Ordinary and Partial Differential Equations and Applications
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Lecture - 01
Introduction to Differential Equations - I

Hello friends. Welcome to the lecture. In today's lecture, we will discuss some basic concept which is very useful in further lectures.

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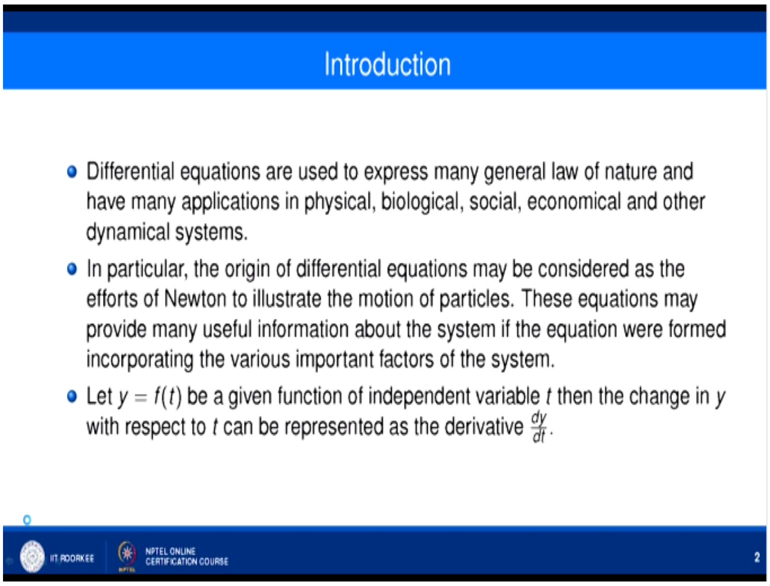

So let us start so differential equation are used to express many general law of nature and have many applications in physical, biological, social, economical and other dynamical systems. In particular, the origin of differential equation may be considered as the effort of Newton to illustrate the motion of particles. This is a very famous example and these equations may provide many useful information about the system if the equation were formed incorporating the various important factors of the system.

So here if we consider let y= f t be a given function of independent variable t then the change in y with respect to t can be represented as the derivative dy/dt.

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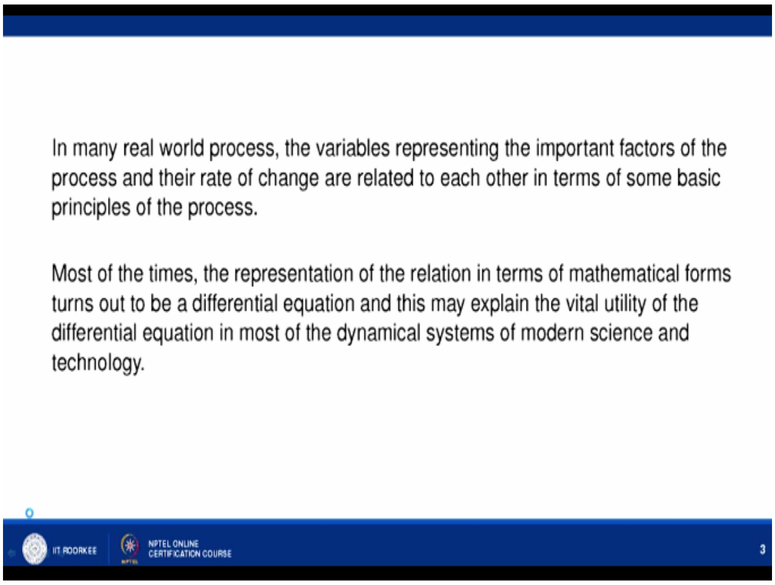

So in many real world process, the variable representing the important factor of the process and their rate of change are related to each other in terms of some basic principle of the process which most of the times the representation of these relations in terms of mathematical forms turns out to be a differential equation and in this way we can explain the vital utility of the differential equation in most of the dynamical system of the modern science and technology.

