Ordinary and Partial Differential Equations and Applications
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Lecture - 12
Solution of Non-Homogeneous Linear System with Constant Coefficients

Hello friends welcome to the lecture on solution of non-homogeneous linear system with constant coefficients.

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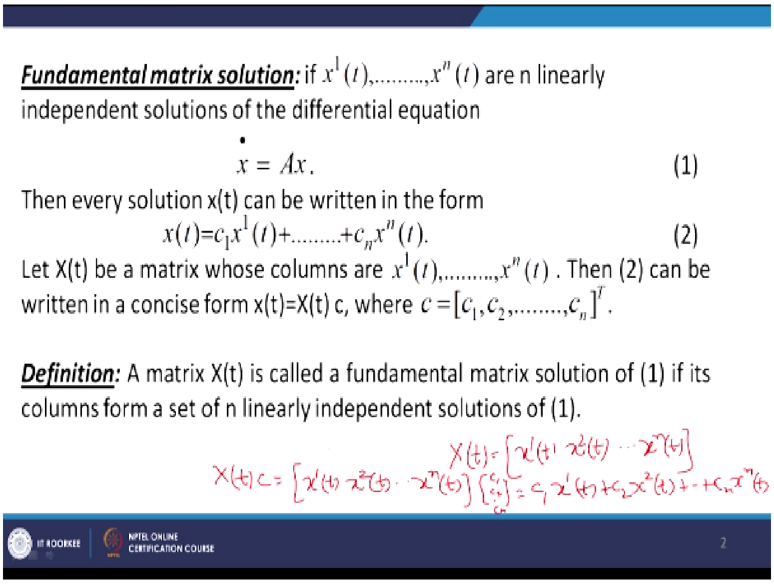

Suppose x1t, x2t, and so on xnt are n linearly independent solutions of the vector differential equation x dot = Ax. Now we have already discussed how to find n linearly independent solutions of this homogeneous vector differential equation so suppose x1t, x2t and so on xnt are n linearly independent solutions of the vector differential equation.

X dot = Ax then every solution xt of the vector differential equation x dot = Ax can be written in the form xt = c1 * x1t + c2 * x2t and so on cn * xnt where c1 c2 cn are constants. Let xt be a matrix whose columns are these n independent solutions x1t x2t and so on xnt. Then 2 can be written in the concise form xt = xt * c, you can write it like this, see Xt = x1t, x2t and so on xnt, okay.

Now when you find out xt * column vector c what do we get xt * c = x1t x2t xnt multiplied by the column vector c1 c2 cn, okay. Now these x1t, x2t, xnt are themselves column vectors okay, so when you multiply c1, c2, c and this is n * n matrix. Xt is n * n matrix whose first column is x1t, second column is x2t and the last column is xnt, okay, so when you multiply this n * n matrix by c1, c2, cn what you get is xt = xt*c, okay.

So this is xt = xt is a column vector okay and it is equal to n * n matrix your xt and multiplied by c vector, okay, so this is nothing but this is = c1 times x1t + c2 times x2t and so on cn times xnt. So it is the general solution of the vector differential equation x dot = x. Now a matrix xt is called a fundamental matrix solution of the vector differential equation x dot = x if it is columns form a set of n linearly independent solutions of 1.

So xt here which comprises of these n linearly independent solutions x1t x2t xnt is the fundamental matrix solution of equation 1. Now you can write these n vectors in any order so there is not a unique fundamental matrix, any matrix okay where the n columns are n linearly independent solutions of the vector differential equation x dot = Ax will always be called as a fundamental matrix solution of the vector differential equation.

So Xt here is the fundamental matrix. Now let us consider a vector differential equation x dot = Ax okay.

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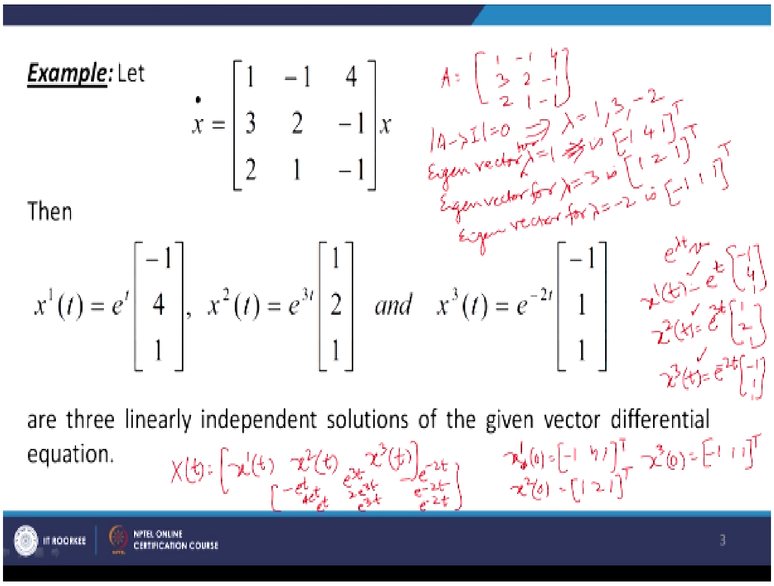

So A is this 3 * 3 matrix 1-14, 32-1, 21-1. In order to find a fundamental matrix of this vector differential equation what we do is, we first find 3 linearly independent solutions of this vector differential equations, so for this we first write the characteristic equation of the matrix A, determinant A-lambda = 0, it will give you a cubic equation in lambda so when you solve that cubic equation in lambda you will get the values of lambda as 1, 3 and -2.

