Ordinary and Partial Differential Equations and Applications
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Lecture – 14
Uniform Convergence of Power Series

Hello friends, welcome to my lecture on uniform convergence of power series. We have already discussed that when the power series converges uniformly some function of the power series is a continuous function. Now as regards the term by term and also we discussed the integration term by term of a power series.

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As regards the term by term differentiation of a power series we have

Theorem: If

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $|x-a| < R$, then the series $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ has precisely the same radius of convergence.

So as regards the term by term differentiation of a power series; we have the following results. If f(x) = sigma n = 0 to infinity, cn(x – a) to power n convergence for mod of x - a < R, then the series sigma n = 1 to infinity nCn(x – a) to the power n - 1 has precisely the same radius of convergence.

So in this series you can see sigma n = 1 to infinity nCn(x – a) to the power n - 1 had been obtained by term by term differentiation of the series sigma n = 0 to infinity Cn(x – a) to the power of n. So, precisely what we are saying is that, if the given series has radius of convergence R, then the differentiated series also has the radius of convergence R. So let us prove this result.

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Proof: Let x_0 be an arbitrary but fixed point such that

$$0 < |x_0 - a| < R$$

and choose x such that $|x - a| < |x_0 - a| < R$.

Then the convergence of the given series for $|x - a| < R$

$$\Rightarrow \sum_{n=0}^{\infty} c_n (x_0 - a)^n \text{ is convergent.}$$

$$\Rightarrow \exists A > 0 \text{ such that } \left| c_n (x_0 - a)^n \right| \le A, \forall n \in \mathbb{N}.$$

Let $\rho = \left| \dfrac{x - a}{x_0 - a} \right|$, then $\rho < 1$

so $\quad \left| n c_n (x - a)^{n-1} \right| \le \dfrac{n A \rho^{n-1}}{|x_0 - a|}.$

$\sum_{n=0}^{\infty} c_n (x-a)^n$

$\Downarrow$

$\sum_{n=0}^{\infty} c_n (x_0-a)^n$ is convergent series

$\Downarrow$

$\lim_{n\to\infty} c_n (x_0-a)^n = 0$

$\Downarrow$

$\{c_n (x_0-a)^n\}_{n=1}^{\infty}$ is a bounded sequence

$$|n c_n (x-a)^{n-1}| = \left| n c_n (x-a)^{n-1} \frac{(x_0-a)^n}{(x_0-a)^n} \right| = \frac{|n c_n (x_0-a)^n|}{|(x_0-a)|} \left| \frac{(x-a)^{n-1}}{(x_0-a)^{n-1}} \right| = \frac{|n c_n (x_0-a)^n|}{|x_0-a|} \rho^{n-1} \le \frac{n A \rho^{n-1}}{|x_0-a|}$$

Let us take an arbitrary point x0 and let us fix it. So x0 be an arbitrary and but fixed point such that 0 < mod of x0 - a < R, and then choose a point x such that mod of x - a < mod of x0 - a < R. Now the convergence of the series, sigma n = 0 to infinity, Cn(x – a) raise to the power n, for mod of x - a < R because we have assumed that series sigma n = 0 to infinity, Cn (x – a) to the power n has the radius of convergence R.

So the reason of convergence for the series is given by the inequality mod of x - a < R, so the convergence of this series for the region mod of x - a < R implies that series sigma n = 0 to infinity Cn(x0 – a) to the power n is convergent. Now let us see how we get this. You have x0 is a point which belongs to this interval okay, which belongs to the region mod of x - a < R. So what we have?

The convergence of this implies that sigma n = 0 to infinity Cn(x0 – a) raise to the power n is a convergent series. The convergence sigma n = 0 to infinity Cn(x – a) to the power n for mod of x - a < R implies that series sigma n = 0 to the infinity Cn(x0 – a) to the power n is convergent because x0 belongs to this interval a – R to a + R by our hypothesis, Okay. Now since the series is convergent.

