Ordinary and Partial Differential Equations and Applications

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Lecture - 17
Regular Singular Points - II

Hello friends, welcome to this lecture. In this lecture, we will discuss the Frobenius series solution method for second order linear differential equation, and we are considering this Frobenius series solution method for the case when we have a point which is a regular singular point.

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$$P_0(t)\,y''(t) + P_1(t)\,y'(t) + P_2(t)\,y = 0$$

$$- \quad t = t_0 \;\checkmark$$

$$P_0(t_0) = 0$$

$$- \quad (t - t_0)\frac{P_1(t)}{P_0(t)} \;\checkmark$$

$$- \quad (t - t_0)^2\frac{P_2(t)}{P_0(t)} \;\checkmark$$

So how we can understand, if you look at the equation like P0(t) y double dash (t) + P1(t) y dash (t) + P2(t)y = 0, so this is a second order linear differential equation where P0(t), P1(t) and P2(t) are some continuous function in terms of t, a point t = t0 is set to be a singular point of this, if P0(t0) = 0. So when we have a singular, then we say that t = t0 is a singular point of this ordinate differential equation.

If in addition to this, if (t-t0) P1(t)/P0(t) and (t-t0) square P2(t)/P0(t), if these 2 functions are, function in the neighbourhood of (t-t0), then we say that t= t0 is a regular singular point and we want to find out the solution in the case of when t = t0 is a regular singular point. So that we are going to do in this lecture here.

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Our aim is to find a class of singular differential equations which is more general than the Euler equation

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0 \qquad (1)$$

then, we rewrite (1) in the form

$$\frac{d^2y}{dt^2} + \frac{\alpha}{t}\frac{dy}{dt} + \frac{\beta}{t^2}y = 0 \qquad (2)$$

Generalization of (2) is the equation

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0 \qquad (3)$$

So here our aim is to find the class of singular differential equation, which is more general than the Euler equation. So 1 example of the t = 0 is a regular singular point for this Euler equation, if you look at t square d2y/dt square + alpha dy/dt + beta y = 0. So if you look at, then I can write this as, that here your t = 0, at t = 0 this t square is vanishing so it is a singular point for this equation, and if you look at this alpha t/t square * t, here t0 = 0.

So here t0 = 0 and if you look at this is nothing but alpha, and if you look at t square beta upon t square, then it is a beta. So we can alpha and beta are constant function and hence we can say that it is having a Taylor series expansion. So it has only one component that Taylor series expansion and we can say that here t0 = 0 is a regular singular point for this Cauchy Euler equation. And here we have already solved this Cauchy Euler equation in previous lecture.

Now we can simplify this equation number 1 in the following form, here we simply divide it by t square, here we are resuming that t>0 or t<0, t not=0, we can say. Then here when we divide by t square, then it is d2y/dt square+alpha/t dy/dt+beta/t square y=0. And here we can generalise this equation in the following form that d2y/dt square+p(t)dy/dt+q(t)y=0. Here p(t) and q(t) can be generalised form of alpha/t and beta/t square in a following way.

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where $p(t)$ and $q(t)$ can be extended in series of the form

$$p(t) = \frac{p_0}{t} + p_1 + p_2 t + p_3 t^2 + \cdots \tag{4}$$

$$q(t) = \frac{q_0}{t^2} + \frac{q_1}{t} + q_2 + q_3 t + q_4 t^2 + \cdots \tag{5}$$

Definition 1

The expression (3) is said to have a regular singular point at $t = 0$ if $p(t)$ and $q(t)$ have series expansions of the form (4). Equivalently, $t = 0$ is a regular singular point of (3) if the function $tp(t)$ and $t^2 q(t)$ are analytic at $t = 0$. Equation (3) is said to have a regular singular point at $t = t_0$ if the functions $(t - t_0)p(t)$ and $(t - t_0)^2 q(t)$ are analytic at $t = t_0$. A singular point of (3) which is not regular is called irregular.