You can see the 3 values of lambda are real and distinct and therefore the corresponding eigen vectors will be linearly independent. For lambda = 1 you can check that for lambda =1 eigen vector for lambda = 1 is -141 transpose okay, and similarly eigen vector for lambda = 3 will come out to be 121 transpose and eigen vector for lambda = -2 is -111 transpose okay.

Now we know that if lambda is an eigen value of the matrix A and v is the corresponding eigen vector then e to the power lambda t * v is the solution of the homogeneous vector differential equation x dot = Ax, so here x1t = e to the power lambda t that is e to the power t * -141 which is an eigen vector corresponding to lambda = 1, this is one solution of the vector differential equation other solution corresponds to the other eigen value.

And the corresponding eigen vector lambda = 3 so e to the power 3t and then you have 121 and x3t = similarly e to the power -2t and then -111, okay, these 3 eigen vectors, these 3 solutions x1t, x2t x3t of the vector differential equations are linearly independent because their values are t = 0, x10 is -141, x20 is 121, x30 is -111, okay, the 3 vectors x10 x20 x30 are linearly independent because they are eigen vectors corresponding to the distinct eigen values 13-2.

So these 3 vectors are linearly independent and therefore x1t, x2t, x3t are linearly independent solutions of the given vector differential equation okay. Now we can write a fundamental matrix. Fundamental matrix xt will be = x1t, let us take first column x1t, x2t, x3t, okay, then x1t, you multiple e to the power t inside so –e to the power t, 4 e to the power t and e to the power t.

So you get –e to the power t, 4 e to the power t, and e to the power t, this is first column. Second column similarly e to the power 3t, 2 e to the power 3t, and then e to the power 3t, this is second column and third column will be here –e to the power -2t, e to the power -2t and e to the power -2t. So this is how we form a fundamental matrix of solution of x dot = Ax, so xt is this matrix.

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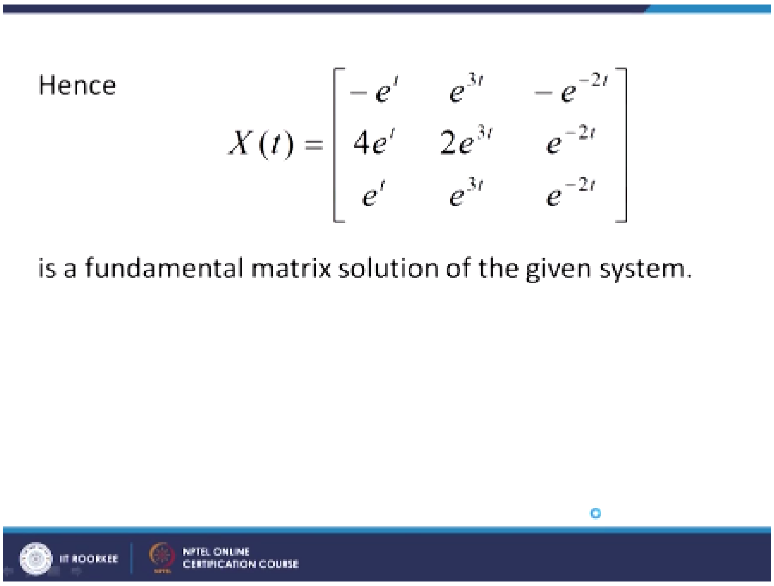

It is a fundamental matrix solution of the given system.

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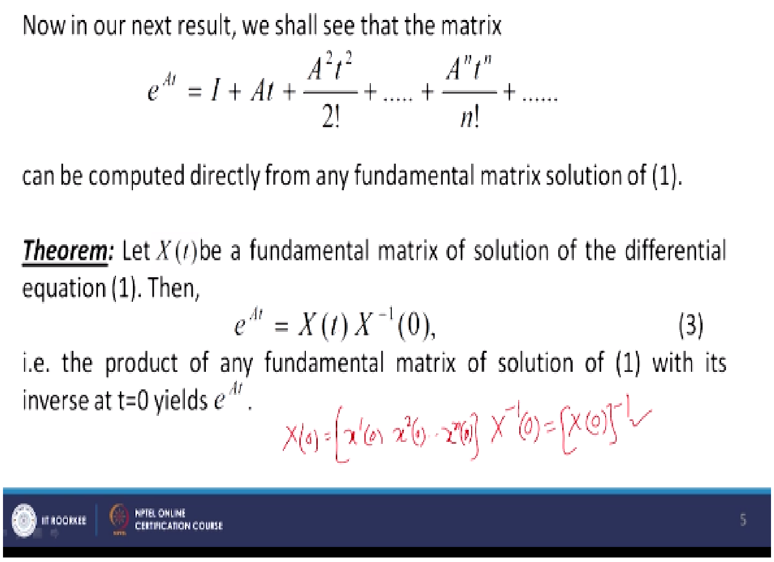

Now we have seen that e to the power At where A is n * n matrix is given by this series I + At a square t square/2 factorial and so on A to the power n t to the power n/n factorial, e to the power At is not easy to compute from this expansion, I + At + A square t square/2 factorial and so on, but we will see in the theorem which we are going to now state that it can be computed directly from any fundamental matrix solution of the vector differential equation x dot = Ax.

The theorem is like this, let Xt be a fundamental matrix solution of differential equation 1 then e to the power At is Xt * X0 whole inverse. X inverse 0 is, I mean to say the matrix X0 inverse okay, so X inverse 0 is the inverse of the matrix X0 that is when you know the matrix

Xt just put t = 0 there so then X0 will be = x10 x20 and so on xn0. So this matrix n/n matrix you will get, you find the inverse of this X0 to get this X inverse 0, (()) (10:22) 0 inverse.