Okay we know that for an infinite series the convergence of an infinite series implies that the nth term of the series goes to 0 so this will imply that limit n tends to infinity Cn(x0 – a) raise to the power n = 0. The nth term of the series goes to 0 that is the necessary condition for the convergence of an infinity series. Now, let us look at the sequence Cn(x0 – a) to the power n the limit of the sequence as n goes to infinity 0 implies that the sequence is bounded.

So we can find the constant A > 0 such that Cn(x0 – a) to the power of n mod of that is < = A okay. So this implies that the sequence (x0 – a) raise to the power n is a bounded sequence. So by the definition of the bounded sequence, we can find the constant a > 0 such that mod of Cn (x0 – a) to the power of n is < = power a for all n belonging to n. Now let us choose real number x such that mod of x – a/mod of x0 – a.

We assume rho this ratio to be rho then from this inequality mod of x - a < mod of x0 – a, we follow that rho is < 1. Now when rho < 1, we get this inequality mod of nCn(x – a) to the power n - 1 is < = to n * a rho to the power n – 1 * x0 - a mod. Okay. So let us see how we get this. So nCn (x –a) to the power of n we can write as nCn(x – a) to power of n * (x0 – a) to power n/x0 - a to the power n and this I can write nCn (x0 – a) to the power n.

And then we have x0 - a here and we get here (x – a) to the power we have nCn (x – a) to power of n nCn (x – a) and we get nCn/ (x0 – a) to the power of n - 1. So we have this nCn (x – a) to the power n - 1 we have okay. So nCn (x – a) to power n - 1 we have. So we take n - 1 here. Okay. Yeah. So we nCn (x – a) to power n - 1 = nCn (x – a) to power n - 1. Now we multiply and divide by the x0 - a to the power n and then collect nCn * (x0 – a) to the power n because for this we have the inequality okay.

We have the inequality that Cn * x0 - a to the power n this is < = a. So what do we get now let us see this will be so mod of this okay mod of this = mod of this okay and then we get mod of this mod of this so let us take what I mean is that you take the mod of nCn(x – a) to the power n - 1 nCn * x - a to the power of n - 1 and then you get mod of nCn(x – a) to the power of (n – 1) (x0 – a) power to n/x0 -a power to n.

And you get mod of nCn x0 - a to the power n upper mod of x0 -a and this is x – a okay. Now this is what this is = mod of nCn (x0 – a) raise to the power n/mod of x0 - a, then this is = and this is a rho okay. So rho to the power n - 1 okay and rho < 1 and the mod of nCn (x0 – a) so mod of nCn (x0 – a) to the power n is < = A.

So this is n * a/mod of x0 -a * rho to the power n - 1 okay. So what we get mod of nCn x0 mod of nCn (x – a) to the power of n - 1 is < n * a < = okay < = mod of n * a rho to the power n – 1/mod of x0 – a okay. This is the rho to the power n - 1okay and mod of Cn (x0 – a) raise to the power n < = so this is < = n * a mod of x0 - a * rho to the power n - 1.

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The convergence of $\sum_{n=1}^{\infty} \frac{nA\rho^{n-1}}{|x_0 - a|}$ implies that the series $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ converges uniformly for $|x-a|<|x_0-a|$.

Since x_0 is arbitrary with $0<|x_0-a|<R$, the series $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ converges uniformly for $|x-a|<R$.

Let R' be the radius of convergence of $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ then $R' \geq R$.

If $R=\infty$ then $R'=\infty$. So, let $R'>R$.

$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)A\rho^n |x_0-a|}{|x_0-a|\, nA\rho^{n-1}}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\rho = \rho < 1$

The limit exists

Now what we do, now let us look at the convergence of this series sigma n = 1 to the infinity n * a rho to the power n - 1 mod of x0 – a okay. The convergence of this implies that now why it is convergent now you can apply ratio test so a limit n tends to infinity let us apply the ratio test. So limit n tends to the infinity mod of cn + 1, so that means mod of n + 1 * a * rho to the power of n/mod of x0 -a * mod of x0 -a/n * a rho to the power n – 1 okay.