So here p(t) is written as p0/t+p1+p2t+p3t square and so on. So basically here, we are adding a kind of power series in terms of t. And if you look at q(t), q(t) is q0/t square now we are adding terms q1/t+q2+q3t+q4t square and so on. So we can say that p(t) and q(t) has these kind of series expansion. So the expression 3 is said to have a regular singular point at t = 0, if p(t) and q(t) have series expansion of the form 4 and 5.

Equivalently, t=0 is a regular singular point of 3 if the function t p(t) and t square q(t) are analytic at t = 0. So here we can define this regular singular point in 2 ways, that either this p(t) and q(t) is having this kind of series expansion like 4 and 5, or you can say that t p(t) and t square q(t) are having Taylor series expansion at t=0, so both have an equivalent meaning. So equation 3 is said to have a regular singular point at t=t0, so this is what we have defined, regular singular point of t = 0.

Now if we have a non zero singular point, then in that case the function (t-t0)*p(t) and (t-t0) square q(t) must be analytic at t=t0, or we can say that (t - t0)*p(t) has a Taylor series expansion in terms of (t-t0), and (t-t0) square q(t) must have a Taylor series expansion around (t-t0) square. And a singular point of 3 which is not regular is called irregular. So the meaning that these analytical at t=t0, means here I can write (t-t0) p(t) as summation ak(t-t0) to power k, k is from 0 to infinity.

Similarly your (t-t0) whole square q(t) = summation bk(t - t0) power k, k=0 to infinity. So it means that t=t0 is said to be a regular singular point, if it is a point of a singular point, and this (t-t0) p(t) and (t-t0) square q(t) must have this power series expansion in terms of t-t0. And we want to find out the solution in which case we have this kind of expression.

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Frobenius method

Now let us generalize the method used to find the general solution of an Euler Cauchy equation to a more general singular differential equation of the form

$$y'' + P(t)y' + Q(t)y = 0, \qquad (6)$$

where one or both of the coefficient functions $P(t)$ and $Q(t)$ is not analytic at $t = t_0$. The ideas developed above suggest that we should try for a formal solution of the following form (known as Frobenious form)

$$y(t) = t^r \sum_{n=0}^{\infty} a_n t^n \quad a_0 \neq 0 \,, |t| > 0 \qquad (7)$$

of (6).

So now let us generalise the method used to find the general solution of the Euler Cauchy equation to a more general singular differential equation of this form y double dash + P(t)y dash + Q(t)y = 0, and here one or both of the coefficient function P(t) and Q(t) is not analytic at t=t0, means first thing is that t=t0 is not a regular point or not an ordinary point. So we can say that here P(t) and Q(t) is not analytic at t=t0.

And the ideas developed above suggest that we should try for a formal solution of this following form, which is known as Frobenius form. If you look at in this case of Cauchy Euler equation here we have take P(t) as alpha/t and Q(t) as beta/t square and we got a solution of t to power r. Now we are generalising our P(t) and Q(t) in a way that P(t) is written as p0/t+p1+p2t and so on.

So we are adding a kind of power series solution to our function P(t) and Q(t). So we can say that here corresponding to solution, if we talk about we must add some kind of t to power r is already a solution, then we can add t to power r + 1, t to power r + 2, and so on, these term if we add in a solution then we might have get a solution. So keeping this in idea, let us take solution as Y(t) = t to power r summation n = 0 to infinity an tn.

And without loss generality we are assuming that a0 is not = 0, because if a0 = 0, then it is what, then we can write t to power r and n is start from 1 to infinity an t to power n. And we can write again it in this form, so it need a without loss of generality we are assuming that the first term of the series is basically nonzero. And here we are assuming that modulus of t > 0.

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Frobenius method

We are assuming that the point $t = 0$ is a regular singular point that is:

(1) $t = 0$ is a singular point,

(2) functions $tP(t)$, $t^2Q(t)$ are analytic functions that is

$$tP(t) = \sum_{n=0}^{\infty} p_n t^n \quad \text{and} \quad t^2Q(t) = \sum_{n=0}^{\infty} q_n t^n. \tag{8}$$

To get a solution we need to know possible values of $'r'$ in(7) for which (7) is a solution and then for each value of $'r'$ we must calculate coefficients $a_0, a_1, a_2, \cdots$. Now (7) implies

$$y(t) = \sum_{n=0}^{\infty} a_n t^{(n+r)}$$

So it means here what we are assuming so for is that t = 0 is a regular singular point and it means what, that t = 0 is a singular point first of all, it must be a singular point and second thing is that function tP(t) and t square Q(t) are analytic function at t = t0, in an aboard of t = t0, means what that tP(t) have Taylor series expansion like n = 0 to infinity Pn t to power n and similarly t square Q(t) is also having Taylor series expansion that is n = 0 to infinity Qn t to power n.