So in order to determine the matrix e to the power At we need to know a fundamental matrix of the vector differential equation and then the inverse of the matrix X0 okay, so the product of the 2 will then give e to the power At.

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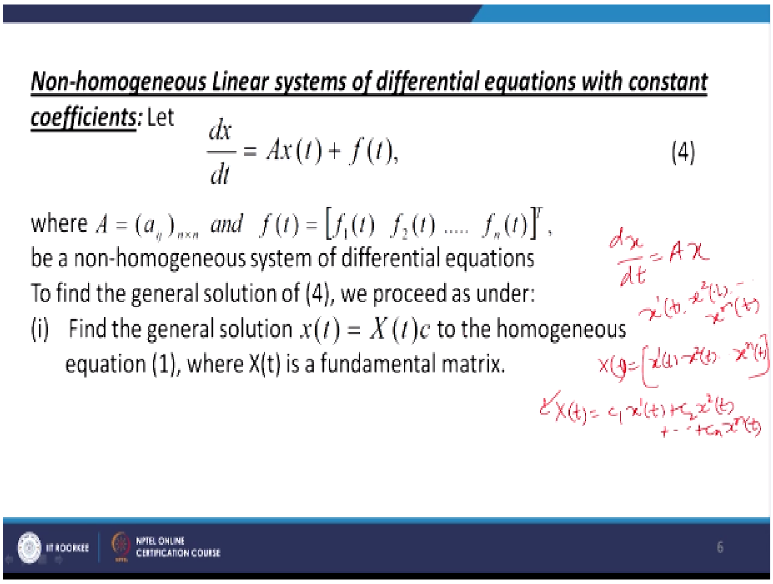

Now let us consider a non-homogenous linear system of differential equation with constant coefficients. So let us say we consider dx/dt = Axt + ft where A is n/n matrix Aij and cross n and ft is a column vector with components x1, f1t, f2t and so on f1t. Earlier we considered only homogenous equations where we took this ft as a 0 vector. Now after that we have tackled the homogenous linear systems of differential equations with constant coefficients.

Now let us consider the case of an non-homogenous linear system of differential equation with constant coefficients. Now so in order to find the general solution of this equation 4 what we do is we proceed in the following manner. First we find the general solution which we write as Xt = Xt * c to the homogeneous equation 1 okay, so homogeneous equation 1 is what dx/dt = Ax okay.

The homogeneous equation corresponding to the nonhomogeneous linear system 4 is dx/dt = Ax so we find general solution of this that means we find n linearly independent solutions x1t, x2t and so on, xnt of this homogeneous linear system and then form the fundamental matrix we have discussed just now fundamental matrix xt = x1t, x2t and so on xnt whose n columns are these n linearly independent solutions of dx/dt = Ax.

So this is fundamental matrix and we then multiply this xt/c okay to get the general solution. So c * xt as we have seen earlier there is nothing but c1 x1t + c2 x2t and so on cn xnt. So first we find the n linearly independent solutions of the vector differential equation dx/dt = Ax and then write the general solution okay, c1 x1t + c2 x2t and so on cn xnt which in a concise form can be written as c * Xt.

So we find this general solution of the homogeneous equation X dot = Ax and then we find particular solution Xpt we write it as Xpt to this nonhomogeneous system okay.

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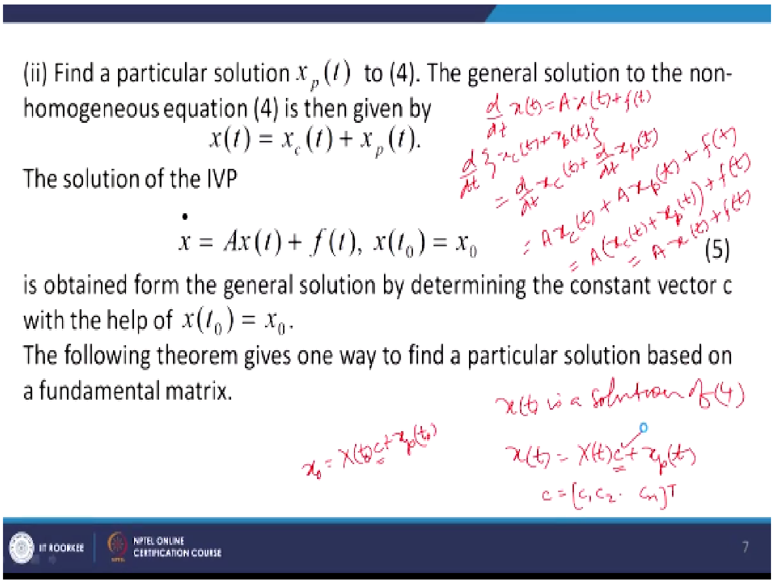

So then the combination of the 2 xct and xpt okay, will give us the general solution of the nonhomogeneous system 4 okay, nonhomogeneous system 4 will have the general solution given by xt = xct + xpt. Now let us see how, first we see that it is a solution of the system 4 and then we see that it is a general solution. So let us first show how it is a solution of 4 okay. So let us in the system 4, let us put xt = Ax ct + xpt.

So this is your dx/dt = Axt + ft let us put in that so we have d/dt this is our system so first we show that xt = Axct + xpt is the solution of this so d/dt let us take the lefthand side, let us put for xt, xct + xpt. Then d/dt of xct + xpt we know if this is d/dt of xct + d/dt of xpt, now xct is the general solution of the homogenous system dx/dt = Axt so we get here d/dt of xct = Axct + xpt is a particular solution of this system 4.

