation n=0 to infinity (-1) to power n factorial n*(n+k) factorial (Y/2) to power (2n+k), remember here, here we are writing k as integer value. But this expression is valid for any real p, such that p>0 and 2p is !=1, so this expression is valid for any p, whether it is integer or non integer.

But if p is an integer, then this gamma (n+p+1) is reduced to factorial (n+p), that is what is written here. So this is the expression of Bessel's function of order k and it is written like this.

(Refer Slide Time: 19:13)



Now find out the second solution, here replace p by -p in 16 we get J(-p)Y=n=0 to infinity (-1) to power n factorial n gamma (n-p+1)(Y/2) to power (2n-p). So here if you look at this is quite symmetric, the only thing is that here we have just replaced p by -p. You can check that actually we are getting this thing. So this is I can say that it is homework that the other solution corresponding to r2=-p you will get the following expression.

So here we say that when p=k, then 17 becomes this and this is known as Bessel's function of order -p or Bessel's second solution.

(Refer Slide Time: 20:02)

Frobenius method

As discussed above, the method of Frobenius may not be able to find the general solution of the differential equation in following two cases.

Case 1. $r_1 = r_2$;

Case 2. When $r_1 - r_2$ is a positive integer.

Case 1. $r_1 = r_2$. In this case (when $r_1 = r_2$) we have only one solution of the form

$$y_1(t) = t^{r_1} \sum_{n=0}^{\infty} a_n t^n. \tag{18}$$

The second linearly independent solution, in this case, is given as

$y_2(t) = y_1(t) \ln t + t^{r_1} \sum_{n=0}^{\infty} b_n t^n$, where $b_n, n \geq 0$ are constants to be determined.

Now as we have discussed earlier, that the method of Frobenius may not be able to find the general solution of the differential equation in following 2 cases. So when r1=r2, then we may not find out 2 Frobenius series method, it means that of course 1 series solution will be given, but the other 2 may not be of the Frobenius series form. Similarly, when r1-r2 is a positive integer, here also we have a possibility that 1 solution is given in terms of series, but other solution may not be given in Frobenius series solution form.

So let us consider these cases, so first let us consider the equality case that when r1=r2, and in this case we have only 1 solution of this form, that is Y1(t)=t to power r1 n=0 to infinity ant to power n. We are not able to find out the second series solution, so the only thing we know is that whatever second solution exists must be linearly independent to this. And here we say that second linearly independent solution is given by Y2(t)=Y1(t)ln t+t to power r1 n=0 to infinity bnt to power n, where bn, n>=0 are constant to be find out.

So here if you remember the case of Cauchy Euler equation, in Cauchy Euler equation if we have a case of equal roots, then we know that solution is given in terms of Y1(t)lnt+Y1(d)lnt.

(Refer Slide Time: 21:45)

Frobenius method

Presence of $ln|t|$ makes the second solution a singular solution and hence most of the times not useful. To find the second solution we adopt the method similar to the method used in finding a second solution of Cauchy-Euler's equation, in the case of equal roots.

Let us re-write (31) in the form $y_1(t,r) = \sum_{n=0}^{\infty} a_n t^{n+r}$ to emphasize that the solution $y(t)$ depends on our choice of r. then

$$L(y(t,r)) = a_0 F(t) t^r + \sum_{n=1}^{\infty} \left(a_{n(r)} F(n+r) + \sum_{k=0}^{n-1} ((k+r) p_{(n-k)} + q_{(n-k)}) a_k \right) t^{(n+r)} \quad (32)$$

So in this if you look at this solution, second solution Y2(t) here this lnt makes the second solution a similar solution in at t=0. And most of the times we are not able to utilise in a full swing, we say that to find out second solution we adopt a method similar to the method used in finding a second solution of Cauchy Euler's equation, that we are going to find out.

So let us rewrite our equation 31 in the form that Y1(t,r) is given as n=0 to infinity ant to power (n+r), to emphasize that the solution Y(t) depend in our choice of r. So here of course this solution is depending on, so when you take r=r1 we have a solution, when you take r=r2 we have another solution. But in this case you write Y1(t,r) as this n=0 to infinity ant to power (n+r). Now using this our expression is written as L(Y(t+r))=a0F(t)t to power r+n=1 to infinity an(r)F(n+r) +k=0 to n-1((k+r)p(n-k)+q(n-k)ak)t to power (n+r).

So that is the simplification of the differential equation when you put Y(t,r) as the possible case of solution. So we need to find out the possibilities of this r and he coefficient in a way such that this expression is coming out to be 0. So we need to find out r and an in a way such that this expression is gives out to be 0 value.

(Refer Slide Time: 23:25)

Frobenius method

Now if we choose $a_n(r)$ as (12) requiring that the coefficients of $t^{(n+r)}$ be zero for $n \geq 1$. Thus

$$a_n(r) = \frac{\sum_{k=0}^{n-1}((k+r)p_{(n-k)} + q_{(n-k)})a_k(r)}{F(n+r)} \tag{20}$$

with this choice of $a_n(r)$, we see that

$$L(y(t,r)) = a_0 F(r) t^r. \tag{21}$$

So here we have to choose an(r) as 12 requiring that the coefficient of t to power (n+r) be 0, so it means that a and +r is given like this. So this is the expression which we have already utilise that F(n+r)*anr is given as k=0 to n-1((k+r)p(n-k)+q(n-k))ak(r). So that is how we choose our coefficients an. So when we choose this coefficient an like this, then your this L(Y) is simplified like a0F(t)t to power r, because we have taken our an in a way such that this expression is going to be 0.