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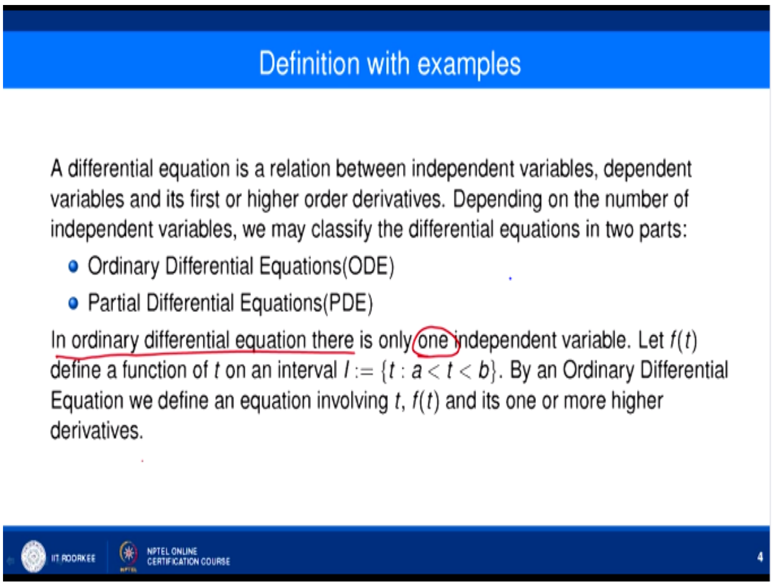

So let us start with proper definition along with some examples. So as the name indicate a differential equation is a relation between independent variables, dependent variables and its first or higher order derivatives. So for example if independent variable is t, dependent variable is y or it means that y is a function of t and we can consider the first higher derivative, it means that y dash t, y double dash t and so on.

So any relation which relate t, y, y dash t, y double dash t and so on is considered to be a differential equation. Now depending on the number of independent variable we may classify the differential equation in 2 parts, ordinary differential equation and partial differential equation. So in ordinary differential equation there is only one independent variable so let f t define a function of t on an interval I, here I is open interval a to b.

And by an ordinary differential equation we define an equation involving t, f t and its one or more higher derivatives.

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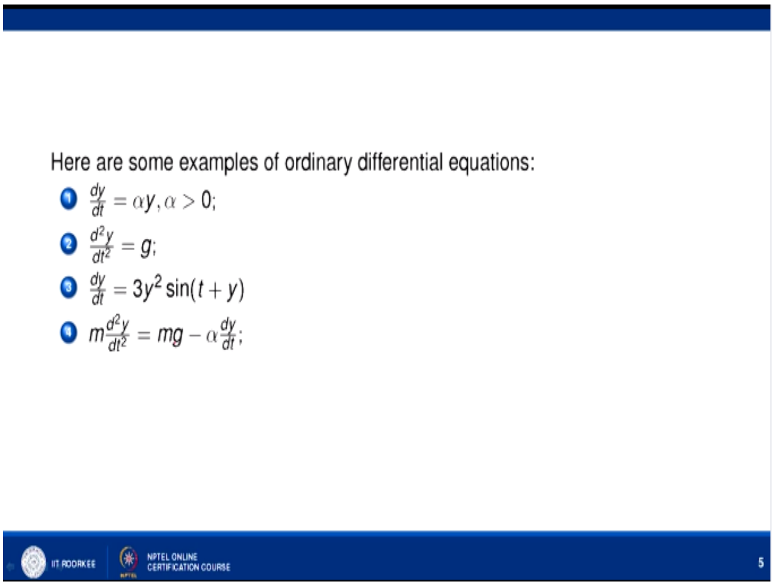

So here are some examples of ordinary differential equation. If you look at this first one, first one is dy/dy= alpha y, alpha is any real constant which is a positive and d2y/dt square as g this is second example and third example is dy/dt=3y square sin t+y and 4th example of differential equation is m d2y/dt square=mg-alpha dy/dt. If you look at the first example, here y is the dependent variable and t is the independent variable.

And dy/dt is the first derivative of y with respect to t and it is related to the dependent variable y by this relation dy/dt=alpha y. This is a very commonly used equation in population dynamics and it says that if the population say grows with the rate of alpha then the population at any given time t can be obtained with the help of this differential equation.