So mod of x0 -a we will cancel the mod of x0 - a; a will cancel okay; rho to the power n - 1 when divides rho to the power n you get rho and n + 1/n okay so what we get is limit n tends to the infinity n + 1/n * rho which = rho and rho is < 1. Okay. So the limit is exists. So the limit exist and is < 1 okay. So this mean that this series by ratio test it is convergent. Now then this series is convergent.

This series sigma n = 1 to the infinity nCn(x – a) to power of n - 1 converges informally okay by (()) (11:14) test for mod of x0 - a < mod of x0 –a okay. So since x0 is an arbitrary okay with 0 < mod of x0 - a < R this series sigma n = 1 to the infinity nCn x - a to the power n - 1 converges informally for mod of x - a < R. Now let us take okay so when this series differentiated series converges informally for a mod of x0 x - a < R.

We can say that if R dash is its radius of convergence then R dash must be at least R. okay. So R dash must be > = R. Now if the radius of the convergence R of the given series is infinity since R dash > R = R. R dash is also infinity and so R = R dash and therefore the theorem is proved okay. So in the other case let us take R dash to be > R okay. Our aim is to show that R dash = R.

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Choose x such that $R < |x-a| < R'$.

Then for this x, $\sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$ converges absolutely

while the series $\sum_{n=0} c_n (x-a)^n$ diverges but

$$\left|c_n (x-a)^n\right| = \left|nc_n (x-a)^{n-1}\right| \frac{|x-a|}{n}$$

$$\leq \left|nc_n (x-a)^{n-1}\right|$$

as soon as $\frac{|x-a|}{n} < 1$.

So when R dash > R, let us choose a point x such that a R < mod of x - a < R dash. Then for this x okay since the radius of the convergence of the differentiated series is R dash okay sigma n = 1 to the infinity nCn x - a to the power n - 1 converges absolutely while the series sigma n = 0 to the infinity nCn(x – a) to the power n diverges because it is the radius of the convergence is R okay and to the point x lies outside.

I mean the circle of the convergence or the region of the convergence because we are assuming the mod of x - a > R so then mod of Cn(x – a) to the power n. So let us look at this we want to arrive at the contradiction okay. So mod of Cn(x – a) to the power of n we can write as nCn(x –

a) power to the n - 1 * mod of x – a/n okay. Now this is < = nCn x - a to the power of n - 1 as soon as mod of x – a/n < 1.

So we have already chosen a point x such that R is < mod of x - a < R. okay. Now let us take n to be so large that mod of x – a/n becomes < 1. Then for such n okay what do we notice? mod of Cn(x - a) to the power n is < = nCn x - a to the power n - 1. So by comparison test okay since the series sigma nCn(x - a) to the power n - 1 is convergent the series sigma Cn(x - a) to the power n is also convergent, but that is clearly a contradiction because the series sigma n = 0 to infinity Cn(x - a) to the power n diverges for x which is mod of x - a. We satisfy mod of x - a > R okay.

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$$\Rightarrow \sum_{n=0}^{\infty} c_n (x_0 - a)^n \text{ converges for this x which is false.}$$

Hence $R' = R$.

The sigma n = 0 to infinity Cn(x0 - a) to the power n converges for this x sigma n = 0 to infinity cn x - a to the power n converges for this x which is false and hence R dash should be = R. So that is how we prove this theorem.

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Corollary: Under the same hypothesis, we have

$$f'(x) = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}.$$

Now under the same hypothesis okay. So what we have now the series sigma n = 0 to infinity Cn(x - a) to the power n we have seen if it has a radius of convergence R then the differential series also has radius of convergence. The sum function of that will then by F dash x okay. So when we assume that if we have an infinite series sigma n = 0 to infinity Cn (x - a) to the power n with radius of convergence are then the differentiated series also has same radius of convergence and the sum function which was f(x) now it will be f dash x okay.

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Corollary: Let

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converge in $|x-a| < R$. Then f has derivatives of all orders, with the k^{th} derivative of f given by differentiating the series for f term by term k times. All the derived series have the same radius of convergence R.