So d/dt of xpt will be equal to Axpt + ft. Now let us combine this can be written as A xct + xpt + ft and this will be = xct + xpt = xt so we have Axt +ft. So xt = xct + xpt satisfies the system 4 and therefore xt is the solution of 4. Now we see that we know that xt = xct + xpt contains n constants in the solution xct of the homogenous system so this solution xt = xct + xpt contains an arbitrary constants c1, c2, cn.

And therefore xt is the general solution of this nonhomogeneous system dx/dt = Axt + ft. Now when we want to find the particular solution of the initial value problem. So let us consider x dot = Axt + ft where at t0 we assume that xt has the value x not then the solution for this initial value problem is obtained by calculating the constant vector c that occurs in xct okay. We have xt = xct that means xt * c + xpt, this constant vector c which we have taken as c1, c2, cn.

So this constant vector c is calculated by using the value of xt at t = t0. When you put t = t0 here we get x != x t0 * c + xp t0, okay, so this c is calculated from here and then we put the value of c in this solution xt = xt * c + xpt to arrive at a particular solution of the initial value problem. Now the question arises how we find particular solution of the system 4 okay.

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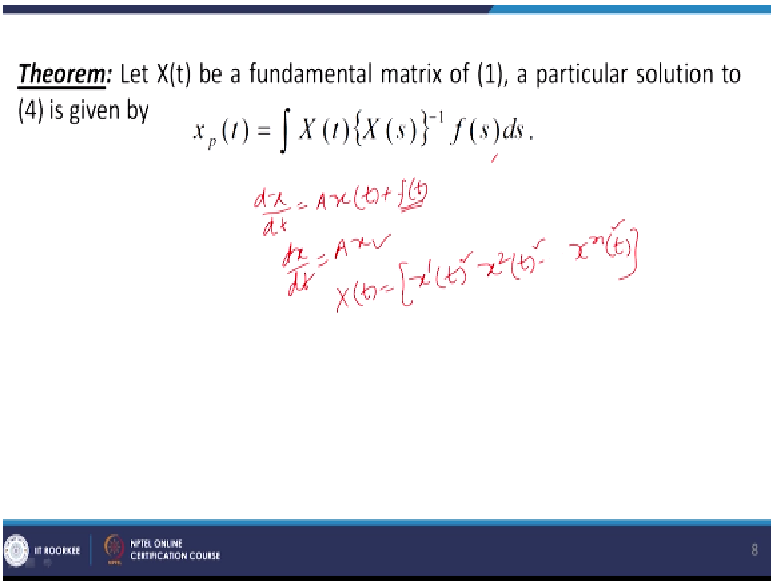

So we state a formula for this particular solution of system 4, let Xt be a fundamental matrix, so in order to find particular solution to the system 4 we need to know the fundamental matrix of 4 that is let us recall we have the system dx/dt = Axt + ft . So this theorem tells us a particular solution xpt can be found once you know a fundamental matrix of the homogenous system dx/dt = Ax.

That is Xt = x1t x2t and so on xnt so if you have a fundamental matrix for the system dx/dt = Ax then that fundamental matrix you put here inside the integral so you have Xt then you find the inverse the matrix xt so that is Xs is the variable of integration here so inverse of the matrix xt you need to find. Now remember x1t, x2t or xnt are n linearly independent solutions of dx/dt = Ax, so xt is a non-similar matrix and therefore we can find it is inverse.

So we find inverse of this matrix xt and put here the value of Xs inverse and then multiply by fs this vector ft is known to us and then integrate with respect to s. So we shall get the matrix, we will get the particular solution of equation 4, okay, so we know xpt and then we add xct the general solution of dx/dt = Axt to that and have the general solution Axt = xct + xpt of the system 4. So let us see how we go about it.

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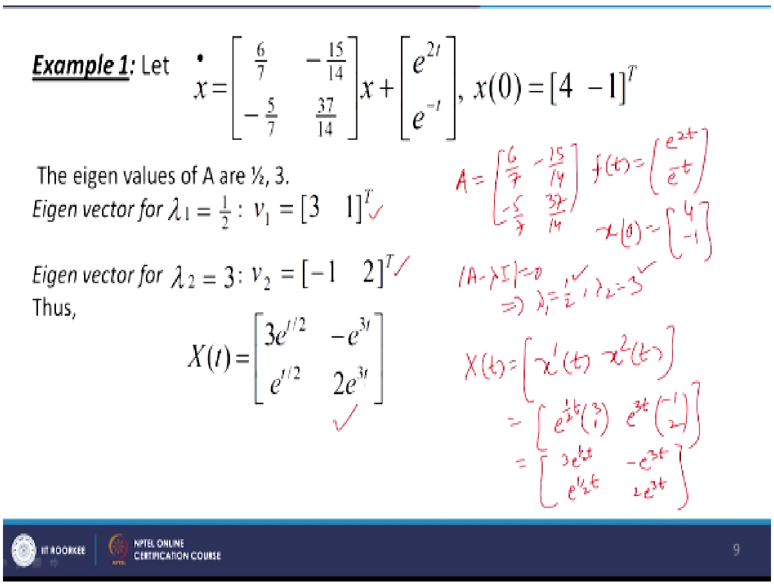

Let us take an example x dot = Ax + ft so A here is this matrix 6/7-15/14 and then -5/7 and we have 37/14 this is matrix A and ft is = the vector e to the power 2t, e to the power –t. Now we have initial value problem so we are given that when t != 0 x t0 is x not that is the vector 4-1 so we have x0 = 4-1. So first we will find the general solution of the nonhomogeneous system x dot = Ax + ft and then use the initial condition x0 = 4-1 to arrive at the solution of this initial value problem.