To get a solution, we need to know possible values so for, for which we have a solution of our differential equation and which 7 is a solution and then for each value of r. So first we have to the find out the values of r for we have a solution and then for each values of r, we need to find out the coefficients. And it means what that let us try for this solution Y(t) = summation n = 0 to infinity an t to power n + r.

So first our attempt is to find out this r for which it is a solution and then we will try to get 2 kinds of values r1 and r2 and corresponding to each r1 and r2 we try to find out the coefficient corresponding to r1 and coefficient corresponding to r2. So this we need to find out and once we are able to find out the r1 and r2 and the corresponding coefficients we can write down solution Y(t) as summation n = 0 to infinity anr1 t to power n+ r1 and similarly the other solution.

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Frobenius method

$$y'(t) = \sum_{n=0}^{\infty}(n+r)a_n t^{(n+r-1)}, \quad y''(t) = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{(n+r-2)}$$

$$\begin{aligned} P(t)y' &= \frac{1}{t}\left[\sum_{n=0}^{\infty} p_n t^n \sum_{n=0}^{\infty}(n+r)a_n t^{(n+r-1)}\right] \\ &= t^{(r-2)}\left[\left(\sum_{n=0}^{\infty} p_n t^n\right)\left(\sum_{n=0}^{\infty}(n+r)a_n t^n\right)\right] \\ &= t^{(r-2)}\left[\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(r+k)\, a_k\, p_{(n-k)}\right)t^n\right] \end{aligned}$$

So let us identify, how to find out the coefficient for these r1 and r2, so for that let us assume that this is a solution, so differentiate it, Y dash(t) = n = to infinity (n + r) ant to power (n+r-1) and differentiate again to get Y double dash(t) and it is = n = 0 to infinity (n+r) (n+r-1)ant to power (n+r-2). So once we have Y(t), Y dash (t), Y double dash (t), we calculate P(t) Y dash here and Q(t)Y.

So P(t)Y dash is what, it is 1 upon t, n = 0 to infinity Pntn, here since we already know that tP(t) is having series solution and = 0 to infinity Pntn. So P(t), I can write 1/t and = 0 to infinity Pntn*Y dash (t) that is, n = 0 to infinity (n+r) ant to power (n+r-1). And which we can write it here as t to power r-2, here we are taking r-1 out and 1/t is already there, so it is t to power r-2*n = o to infinity Pnt to power n*n = 0 to infinity (n+r) antn.

Now here we multiply 2 power series to get the following form that is n= 0 to infinity k = 0 to n (r + k) akp(n-k). Here how we multiply this power series, if you look at, if we have 2 power series like this ak t to power k * summation bk t to power k, then we will have a series like ck t to power k where ck is given as summation n = 0 to k here and ak and bk-n here. Here we have assumed like this.

So we are using this formula that ck must have, if you look at a term here, then sum is going to be, sorry here it is an so it is an bk-n, if you look at this sum is going to be k-n+n is going to be k here. So here ck must have this kind of form, using this formula here we are writing that n = 0 to infinity Pntn * n = 0 to infinity n+r antn. Now I want a coefficient of t to power n, so it means that I am finding cn. So cn is going to be what?

Summation if k is from 0 to n here and then here you find out in this we find out pk, so here we can say that this is bk and this is ak. So here we can write that n, here it is (r+k)ak, this we are writing and other one is (n-k), so it is P(n-k). So if you look at this k and (n-k) is going to be, if you sum them it is going to be n here. So cn is going to be k = 0 to n, here you can write bk and a(n-k), it is kind of symmetric thing.