Using mathematical induction on n, we get this result

Let f(x) be sigma n = 0 to infinity cn(x - a) to the power n it converges in the region mod of x - a < R then by induction we can say because we have seen that f dash x = sigma n = 0 to infinity cn(x - a) to the power n okay. F has derivatives of all orders and the kth derivative of f is obtained by differentiating the infinite series sigma n = 0 to infinity cn(x - a) to the power n term by term k times. All the derived series have the same radius of convergence. So using mathematical induction on n we get this corollary.

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Remark: If the radius of convergence of the power series is infinite then the convergence is not necessarily uniform on the entire line, but it is uniform on any closed interval.

Definition: A function f defined in some interval by a convergent power series is called an ***analytic function***.
If f is analytic, then the power series representing f is known as the ***Taylor series***.

Okay, so now let us give a remark if the radius of convergence of the power series is infinite. Suppose the series converges for all values of x okay then the convergence is not necessarily uniform on the entire line. We have said that the series had radius of convergence R and then the region of convergence given by mod of x - a < R then the differentiated series also has the same radius of convergence that it says that it also converges in the same region mod of x - a < R.

So this convergence is not uniform on the entire line okay, but it will be uniform on a any closed interval okay. So this remark which we have to notice now a function f defined in some interval by a convergent power series is called an analytic function okay. So the definition of an analytic function here is that it should be defined in some interval by a convergent power series. Now if f is analytic, then the power series representing f is known as the Taylor series.

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Theorem: If two functions $f(x)=\sum_{n=0}^{\infty} a_n x^n$ and $g(x)=\sum_{n=0}^{\infty} b_n x^n$ are analytic, and $f^{(n)}(0)=g^{(n)}(0)$ for all n, then $f(x)=g(x)$.

Then $\sum_{n=0}^{\infty} a_n x^n$ is the Taylor series of f

So $a_n = \frac{f^{(n)}(0)}{n!}$

& $\sum_{n=0}^{\infty} b_n x^n$ is the Taylor series of g

so $b_n = \frac{g^{(n)}(0)}{n!}$

If $f^{(n)}(0)=g^{(n)}(0)$ then

$a_n = b_n \ \forall n=0,1,\ldots$

Let us say if we have 2 functions f(x) sigma n = 0 to infinity an x to the power n and g(x) sigma n = 0 to infinity bn x to the power n are analytic okay, then the sigma n = 0 to infinity an x to the power n is the Taylor series of f okay then sigma n = 0 to infinity an x to the power n is the Taylor series of f so an must be = f(n) 0/n factorial okay and sigma n = 0 to infinity bn x to the power n is the Taylor series of the function g so bn must be = g(n)0/n factorial.

So if 2 functions f(x) and g(x) are analytic and f(n)0 = g(n)0. So when f(n)0 = g(n)0 okay then an will be = bn for all n okay = 0, 1, 2, 3, and so on okay and so the 2 functions f(x) and g(x) will be same.

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Vanishing of all coefficients: If a power series has a positive radius of convergence and a sum that is identically zero throughout its interval of convergence, then each coefficient of the series must be zero.

$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n, \quad |x-a|<R$

We are given $f(x)=0, \forall x$ satisfying $|x-a|<R$

Now, $c_n = \frac{f^{(n)}(a)}{n!}, \forall n=0,1,2,\ldots$

Since $f(x)=0, \forall x$, we have

$f^{(n)}(a)=0, \forall n=0,1,2,\ldots$

Therefore $c_n=0, \forall n=0,1,2,\ldots$

Now vanishing of all coefficients. If a power series has a positive radius of convergence and a sum that is identically 0 throughout the interval of convergence. So let us consider the power series sigma n = 0 to infinity cn(x - a) raise to the power n which has a positive radius of convergence say R and its region of convergence let us say is given by mod of x - a < R. The sum of the series is suppose f(x) then we have f(x) = sigma n = 0 to infinity cn(x - a) raise to the power n.