Now it is easy to see that determinant of A-lambda = 0 gives us lambda = 1/2 and lambda = 3. So lambda 1 = 1/2 lambda 2 = 3. So we get the 2 eigen values which are real and distinct, okay, so the eigen values are real and distinct. We can find eigen vectors corresponding to lambda 1 = 1/2 and lambda 2 = 3 since the eigen values are real and distinct the corresponding eigen vectors are orthogonal.

So v1 and v2 we can find they are orthogonal, they are linearly independent okay. Now what we have xt, xt is the fundamental matrix, so xt = x1t x2t. So these eigen values are real and distinct, so x1t will be e to the power lambda 1t that is e to the power 1/2 t * 31, this is x1t and x2t will be e to the power 3t, e to the power lambda 2t * v2, so -12 and you can see this will be equal to 3 e to the power 1/2 t, e to the power 1/2 t, –e to the power 3t and 2 e to the power 3t.

So we get this fundamental matrix, okay now what we do.

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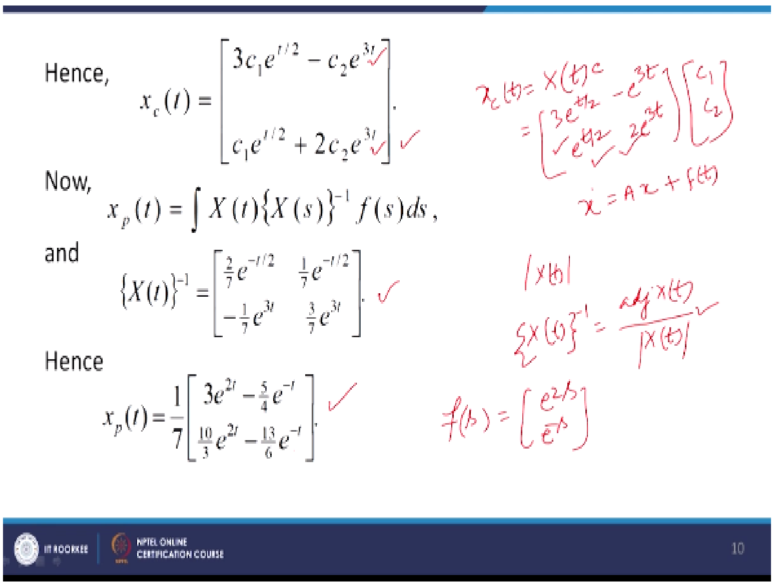

Once we have xt with us, fundamental matrix with us we can write xct. So xct = xt*c where xt is 3 e to the power t/2 and second column is this one what is that –e to the power 3t, 2 e to the power 3t multiplied by the vector c1 c2 okay, so when you multiply this column vector to this 2/2 matrix you get 3 c1 e to the power t/2-c2 e to the power 3t, this one first element and second element is c1 e to the power t/2 and then 2c2 e to the power 3t.

So we get this general solution of the vector equation x dot = Ax. Now let us find a particular solution of x dot = Ax + ft so particular solution by our theorem is given by Xt * Xs inverse * fs. Now Xt, Xt we know this is our Xt, this is our Xt, we can find it is determinant since the vectors x1t and x2t are linearly independent this matrix is determinant is nonzero okay, so we can find determinant of this matrix xt.

We can find adjoint of the matrix xt and then find xt inverse. So xt inverse is adjoint of the matrix xt/determinant of xt. Now when you do this you will get this matrix. Xt inverse = 2/7 e to the power –t/2 1/7 e to the power –t/2 -1/7 e to the power 3t 3/7 e to the power 3t and hence now you put this xt inverse here in this okay. So you have xt with you, this is xt then xs inverse.

So when Xs inverse will be 2/7 e to the power –s/2 1/7 e to the power –s/2 -1/7 e to the power 3s 13/7 e to the power 3s so put here an fs, fs is given to us as e to the power 2s, e to the power –s. So we put the value of fs thus inside this then we have to integrate with respect to s. Remember this t and s are independent variables so Xt you can even write outside the integral okay.

We just need to find the product of Xs inverse and fs and then integrate with respect to s and then multiply ultimately by the Xt matrix that is the fundamental matrix this one. So after these competitions what you get is xpt = this matrix 1/7 3 to the power 2t – 5/4 e to the power –t 10/3 e to the power 2t – 13/6 e to the power –t.

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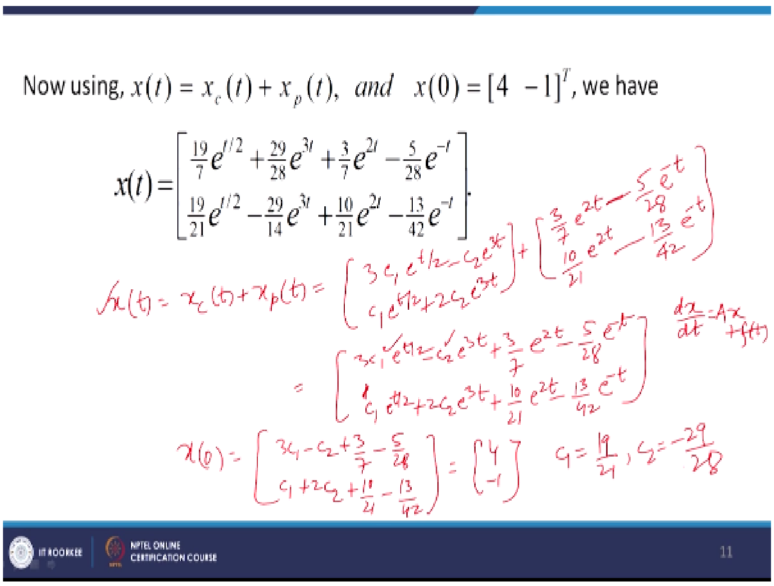