So here it is, I will write it, this is your ak and this is bk if you write it like this, then here we can write it this product formula as n = 0 to infinity k = 0 to n (r+k) ak, consider this as 1nk, and this as b(n-k), so that is what how we write it, this product. So P(t)Y dash is written as t to power (r-2) summation n = 0 to infinity and coefficient of t to power n is given as k = 0 to n (r+k) akp(n-k)t to power n.

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Frobenius method

Similarly

$$
\begin{aligned}
Q(t)y &= \frac{1}{t^2}\left[\left(\sum q_n t^n\right)\left(\sum a_n t^{(n+r)}\right)\right] \\
&= t^{r-2}\left[\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} q_{n-k}a_k\right)\right]t^n \\
&= t^{r-2}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n-1} q_{n-k}a_k + q_0 a_n\right]t^n.
\end{aligned}
$$

Similarly, we can calculate Q(t)Y, so Q(t)Y is 1 upon t square summation n = 0 to infinity qntn * summation n = 0 to infinity an t to power (n+r). Here again we apply the same formula, so here this t to power r we have taken out, so we can write t to power (r-2) summation n = 0 to infinity, and here again we will find the coefficient of t to power n. And how we can find out coefficient t to power n, this is your ak and this is your bk.

So ak and b(n-k), b is what, qn here, so q(n-k) ak, so k = 0 to n, q(n-k)ak. Now we can simplify this further. We are writing k = 0 to n-1, so we are taking the term corresponding to n out. So we can write q(n-k) ak + if you take k=n, then it is what q0antn, will understand why we are writing this. So once we have this Y dash(t), Y double dash(t) calculate P(t)Y dash and Q(t)Y, then we plug in all these value in your differential equation.

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Frobenius method

Using the expressions for y'', $P(t)y'$ and $Q(t)y$ in (6), we have the following differential equation by taking out the common factor t^{r-2} :

$$\sum_{n=0}^{\infty}\left\{(n+r)(n+r-1)+(r+n)p_0+q_0+\sum_{k=0}^{n-1}a_k\left[(r+k)p_{n-k}+q_{n-k}\right]\right\}t^n=0,$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^n+\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n-1}(r+k)a_kp_{n-k}+a_np_0(r+n)\right)t^n$$

$$+\sum_{n=0}^{\infty}\left(\left(\sum_{k=0}^{n-1}a_kq_{n-k}\right)+q_0a_n\right)t^n=0.$$

So here using this expression for Y double dash, P(t)Y dash and Q(t)Y in 6, we have the following differential equation by taking out the common factor t to power n-2. So it means that here t to power r-2, I am writing here, so t to power r-2 * this thing. So here in bracket it is what, n=0 to infinity (n+r)(n+r-1)+(r+n)p0+q0+k=0 to n-1(ak)(r+k)p(n-k)+q(n-k)*t to power n = 0.

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$$y''(t)+P(t)y'+Q(t)y=0$$

$$y(t)=\sum_{n=0}^{\infty}a_nt^{n+r},\quad y''(t)=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n+r-2}$$

$$P(t)y'=t^{r-2}\left[\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(k+r)a_kp_{n-k}\right)t^n\right]$$

$$Q(t)y=t^{r-2}\left[\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}b_kq_{n-k}\right)t^n\right]$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n+r-2}+\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(r+k)a_kp_{n-k}\right)t^{n+r-2}+\sum t^{n+r-2}\left(\sum_{k=0}^{n}b_kq_{n-k}\right)t$$

If you look at this I can write it like this, so here we have what, we have Y double dash(t) + P(t)Y dash + Q(t)Y = 0. So Y double dash(t) is what, Y(t) is our summation an t to power (n+r), n is from 0 to infinity, so we can easily find out Y dash(t) = summation (n+r) (n+r-1) an t to power (n+r-2), sorry this is Y double dash(t), so n = 0 to infinity. So here we have just calculated the Y double dash(t) and P(t)Y dash is basically what?