And if you look at the second equation d2y/dt square=g, basically this is what when you consider a particle say falling from a particular point then this will represent the equation governing the motion of that particle and it is given as d2y/dt square=g. Here this g represents the gravitational force and third one is dy/dt=3y square sin t+y. Here this represent a particular say motion of some particle.

And in 4 if you look at m d2y/dt square=mg-alpha dy/dt and if you look at this is similar to equation number 2. In fact, in equation number 2 we have ignored that there is any kind of friction due to air on this particle. So here we are assuming that this particle is moving under only the influence of gravitational force and there is no other say restriction on that particle is put.

Now but if we assume that this air if this air put some kind of force upward then we can modify the equation number 2 in this way. Here this alpha dy/dt represent the friction due to air so here the second can be modified in this way. Here we can write m d2y/dt square=mg-alpha dy/dt. Here alpha represent the constant which says that the restriction or the force due to air is proportional to the velocity of the particle.

So if we assume alpha is 0 then this 4th equation is nothing but the second equation. So these are some example of ordinary differential equation which most of the time we came across solving these equations.

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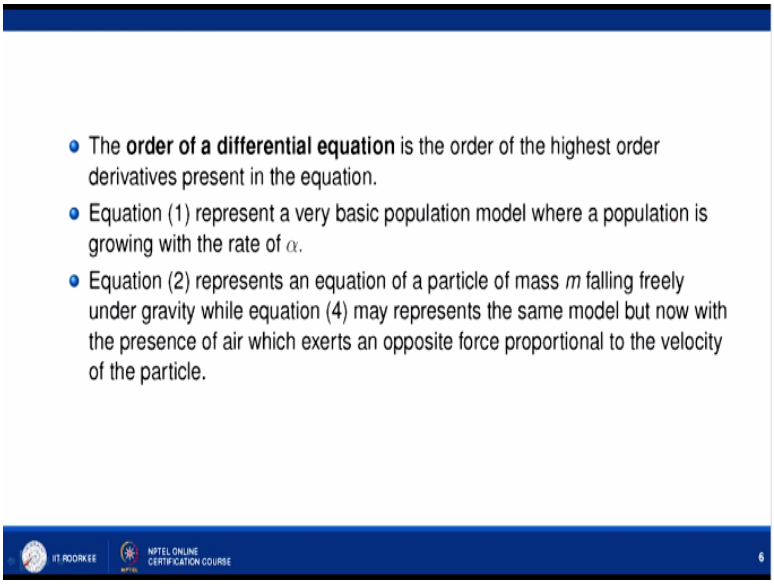

So once we have the definition and the example of differential equation let us discuss more basic thing about the differential equation. So first thing is the order of a differential equation, so the order of a differential equation is the order of the highest order derivative present in the equation. For example, if you look at here in first equation the order of the highest order derivative is 1.

Here we have only one derivative and that is dy/dt=alpha y, so here we have only one derivative and the order is 1. So equation 1 is the differential equation of order 1. If you look at the second equation d2y/dt square=g, so here we have only one derivative and that is of highest order. So here the order of this differential equation 2 is 2 and similarly equation number 3 is a differential equation of order 1 and in 4 we have 2 terms of derivative exist, first one is dy/dt and another one is d2y/dt square.

But if you look at this is first order and this is the second order so highest order is 2, so we can say that this differential equation is of second order and as we have pointed out this equation 1 represent a very basic population model where a population is growing with the rate of alpha.

Equation 2 represent an equation of a particle of mass m falling freely under gravity while equation 4 may represent the same model but now with the presence of air which exert an opposite force proportional to the velocity of the particle that we have already seen.