So we can write general solution xt = xct + xpt gives us now let us see xct is what this is 3 c1 e to the power t/2 –c2 e to the power 3t and then we have c1 e to the power t/2 + 2 c2 e to the power 3t and to this we add this matrix, 3/7 e to the power 2t then we have 10/21 e to the power 2t and then we have -5/28 e to the power –t and then we have -13/42 e to the power –t, okay, so we aid the 2 matrix.

We add the 2 matrix xct okay, xct is this and xpt is this, okay, we get then 3c1 e to the power t/2-c2 e to the power 3t + 3/7 we add component wise, e to the power 2t-5/28 e to the power –t and then c1 e to the power t/2 + 2 c2 e to the power 3t and then we get 10/21 e to the power 2 t and then -13/42 e to the power –t.

So this is the general solution of dx/dt = Ax + ft, this is the general solution of this. Now let us apply the initial condition. So we put t = 0 in this and so x0 will be = put t = 0 here, 3c1-c2 + 3/7 – 5/28 and here we get c1+2c2 and 10/21 and we get -13/42 this is = we are given 4-1. So this will give you 2 equations, 3c1-c2 + 3/7 -5/28 = 4 c1+2c2 +10/21-13/42 = -1 so these are 2 linear equations in 2 unknowns c1 and c2 we can determine the value of c1 and c2 here.

And then put in the general solution okay, so c1 and c2 we can put here and when you put the values of c1 and c2 you arrive at this solution, so xt = 19/7 e to the power, so you can see here the coefficient of e to the power t/2 is 19/7 that means 3 c1 = 19/7 so c1 will come out to be 19/21 when you find c1 = this and the coefficient of e to the power 3t is –c2 –c2 is 29/28 so c2 will come out to be -29/28.

So put these values okay, of c1 c2 here in this general solution you get the solution of the initial value problem.

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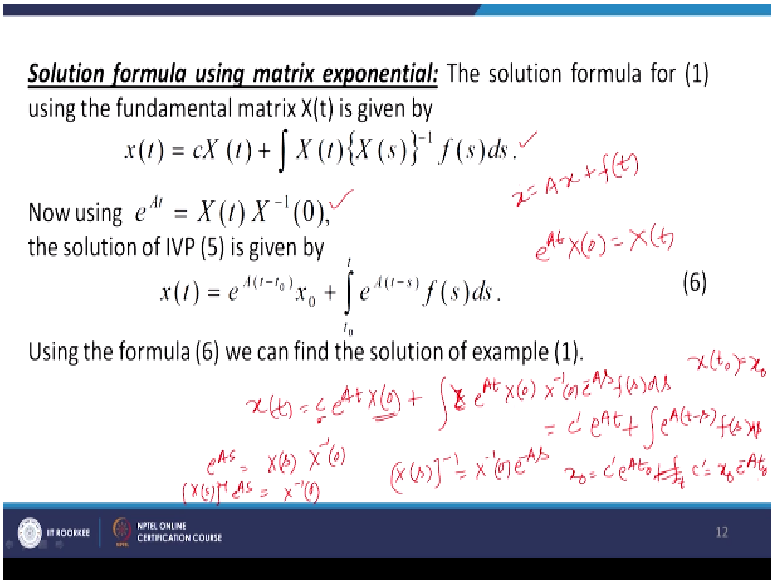

Now the solution formula for the nonhomogeneous system okay, the solution formula for the nonhomogeneous system x dot = Ax +ft using the fundamental matrix we know is given by this okay, now let us apply we will find the another formula, alternate formula for the matrix solution e to the power At we know e to the power At is Xt * X inverse 0 so if you put here the value of Xt we get this following.

Xt = we can find this formula c then Xt you can post multiply by x0, so e to the power At * X0 this is = Xt. So we will get the value of Xt here so e to the power At * X0 + integral Xt, Xt = e to the power At * x0 okay, xs inverse, xs will be e to the power As = xs * x inverse 0. Now we want to find xs inverse so premultiply by xs inverse, so xs inverse e to the power As = x inverse 0 okay.

And xs inverse is therefore = post multiply by e to the power –s, so we get x inverse 0 e to the power –As. So let us put here so x inverse 0 * e to the power –As fsds and we arrive at, now c * x0, where 0 is a constant, c is a constant so we can put another constant c dash, e to the power At + x0 * x inverse 0 is identity matrix so e to the power At * e to the power –As is e to the power A times t-s * fsds.

Okay, so this is the formula and when you have to use xt != x not then to determine this c dash okay, what you do put here that value so x t != x not, x != c dash, e to the power At0 and here you get the integral okay, t0 to t, okay, x not when we put t = t0 okay yeah this integral will be 0 so Xt = X0 = c dash A t0 and therefore c dash = x not * e to the power –A t0.

So you put here c dash = x not * e to the power –A t0 so we will get e to the power A times t-t0 * x not + this integral. You can put here t = t0 this integral vanishes and what we get is x t ! = x not. Now this formula we can also use to determine the solution of example 1 okay, particular solution of example 1, here we need to find the matrix e to the power At.