P(t) is t to power r-2 summation here we have calculated (k+r)ak and p(n-k) and it is what summation n is from 0 to infinity and is k from 0 to n, and here we have t to power n. So that is how we have calculated P(t)Y dash, if you look at here we have calculated P(t)Y dash is this t to power (r-2) n = 0 to infinity k = 0 to n (r+k)ak p(n-k) * t to power n, that is what I am writing here, that n = 0 to infinity k = 0 to n (k+r)ak p(n-k)t to power n.

Similarly, we can calculate Q(t)Y, Q(t)Y is again t to power r-2 summation n = 0 to infinity summation k = 0 to n and here we have bk and q(n-k) and t to power n here. So when you plug in this value, then what you will get, here we will get summation (n+r)(n+r-1)an t to power (n+r-2) n is from 0 to infinity + P(t)Y dash is again if you multiply t to power r-2 inside you will get what, summation n is = 0 to infinity, summation k = 0 to n (r+k)ak p(n-k) t to power (n+r-2).

Similarly, you can write it summation t to power r-2, n-r-2, so n+r-2 * summation k = 0 to n bk q(n-k) and, we have already written t to power n+r-2 and it is = 0. So here if you simplify this, then this is what we are getting here, that t to power r-2 if you take it out, so n = 0 to infinity (n+r)(n+r-1)+ here corresponding to this, we are taking the term here, if you put k = n value here, then I can write this k = 0 to n-1 + the term corresponding to k = n.

Similarly, here also corresponding to k = n we can take the term out. There is a small problem here, I think it is ak not bk, so it is ak here, similarly here it is ak, so this is the correction here. So here we are writing this in 2 terms k = 0 to n-1 and k = n term, so and here also we are writing the same thing. So in this way we are getting the following formula n = 0 to infinity (n+r)(n+r-1) + (r+np0+q0) + k =0 to n-1 (ak) and it is r+k p(n-k) because of P(t)Y dash and + ak*q(n-k) because of Q(t)Y * t to power n = 0.

Now we can simplify it further, we can write n = 0 to infinity (n+r)(n+r-1) an tn+n = 0 to infinity k = 0 to n-1 (r+k)ak p(n-k) + anp0(r+n) that is what we have written already here t to power n + n = 0 to infinity k = 0 to n-1 akq(n-k) + q0an, this is corresponding to k = n term, t to power n that is, this is the expression which we have simplified in this manner. So now since if you look at the left hand side and right hand side and since in left hand side we have this expression and in right hand side we have 0 so it means that the corresponding coefficient of t to power n is going to be 0.

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Frobenius method

Equating the coefficients of t^n to zero, we get

$$a_n\left[(n+r)(n+r-1)+(r+n)p_0+q_0\right]+\sum_{k=0}^{n-1}a_k(q_{n-k}+(r+k)p_{n-k})=0. \quad (9)$$

Coefficient of t^0 is zero

$$a_0(r(r-1)+rp_0+q_0)=0.$$

Let $F(r)=r(r-1)+rp_0+q_0$ (indicial equation same as in Euler-Cauchy equation). Then we may re-write the last equation as

$$a_0F(r)=0 \quad (10)$$

So using this, equating the coefficient of t to power n to 0 what we get an (n+r)(n+r-1), so here if you look at here we just calculate the coefficient of t to power n here. So what you will get (n+r)(n+r-1)an and here coefficient of t to power n is anp0(r+n), here what you will get q0an. So here we are writing an*(n+r)(n+r-1)+(r+n)p0+q0+ now we look at this term, here k=0 to n-1 (r+k)akp(n-k)+ k=0 to n-1 akq(n-k) that we are writing here.

So here we are writing an n+r (n+r-1) + (r+n) p0+q0+k=0 to n-1 akq(n-k)+(r+k)p(n-k) that is = 0. So here we are trying to show that equating the coefficient of t to power n to 0, we get this following form, that an*(n+r) (n+r-1)+(r+n)p0+q0+k=0 to n-1 akq(n-k)+(r+k)p(n-k)=0, and we can work it out here, we can work it out like this here we have already calculated Y(t), P(t)Y dash, Q(t)Y dash, Q(t)Y and we have plugged in all these values.