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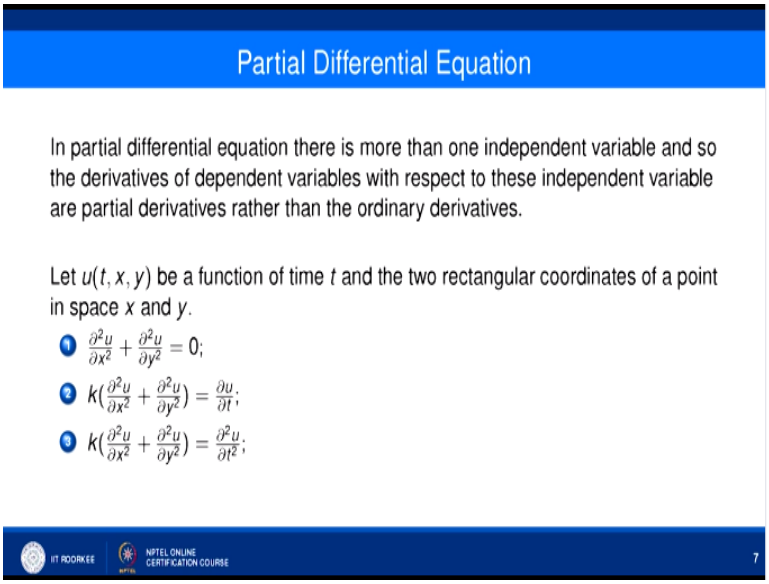

Now moving on next, so now we define what is partial differential equation. We have seen that the number of independent variable simply classify the differential equation into 2 parts, one is ordinary differential equation and another one is partial differential equation. In ordinary differential equation, we have seen that independent variable is only 1. So there is only one independent variable.

And we are considering all the function of that independent variable but in partial differential equation we may have so in partial differential equation there is more than one independent variable and so the derivative of dependent variable with respect to these independent variables are partial derivative rather than the ordinary derivative. So here since number of independent variables are more.

So if we consider any function of these independent variables then if we consider the derivatives then it is coming out to be partial derivative rather than the ordinary derivative. So that is why we call the equation which involve the function in dependent variables and the derivative of function with respect to these independent variable as partial differential equation rather than ordinary differential equation.

So let us consider this example. Let u t, x, y be a function of time t. This is a time variable and 2 rectangular coordinates of a point in a space x and y. So here x and y are space variable and t is a time variable. So here let consider u is a function of t, x and y where t is a time variable and x and y are space variable. So let u t, x, y be a function of time t and the 2 rectangular coordinates of a point in a space x and y.

Then consider the following 3 equations. First equation is dou 2u/dou x square+dou 2 u/dou y square=0. So this is the relation where unknown function u is a function of x and y and there is no partial derivative of u present in terms of t. So it is an example of a partial differential equation. Now consider the second example where all these partial derivatives present. So here this is k times dou 2u/dou x square +dou 2u/dou y square=dou u/dou t and third one is k times dou 2u/dou x square+dou 2u/dou y square=dou 2u/dou t square.

In these last 2 problems, this k is a constant and depending on this problem this k may have some values and these are very famous examples of partial differential equation.

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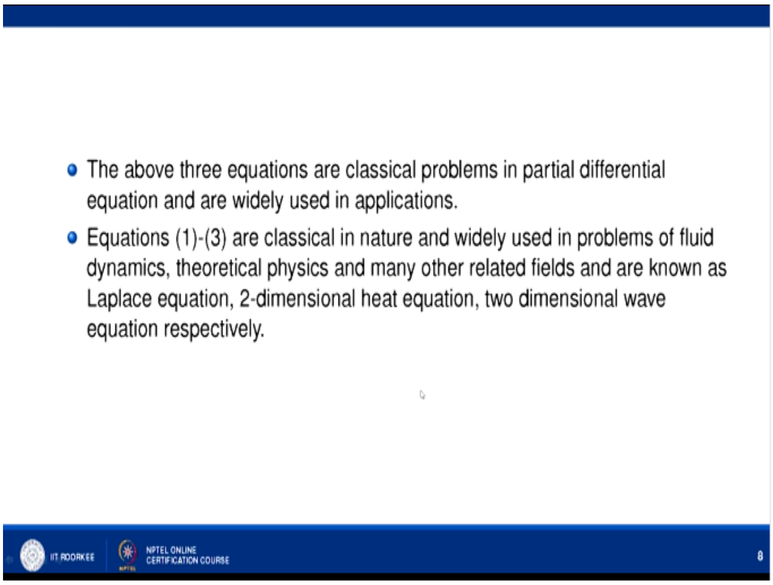

So the above 3 equations are classical problems in partial differential equation and are widely used in application. So these equations given in 1, 2, 3 are classical in nature and widely used in problems of fluid dynamics, theoretical physics and many other related fields and these equations are known as the first equation is known as Laplace equation, the second one is heat equation and third one is wave equation.