E to the power At can then be found by xt * x inverse 0 now to determine x inverse 0 what you do, you have the matrix xt with you, put t = 0 in that you will have x0 3/3 matrix X0 find the inverse of the matrix you will have x inverse 0. So then we will take the product of Xt with x inverse 0 to determine e to the power At and put here in this formula will get the solution. Now there is one more method, solution method by decoupling.

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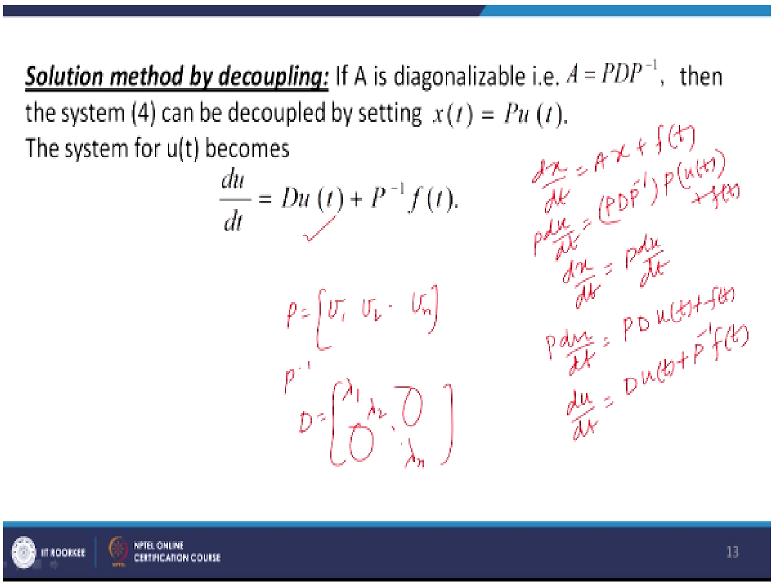

If A is diagonalizable we know that they are just non-singular matrix P such that A = pdp inverse and then the system 4, okay, dx/dt = Ax + ft can be expressed like this. If you assume xt = Put, suppose you assume xt = Put then dx/dt where P is this matrix, so Pdu/dt so what you get Pdu/dt = A is PDP inverse * x is Put + ft, okay so Pdu/dt = now P inverse P is the identity matrix so you get PD ut + ft.

Now pre-multiply this equation by P inverse. So P inverse P will be identity matrix and we will get du/dt = Dut + P inverse ft. So this is how we get this formula okay, so if the matrix A is diagonalizable okay, then we can use this formula also. We will solve this equation we know the matrix P with P is the matrix which is found from the eigen vectors of A okay, so we find the matrix P from the eigen vectors of A.

Let us say they are v1, v2, vn since they are linearly independent we will be able to find P inverse also okay. D matrix will support v1 eigen vector corresponding to lambda 1, v2 eigen vector corresponding to lambda 2 and vn eigen vector corresponding to lambda n then this is your D matrix, okay. So we have A = PDP inverse. We know the D matrix we know D matrix P inverse because we know P.

We can substitute in this and determine solve this equation. Okay, let us see for example 1, in the case of example 1 we have seen the eigen vectors corresponding to lambda = 1/2n3 are 31-12.

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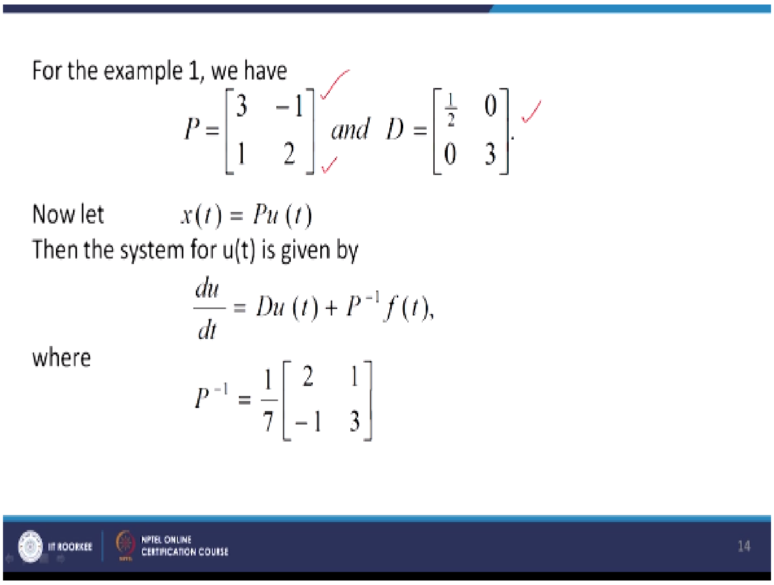

So this is your P matrix, D matrix is lambda 1 = 1/2 D2 = lambda 2 = 3 so this is D matrix xt = Put let us assume, so then du/dt = Dut + P inverse ft. P inverse if you find for this matrix P it comes out to be this matrix 2/7 1/7 -1/7 and 3/7 okay. We know the matrix D now okay. We know the matrix P inverse and so we can find P inverse * ft, ft is also know to us. Okay.