And we have this following thing, n=0 to infinity (n+r)(n+r-1)an t to power (n+r-2) that is the expression for Y double dash(t) that we have calculated here + expression for p(t)Y dash that is n=0 to infinity k=0 to n (r+k)akp(n-1)t to power (n+r-2), so that is the expression for P(t)Y dash and then we write down the coefficient expression for Q(t)Y that is t to power (n+r-2) k=0 to n akq(n-k)=0.

Now if you further simplify this then here we have n=0 to infinity (n+r)(n+r-1)an t to power(n+r-2) so this is nothing but Y double dash(t) + n=0 to infinity, now we simplify the coefficient of this t to power (n+r-2) in P(t)Y dash as follows, you write k=0 to n-1 and this is (r+k)akp(n-k) and the corresponding term k=n we are writing here, that (n+r)anp0 that we are taking out + n=0 to infinity.

Similarly, we are doing for Q(t)Y, so k=0 to n-1 here and akq(n-k) and this is the term corresponding to k=n here. So this for k=n and here also it is k=n, writing anq0*t to power(n+r-2) = 0. Idea is to collect the term an at one side and term which is < a0 to a(n-1) in other side. So if we simplify, then we are writing t to power r-2 * n=0 to infinity, now collect the coefficient of an t to power n.

If you look at what you will have, here we have (n+r)(n+r-1) and here what do you have, here we have (n+r)p0 so that is I am writing here, and here we have q0 so here we writing q0 ant to power n + now we are writing the remaining term that is n=0 to infinity and term which are left here that is k=o to n-1 (r+k)akp(n-k) here we have this thing and the term here that is akq(n-k)t to power n = 0.

So now again simplifying this t to power r-2 summation n=0 to infinity, now (n+r)(n+r-1)+(n+r)p0+q0*an+ what we are doing, we are taking out this t to power n term out + summation k=0 to n-1 (r+k)p(n-k)+q(n-k)akt to power n. Here if you look at in this term, if you start from k is 0 to n-1, then here we are getting term from a0 to a(n-1). So here we are just collecting the term coefficient of an and here it is the coefficient of ak where k is from 0 to n-1, so this is the coefficient of t to power n.

So it means that coefficient of t to power n has to be 0, so this quantity has to be 0 for each n. So here we say that equating the coefficient of t to power n to 0 we get the following expression that we have just looked at. Now we will start from coefficient of t to power 0, coefficient of t to power 1 and so on. So if you look at the coefficient of t to power 0 is going to be a0(r(r-1)+rp0+q0) that we can look at here in this way.

Here if you look at coefficient of, here if you put n=0, then you will get what r*(r-1)+ here we will get p0r+q0 here. So this is the coefficient of, a0(r(r-1)+rp0+q0)=0, so here we will not get any contribution from this. So here we are just using this expression for n=0, so here we get a0 into this expression. Now if we call this as F(r), that is F(r) is denoted as r(r-1)+rp0+q0 and this expression we call as indicial equation same as in Euler-Cauchy equation.

If you remember in Euler-Cauchy equation, we also have the similar form. So here F(r) represented by r(r-1)+rp0+q0, so if we take this expression as F(r), then we can write now this as a0*F(r)=0.So here we have already assumed that this a0 is non zero so that coefficient of t to power 0 has to be 0 provided that F(r) has to be 0, so that gives you the following thing.

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Frobenius method

Now since $a_0 \neq 0$ so (10) is true only if $F(r) = 0$, $\Rightarrow r(r-1) + rp_0 + q_0 = 0$, which is a quadratic equation in r and will give two roots r_1 and r_2, $r_1 \geq r_2$. These roots, called the exponents of the differential equation (6) at the regular singular point $t = 0$, will give two values of r for which (7) may work as a possible solution of (6). Coefficient of t^1 is zero

$$a_1[(r+1)r + (r+1)p_0 + q_0] + a_0[rp_1 + q_1] = 0,$$

or

$$a_1 F(r+1) + a_0[rp_1 + q_1] = 0. \quad (11)$$

Now since a0 is non zero so 10 is true only when F(r)= 0, that is r(r-1)+rp0+q0=0, now this is a quadratic equation in r and will give 2 roots r1 and r2, and here just for sake of simplicity, we are assuming that whatever be the value of r1 and r2 we assuming that r1 is the larger of these 2 roots, so r1 >= r2 that we are assuming here. So these roots, called the exponents of the differential equation 6 at the regular singular point t=0