So this is given here, Laplace equation, 2-dimensional heat equation, here space variables are 2 so this dimension is depending on the space dimension. So here we have 2 space coordinates x and y so it is 2-dimensional heat equation. Similarly, this is 2-dimensional wave equation.

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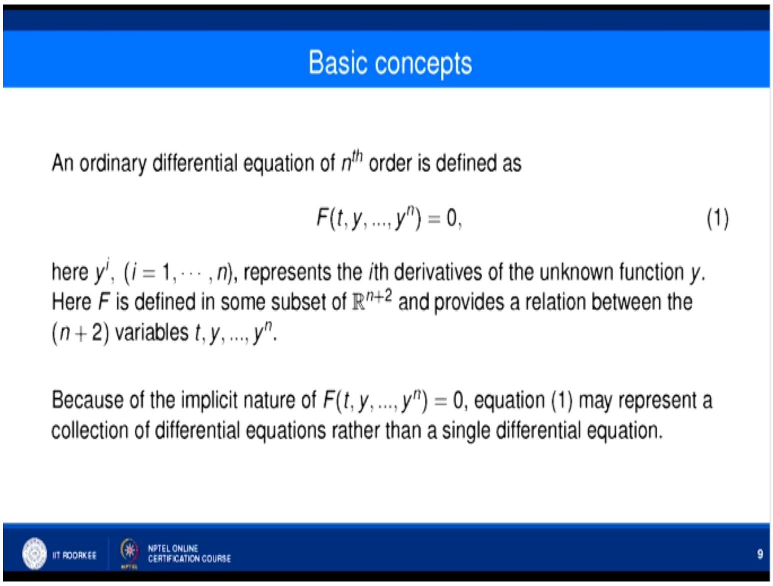

Now coming back to differential equation so what we try to do here first we discuss the ordinary differential equation, we discuss all the basic concept, existing uniqueness and related problems in ordinary differential equation and then we will start with partial differential equation also. So first let us start with ordinary differential equation. So in ordinary differential equation of nth order is defined as follows.

Here F is some smooth function and it is a function of t, y and yn. Here t is independent variable as we have already pointed out that ordinary differential equation contains only one independent variable. So here y is a function of this independent variable t, so F is the relation between t, y and yn. Here yn represent the nth derivative of y with respect to t. So here we can say that here yi where i is from 1 to n represent the ith derivative of the unknown function y.

And here F is defined in some subset of R n+2 and provide a relation between the n+2 variables, what are n+2 variables t, y, y dash, y double dash and yn but this is the general definition of ordinary differential equation of nth order but because of the implicit nature of this function F this may represent more than one differential equation or we can say that it may represent the collection of differential equation rather than a single differential equation.

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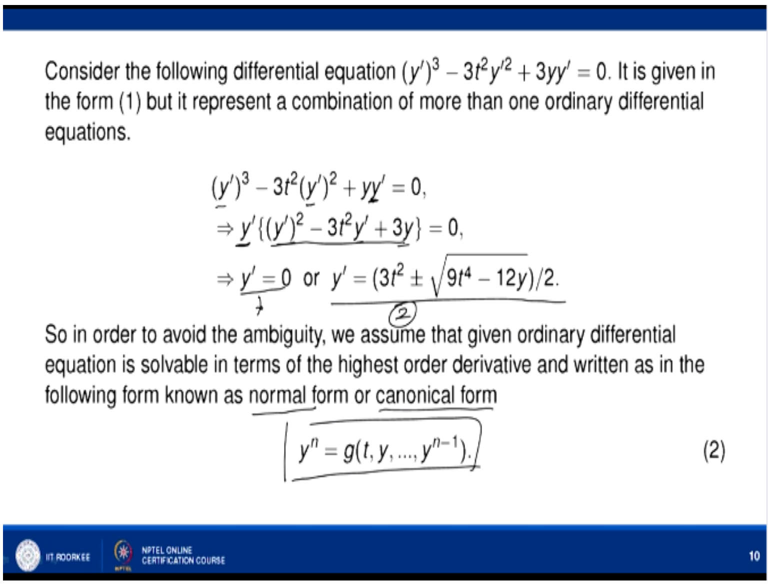

Let us explore this possibility. Consider this differential equation y dash cube-3t square y dash square+yy dash=0. If you look at this is a relation between what here t is present, y dash is here, y is here so here this is the relation between t, y and y dash. So it is a differential equation of order 1 but if you look at closely then it gives not only one differential equation but it is giving basically 3 differential equations at once.