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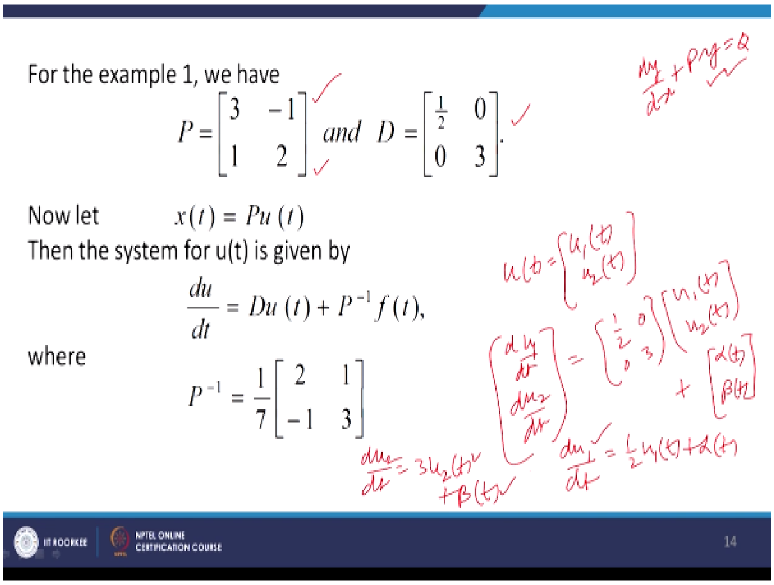

So now what will happen, du/dt is let us say Ut = U1t U2t okay, so then du/dt will be du1/dt or du2/dt, this is du/dt. D is your matrix, 1/20 and 03, okay. U2 is u is u1t u2t then you know P inverse * ft which is again a column matrix, okay, you will have du1/dt = you can multiply here 1/2u1t + let us say p inverse ft is alpha t beta t. So you will get + alpha t this is one equation and the similarly second equation is du/dt = 3u2t + beta t.

Now this equation and this equation they are linear equations in u1 okay. We know how to solve the equation dy/dx + Py = Q where P and Q are functions of x. so you can solve du1/dt – 1/2 U1 = alpha t from this and Dut/dt -3 ut = beta t also from this we can find the integrating factor and then find the solution u1t and u2t. So we will find that.

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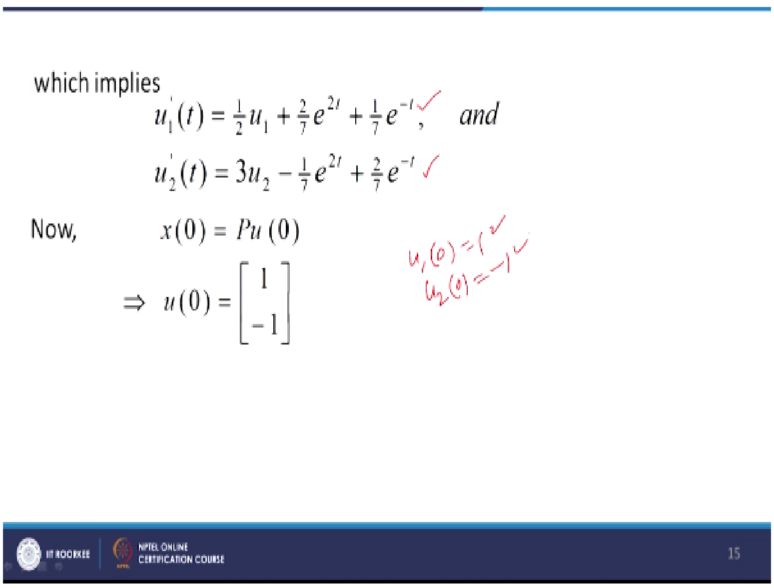

So these are the equations for u1 and u2 differential equations when you solve them you get the values of u1t and u2t and then you apply this initial condition. Initial condition we have assumed that xt = Put, so x0 = pu0 and then u0 = p inverse x0. P inverse x0 if you find okay, you will get 1-1, so u10 = 1 and u20 = -1. So we have these 2 linear equations in u1 and u2 with the initial conditions u10 = 1 u20 = -1. We can solve them easily to get the value of u1t and u2t.

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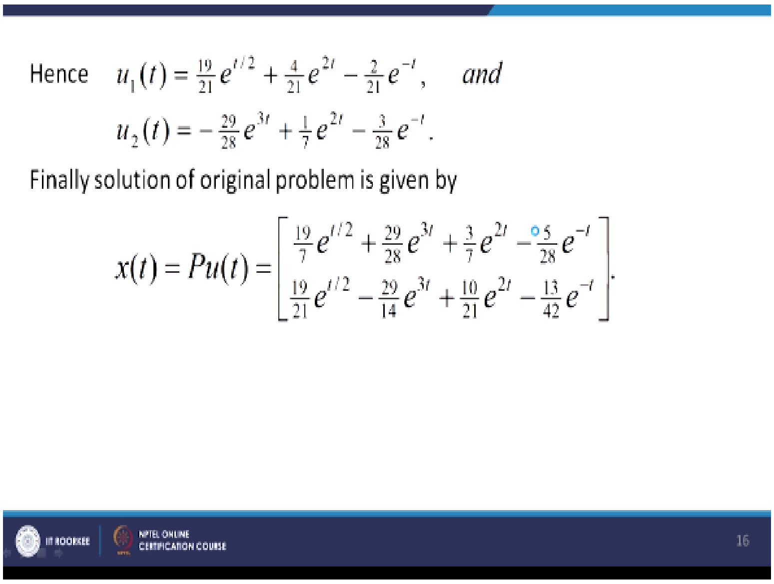

Now we know the vector ut so xt will be p*ut okay. So multiply this vector ut whose components are u1t and u2t by the matrix p will get the solution of the example 1. So this is another method which we use in case the matrix A is diagonalizable. So with that I come to the end of my lecture. Thank you very much for your attention.