So r1 and r2 are known as exponent of differential equation at regular singular point t=0. So it means that that will give you exponents, once you have exponents, then look at the coefficient of t to power n and equate it to 0. So here we again use this thing a1(r+1), so here what we are using, here we are using this expression again, so here n=1, if you put a1*(1+r) here it is (1+r-1)+(1+n)p0+q0 and here when you put n=1, then you will get term corresponding to k=0, so here you will get a0qn+pn.

So here we are getting this we are using n=1 and here we are using expression for n=0. We are using 9 for n=1, so we are getting this equation a1*(r+1)r+(r+1)p0+q0+a0rp1+q1=0. And we already know that this a0 is non zero, so we can write a1 and if you use this expression for

F(r), then this is nothing but written as F(r+1). So we can write this equation as a1*F(r+1)+a0rp1+q1=0. So once a0 is non zero you can calculate the value of a1 in the following way provided that F(r+1) is non zero.

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Frobenius method

Coefficient of t^2 is zero

$$a_2[(r+2)(r+1)+(r+2)p_0+q_0]+a_0[rp_2+q_2]$$
$$+a_1[(r+1)p_1+q_1]=0,$$
$$a_2F(r+2)+a_0[rp_2+q_2]+a_1[(r+1)p_1+q_1]=0.$$

Similarly, equating the coefficient of t^n is zero, we have

$$a_n\Big((r+n)(r+n-1)+(r+n)p_0+q_0\Big)+a_0(q_n+(r+n)p_n)$$
$$+a_1\Big((r+1)p_{n-1}+q_{n-1}\Big)+a_{n-1}\Big((r+n-1)p_1+q_1\Big)=0,$$

So we can write in this way we can look at the next coefficient of t2 is 0, and we can write a2[(r+2)(r+1)+(r+2)p0+q0]+a0[rp2+q2]+a1[(r+1)p1+q1]=0. Here please note down, this is again this equation number 9 for n=2. So here we are looking at equation number 9 and for coefficient of t to power 0, put n = 0 and for coefficient of t to power 1, put n = 1 and coefficient of t square we are putting n = 2.

So we are looking at this equation number 9 again and again, and we are getting equation number 11 and similarly we can get this expression. So this we can write in terms of F(r) as a2F(r+2)+a0(rp2+q2)+a1[(r+1)p1+q1]=0, so if we know the values of a0, a1 we can get the value of a2 from this. Similarly equating the coefficient of t to power n is 0 we have this following expression, an[(r+n)(r+n-1)+(r+n)p0+q0] and this is the expansion form of the formula this term.

So here we can expand in the following way, so a0(qn+(r+n)pn) + a1 and so on, so that is = 0. So idea is that using this equation number 9 and the fact that coefficient of t to power n = 0, we are trying to find out the coefficient a0, a1, a2 and so on and provided that this can be possible provided that F(r+n) is non zero.

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Frobenius method

$$a_nF(r+n)+\sum_{k=0}^{n-1}a_k(q_{n-k}+(r+k)p_{n-k})=0 \qquad (12)$$

The equations (11), (12) indicates that coefficients a_1 is given in terms of a_0, a_2 is given in terms of a_0 , a_1 and so on. The a_n's are depends on r and $a_0, a_1, \ldots, a_{n-1}$ provided $f(r+n)\neq 0$ for some positive n. In that case we can not utilize (12) to find a_n's.
Thus if $r_1=r_2+n$ for some integer $n\geqslant 1$, the choice $r=r_1$ gives a formal solution but in general $r=r_2$ does not, since $F(r_2+n)=F(r_1)=0$.
Another case where we obtain only one formal solution of the Frobenius form is the case when $r_1=r_2$.