So it is given in the form 1 but it represents a combination of more than one ordinary differential equation. If you can solve this let us right now this is given in a very nice form and we can solve it. So here we can solve it y dash cube-3t square y dash square+yy dash=0. So here if you look at in all the terms your y dash is there, here y dash is there, here y dash is there, here y dash.

So what we can do, we can take out this y dash out and if you take y dash out then first one is y dash square-3t square y dash+3y. Now here y dash and if you look at in the bracket we have a quadratic equation in terms of y dash. So we can solve using the theory of quadratic equation and we can write this as y dash=0 or y dash=3t square+- under root 9t to power 4-12y/2.

So if you look at this represent one differential equation and this is the say basically 2 differential equations. So here we have this differential equation is not only one differential equation but it is giving you the possibility of 3 differential equations. So in order to avoid this ambiguity we assume that given ordinary differential equation is solvable in terms of the highest order derivative.

So rather than considering the implicit nature like here this is the implicit nature, so here we do not know that we are considering only one differential equation or a collection of differential equation. So to avoid this possibility what we assume that this yn is solvable, we can solve this equation in terms of yn and we can write our equation in this form yn=g t, y, up to y n-1.

Here g is some kind of function given in terms of n+1 variable that is t y, y dash up to y n-1 and this form is known as normal form or canonical form. So now onwards whenever we try to discuss we assume that our differential equation given in terms of normal form or canonical form. Actually, this will avoid the possibility of that multiple differential equation considered at one time okay.

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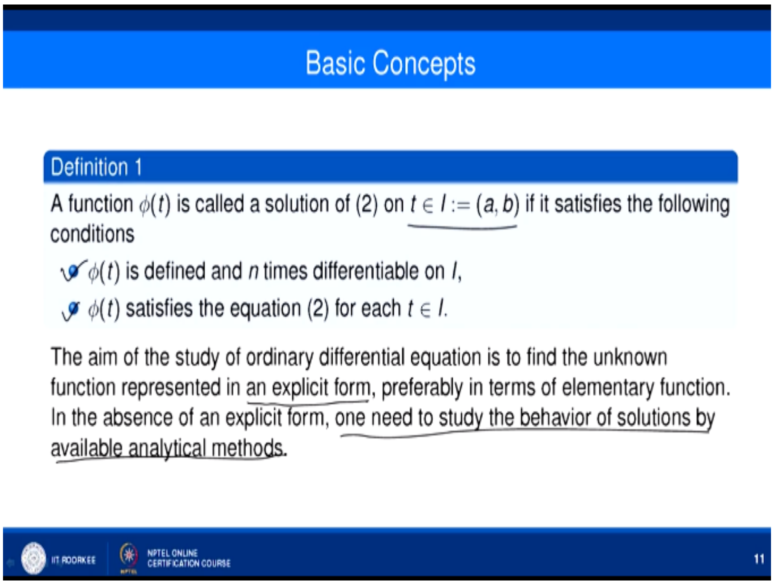

Now coming to basic concepts, once we define what is differential equation we need to define what is solution of differential equation. Similarly, we can define the solution of this differential equation in the following manner. So a function phi t is called a solution of 2, 2 means this equation yn=g t, y up to y n-1 if it satisfies the following condition. Here we need to consider that this t is belonging to at least some interval.

Right now I am considering this as an open interval a, b. So if it satisfies the following condition, first condition is that phi t is defined and it is n times differentiable on I so that all these terms exist, this nth derivative of y exist and all the given derivative exist right and second is that this phi t satisfy the equation 2 for each t in I. So it means that it is not only all the derivatives exist but also satisfy is a question for all t in that particular interval.