So choosing n in a way such that this expression is 0, then what we left is L(Y(t,r))=a0F(t)tr, there is a small correction here that it is F(r)tr. So once we have made this choice, then we have L(Y(t,r)) as =a0F(r)*t to power r.

(Refer Slide Time: 24:29)

Frobenius method

In the case of equal root $F(r) = (r - r_1)^2$, so that (21) implies

$$L(y(t,r)) = a_0(r - r_1)^2 t^r.$$

Since

$$L(y(t,r)) = 0 \Rightarrow \quad y_1(t) = t^{r_1}\left[a_0 + \sum a_n(r_1)t^n\right].$$

One solution

$$\frac{\partial L(y(t,r))}{\partial r} = L\left(\frac{\partial y}{\partial r}\right)(t,r) = a_0 \frac{\partial}{\partial r}\left[(r - r_1)^2 t^r\right]$$

$$\Rightarrow L\frac{\partial y}{\partial r}(t,y) = a_0\left[2(r - r_1)t^r + (r - r_1)^2(\ln t)t^r\right]$$

also vanish at $r = r_1$

Now we know that in case of equal root F(r) is given as (r-r1) whole square, so it means that I can write our previous equation as L(Y(t,r))=a0(r-r1) square t to power r. Now again we already know that when Y=Y1(t), then we have a 0 of this, so it means that for r=r1 you already have a solution. So it means that L(Y(t,r))=0 and then we have a solution Y1(t)=t to power r1 summation a0+ summation n=1 to infinity an(r1)t to power n, so that we already know.

So one solution is given by this, to find out the other solution we already know let us differentiate this equation corresponding to r, with respect to r, so apply del L/del r of L(Y(t,r))=L(del y/del r)(t,r). So here we already discuss that L is a differential operator in terms of t. So if you apply differential operator corresponding to r it will go inside. So here we write L(dy/dr)(t,r) so if you differentiate (dy/dr) it is a0(d/dr)(r-r1) square t to power r, because in right hand side we have this.

So here we have L(dy/dr)(t,y)=a0[2(r-r1)t to power r+(r-r1)square (ln t)t to power r], and if you put r-r1, then right hand side =0. So it means that dy/dr at r=r1 will give you a second solution.

(Refer Slide Time: 26:11)

Frobenius method

showing that $\left(\frac{\partial y}{\partial r}\right)_{r=r_1}$ is also a solution

$$\begin{aligned} y_2(t) &= \frac{\partial}{\partial r}\left[\sum_{n=0}^{\infty} a_n(r) t^{n+r}\right]_{r=r_1} \\ &= \sum_{n=0}^{\infty} a_n(r_1) t^{n+r_1}(\ln t) \\ &+ \sum_{n=0}^{\infty} a_n'(r_1) t^{n+r} \\ y_2(t) &= y_1 \ln t + \sum_{n=0}^{\infty} a_n'(r_1) t^{n+r_1} \qquad (22) \end{aligned}$$

is a solution of (6).

So how to find out a second solution, so dy/dr at r=r1 is also a solution, so it means that Y2(t) which is given as d/dr(y), y1 is already given, n=0 to infinity an(r)t to power(n+r) at r=r1 will also give a solution. So simplify this expression, so what you will get n=0 to infinity, now an is also depending on r, so here when you differentiate this you will get what, an(r1)t to power (n+r1)(lnt). Here I am just differentiating this t to power (n+r).

So here we are getting to power (n+r1)(lnt), and then differentiate this an(r), so we get n=0 to infinity an dash (r1)t power (n+r). So here what you will get this you write as n=0 to infinity, so an(r)t to power n you are taking out and differentiating t to power r, so differentiating t to power r means t to power r (lnt)+summation n=0 to infinity t to power (n+r) as it is, then differentiate an that is an dash (r), so that is what we are writing here.

So we can write this as here if you simplify what is this, if you look at this is what an(r1) t to power (n+r1) if you look at this then this is nothing but Y1. So the first term is written as Y1lnt and second term is written as n=0 to infinity an dash (r1) t to power (n+r), so it means that the bn is nothing but an dash(r1). So it means that here second solution is given by Y1lnt+n=0 to infinity an dash (r1)t to power (n+r), and here this we are able to get because we have already discussed the similar case in case of Cauchy Euler equation.

So here we have seen that 1 solution is given by Y1(t), then another solution is given by Y2(t) as dy/dr, r=r1 and it is having this expression Y2(t)=Y1lnt+n=0 to infinity an dash (r1)*t to power (n+r). So as claimed this is having the form which we have already claimed, the only thing is that bn is now replaced by an dash(r). So here I will stop here and we say that in case of equal roots we are not able to find out both the solution as Frobenius series solution form.

So 1 solution is given in terms of Frobenius series solution, but the other solution is not the Frobenius series solution form. So in equal roots we have second solution given by Y2(t), and general solution we can find out using Y1 and Y2, and in next class, we will discuss an example based on this and the other case, that is when r1 and r2 differ by an integer, so in next class we will continue this lecture. Thank you very much for listening us.