So here we can simplify in this way that anF(r+n)+k=0 to n-1 ak(q(n-k)+(r+k)p(n-k))=0. So this equation 11, 12 indicate that the coefficient of a1 is given in terms of a0, a2 is given in terms of a0 and a1, and so on. So the an's are depending on r, if you look at here, here r is involved, so we can say that an's are depending on r and the coefficient, which we have already obtained that a0, a1 up to a(n-1).

This can be possible when this coefficient of an is non zero, provided that F(r+n) is nonzero, for some positive n. In that case, we cannot utilise 12, if this F(r+n) = 0, then I cannot utilise this equation number 12 to find out the value of an. So it means that here if r1=r2+n, here I am assuming that r1 is bigger than r2, so if r1=r2+n for some integer, which is n>=1, the choice r=r1 will give a formal solution.

The choice corresponding to a bigger root will give you a formal solution, but choice corresponding to the smaller root does not give a solution, why because when you apply r=r2, then F(r2+n) is given as F(r1), which is = 0, so it means that corresponding to r-r2, you cannot find out an using the formula 12. Because an*0=something. So we will see that under what condition this will also give a solution, that we are going to discuss this case.

So another case where we obtain only one formal solution from this form is the case when r1=r2. So here you can find out the coefficient an using this formula, equation 12 provided that the F(r+n) is = 0 and it is happening in what case. Here if r+n = 0. So it means that here we have 2 values r1 and r2 and if r1=r2+n, it means that when you are looking a solution corresponding to r=r2, then when you put an, then an*F(r2+n)is going to be F(r1), that is going to be 0, then this expression.

So if this expression is non zero, you cannot find out a solution, but if this expression is 0, then we can look at some possible way to find out the solution an. Another instance where we may not get the other formal solution in the Frobenius series solution form is the case when both roots are equal roots, so it means that when r1=r2 because I can have only one an, using this formula, we can calculate and we do not have any method to find out the other formal solution.

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Frobenius method

In all other cases where r_1 and r_2 are real numbers the above presented procedure gives two independent real solution of (6).
Consequently, if $F(r) = 0$, has two real roots r_1 and $r_2 : r_1 > r_2$ and they do not differ by an integer, then equation (6) has two linearly independent solutions of the form

$$y_1(t) = t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) t^n$$
$$y_2(t) = t^{r_2} \sum_{n=0}^{\infty} a_n(r_2) t^n. \quad (13)$$

and these solutions have radius of convergence $R = \min\{R_1, R_2\}$ where R_1 and R_2 are radius of convergence of $tP(t)$ and $t^2Q(t)$ respectively.

So we will discuss these 2 cases in a separate way. So in all other cases where r1 and r2 are real numbers, the above presented procedure give 2 independent solution of 6 and consequently if F(r)=0 has 2 real roots r1 and r2, when r1>r2 and they do not differ by an integer. It means that your r1 is bigger than r2, but r1=r2+some n, where n is some integer. In this case, we have 2 linearly independent solutions and corresponding to r1 and r2, we can have a solution like this.

Where with t=t2 power r1 summation n=0 to infinity an(r1)t to power n and corresponding to r2, we have Y2t, that is t to power r2 from n=0 to infinity an, r2 t to the power n and these solutions have radius of convergence r, minimum of r1 and r2 where r1 and r2 represent the radius of convergence for tpt and t square qt, because tpt and t square qt are having power series expansion, so it must have a radius of convergence.

So if you take the minimum, then that is going to be the radius of convergent of the solution here, that is the guarantee of this method. We are not going to prove the following statement, but you take this as a given thing. If F(r)=0 has 2 roots, r1 and r2 and these are different and they are not different by an integer. In that case, we have Frobenius series solution given in the following form, y1t=t to power r1, n=0 to infinity an(r1)tn, and y2t=t to power r2 n=0 to infinity an(r2)tn.

And this procedure gives you that this series solution is valid in a reason having radius of convergence r, minimum of r1 and r2 where r1 and r2 are radius of convergence of tpt and t square qt respectively. So in this case, we have 2 Frobenius series solution method guaranteed. So here I will finish. We will continue this in next class. Thank you very much.