Then in this case we say that the function phi t is a solution of the differential equation. If any of these 2 fail, then we cannot say that phi t is a solution of equation 2. Now in this particular course, the aim of the study of the ordinary differential equation is to find the unknown function represented in an explicit form. So our first important aim is to find the solution. Solution means dependent variable in terms of t in explicit form.

It means that your phi t is given in terms of t in elementary form or we can say that in terms of elementary functions. Elementary functions are like for example polynomials, trigonometry functions, exponential functions all combinations of these elementary functions and if you are not able to find out the solution in explicit form, then we try to find the solution in terms of implicit form or we need to go further to study the behaviour of a solution by available analytical methods.

So suppose we are not able to find out the solution in explicit form means your a is given in terms of some function of t, if you are not able to find out that then we try to find out a relation between t and y and in terms of like f t, y t kind of thing and it is known as implicit form but suppose we are not able to find these kind of explicit form, we try to find out the behaviour of the solution by given analytical methods.

We will discuss some of these analytical methods in due course of time. Now before proceeding further we try to discuss the classification of the ordinary differential equation.

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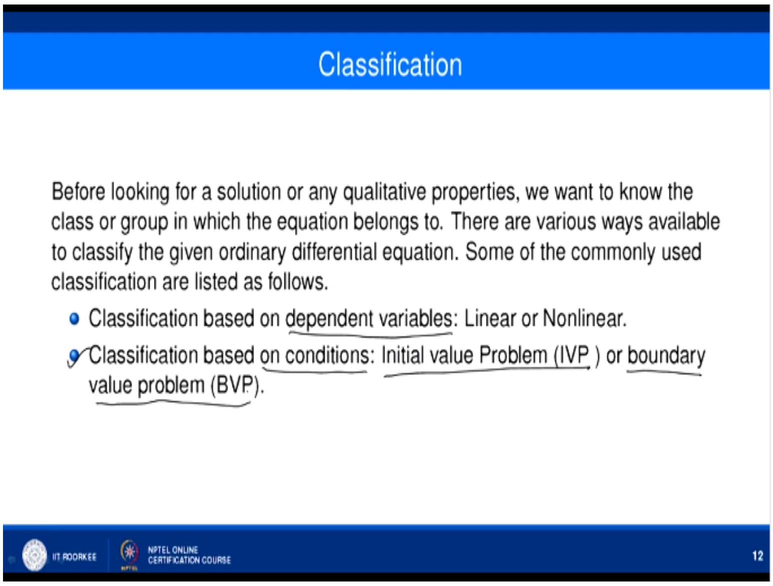

Or we say that before looking for a solution or any qualitative properties, we want to know the class or group in which the equation belongs to right. So what are these class or groups, let us define that. There are various ways available to classify the given ordinary differential equation and some of the commonly used classification are listed as follows. First based on the dependent variable okay.

So what it means that here dependent variables coming in linear manner or nonlinear manner and second classification is based on conditions. So if initial conditions are given then we call our problem as initial value problem or IVP denoted in a short form or if boundary conditions are given then we say that our equation is boundary value problem, in short we are writing BVP.

What are these initial and boundary condition? we will come to know. So we say that we first classify depending on the number of independent variable that is ordinary differential equation or partial differential equation and then we try to define further classify in terms of dependent variable or conditions. So let us consider the first case that classification depending on the dependent variable.