Now it is given that the sum is identically 0 throughout its interval of convergence, so f(x) = 0 we are given f(x) = 0 for all x satisfying mod of x - a < R okay. Now cn are given by f(n)(A)/n factorial for all n = 0, 1, 2, and so on to infinity okay. So since f(x) = 0 okay for all x we have f(n)(a) = 0 for all n because f(x) is identically 0 so all other derivatives of f(x) that x = a are also 0 and therefore cn = 0, cn is f(n)/n factorial. So cn = 0 for all n and so on.

So each coefficient of the series the coefficients cn each coefficient of the series is also 0.

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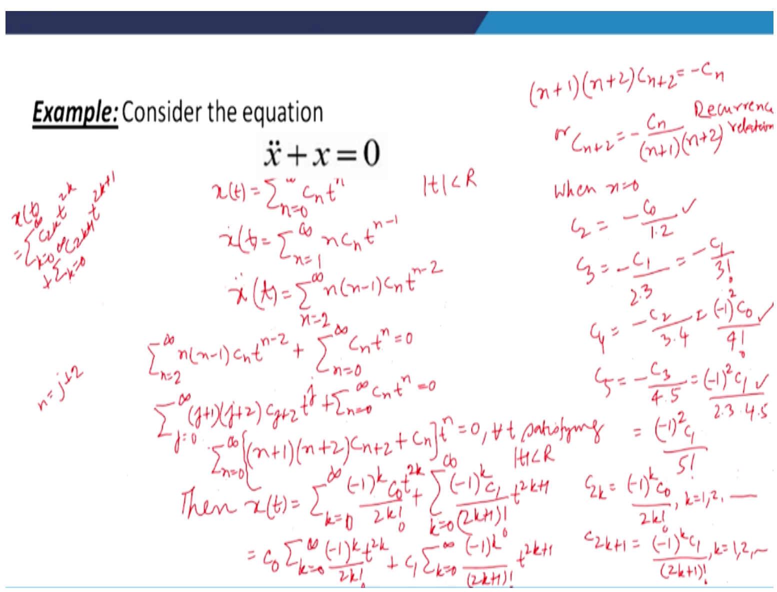

Let us consider the second order differential equation x double dot + x = 0 or we can say d square x/dt square + x = 0 and so how we can find the power series solution of this differential equation so let us assume the solution of this differential equation to be x(t) = sigma n = 0 to infinity cn t raise to the power n okay. So let us assume that this power series is a summation of this given differential equation which has got the radius of convergence R.

So the series converges for mod of t < R and R is positive. So then what we will have? X dot t will be = sigma. Now the first term here is c0 so when we differentiate that it will vanish and term series start with n = 1 onwards. So n = 1 to infinity n * cn * t to the power n - 1. Now we have seen that the series can be differentiated term by term in the region mod of t < R so n has the same radius of convergence x double dot will be = sigma.

Now n * n - 1 * cn * t raise to the power n - 2 okay and now n I am going to start with n = 2 to infinity. So now let us substitute because x(t) = sigma n = 0 to infinity cn * t to the power n we have assumed to the solution of the given equation so let us substitute these values in the given equation then what we will have. Sigma n = 2 to infinity n(n - 1) Cn t raise to the power n - 2 + sigma n = 0 to infinity cn * t raise to the power n = 0.

Here now what we will do let us replace n by say j + 2 okay. n = j + 2 then this will be = sigma j = 0 to infinity and then we will have j + 2, j + 1. So I can write j + 1, j + 2, cj + 2 t to the power j + sigma n = 0 to infinity cn t raise to the power n = 0. I can now change the summation/j2 summation/m okay. So sigma n = 0 to infinity and I can collect the terms so (n + 1) (n + 2) cn + 2 okay and then I can write like this okay. So what we will have.

Now we know that when the power series I mean some function of the power series is identically 0 over an interval so here this is identically 0. This is valid for all t satisfying mod of t < R. So every term of the series must be 0 okay. So when we take okay this will give you okay and t to the power n and the coefficient of t to the power n = 0 gives (n + 1) (n + 2) cn + 2 = - cn or I can say cn + 2 = - cn/(n +1) (n + 2).

So we can start with n = 0 when n = 0 what we get? C2 = - c0 and then n = 0 gives 1 * 2 then I can find c3 this relation is known as recurrence relation okay. It can be recursively used to determine the values of the coefficient cn so for different values of n so when n = 0 c2 we can find from here. This is - c0/1 * 2 then c3 will be - c1 n = 1 so we have 2 * 3 and then we can find c4. C4 will be = - c2 we are putting n = 2 so 3 * 4 and is further = c2 = - c0/1 * 2.

So we have - 1 whole square * c0 and then 1, 2, 3, 4. So we have 4 factorial okay. Then C5 we can find. C5 will be = when you put n = 3 - c3/3 + 1 that is 4 and then 3 + 2 = 5. So we have and then we can get 5 in terms of c1 from using this relation so we get - 1 to the power 2 and then c3 is - c1/2 * 3. So c1/2, 3, 4, 5 okay. So what we get? This will be = 5 factorial right. - 1 to the power 2 c1/5 factorial.

Now so what we get is if we go on like this we can see that c2k okay c2k will be = - 1 to the power k * c0/2k factorial and k will take values and k can take values 1, 2, 3, and so on and c2k + 1 will come out to be - 1 to the power k * c1/2k + 1 factorial. So let us verify. We have written the values of c2k and c2k + 1 okay. So you can put k = 1 where c2 should be = - c0/2 factorial okay. So this is - c0/2 factorial then you put k = 2 then we have c4.

C4 will be - 1 to the power 2 c0/4 factorial. So we get this is c4 okay and here when we put k = 1 you get c3 as - c1/3 factorial. So this is - c1/3 factorial and when you put k = 2 then you get c5 as - 1 to the power 2 c1/5 factorial so you get this. So in general we can write the value of c2k and c2k + 1 from here and once we have that okay what we have? xt will be = so then xt sigma n = 0 to infinity cn t to the power n okay.

So we will have c0. So this series will be = sigma k = 1 to infinity - 1 to the power k. If I take k = 0 to infinity - 1 to the power k. c0/2k factorial + sigma k = when you take c0 and c1 are arbitrary okay. So k = 0 to infinity - 1 to the power k is t to the power 2k here okay because we are writing the value of c2k okay so - 1 to the power k * c1/2k + 1 factorial * t raise to the power 2k + 1. Luckily what we are doing is that xt you can split into 2 parts in 2 series sigma k = 0 to infinity c2k t to the power 2k + sigma k = 0 to infinity c2k + 1 t to the power 2k + 1 okay.

One series will contain even powers of t, the other series will contain odd powers of t. So we can write the series representing xt function in this manner and then put the values of c2k and c2k + 1. So we get this okay. now what we have, this is c0 * sigma k = 0 to infinity - 1 to the power k t to the power 2k/2k factorial and then we have c1 * sigma k = 0 to infinity - 1 to the power k/2k + 1 factorial t to the power 2k + 1 okay and we can see that.

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Hence,

$$x(t) = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}$$

$$\Rightarrow x(t) = c_0 \cos t + c_1 \sin t.$$

In a general situation we will obtain power series that do not necessarily represent known functions.

This series sigma k = 0 to infinity - 1 to the power k * t to the 2k/2 factorial represents the cosine t function and sigma k = 0 to infinity -1 to the power k * t to the power 2k + 1/2k + 1 factorial represents the sine function okay so xt = c0 cos t + c1 sine t. Now you note that in this example in this particular example we are able to write the solution of the differential equation in a closed form.

We are getting xt = c0 cos t + c1 sin t since it is a second order differential equation then (()) (32:16) will have 2 arbitrary constants and we have 2 arbitrary constants here c0 and c1 so it is the general solution. Now in this particular example of course we have got the solution in the closed form, but in a general situation we will obtain power series that do not necessarily represent known functions.

So that is the remark I want to make. In our next lecture, we shall see more examples of differential equations where we will see that the solution cannot be represented in a closed form that is the solution of the differential equation general solution comes in the form of infinite series which do not represent known function, known elementary functions. Thank you very much for your attention.