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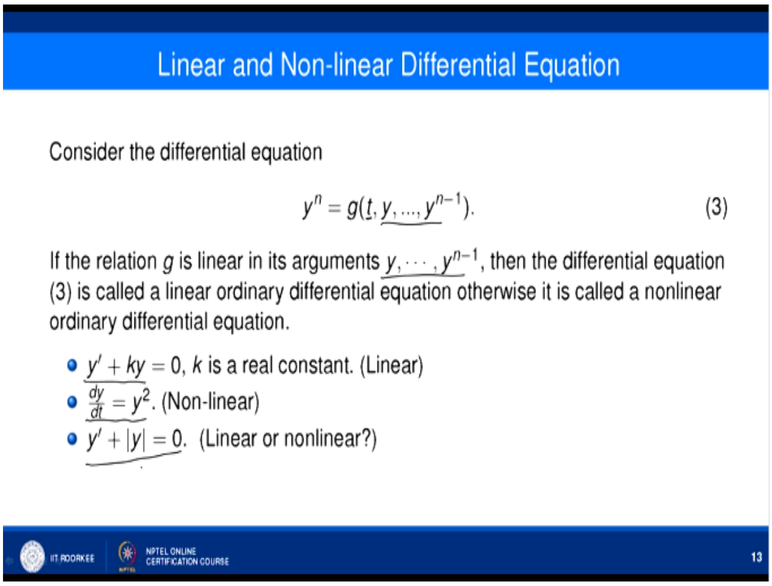

So here we try to find out what is linear and nonlinear differential equation. So let us consider the equation this yn=g t, y, y dash up to y n-1. So if the relation g is linear in its argument y to y n-1 so it means that look at the variable of g, here its variable is t, y and y n-1. These are the argument of this g. So t represents the independent variable, so we need not check, we need not to bother about g is linear with respect to t or not.

We need to consider the argument which are dependent variables so it means that if g is linear in terms of y to y n-1 then we call this equation 3 as linear ordinary differential equation otherwise it is called a nonlinear ordinary differential equation. For example, if you look at this first problem, it is y dash+ky=0. If you look at arguments of what it is y dash and y, so here y dash is coming linearly, y is also coming linearly so we can say that this equation is a linear differential equation, here k is a real constant.

And if you look at this second equation dy/dt=y square, if you look at here y is not coming in a linear manner, so we can call this differential equation dy/dt=y square. By the way here I am writing that y and y dash are coming in linear manner and here y is not coming in a linear manner what do you mean by that we are going to explain because here I can see easily that okay this is linear differential equation.

And here since y is coming in terms of powers of y so we say that it is a nonlinear differential equation but if you look at this equation y dash+mod of y=0. It looks that all these y and y dash is coming in a linear manner but really it is quite difficult to say whether it is linear or a nonlinear ordinary differential equation. So we need to devise a method by which we can check that okay the first one is linear one, second one is nonlinear one and third one is whether linear or nonlinear one.

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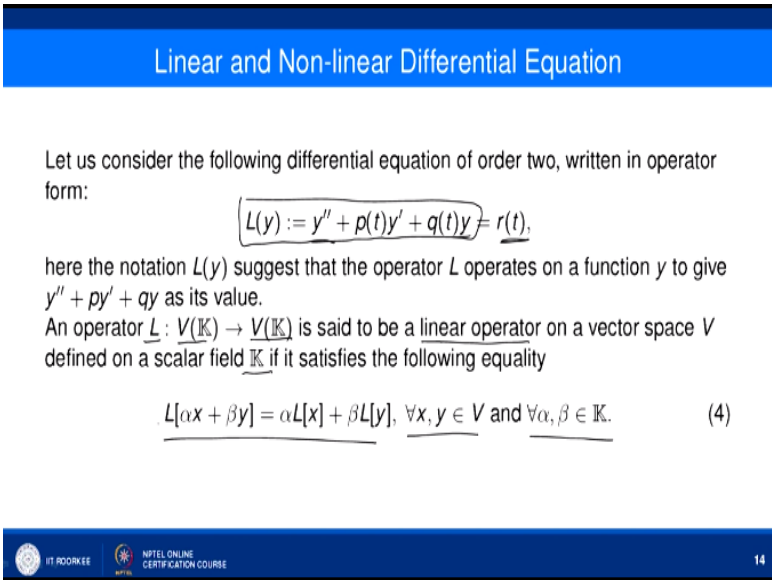

So let us devise a method so here we consider the following equation. So let us consider following differential equation of order 2. We can do this process for any kind of differential equation of any order. So for the time being let us consider the differential equation of order 2. So we define L y as the following operator. Here L y=y double dash+p t y dash+ q t y. It is kind of an operator which operate on y and give rise to this r t.

So it means that you put this y then this will give you the value r t. So it is an operator so this operator is defined like this L y=y double dash+p t y dash+q t y. Now if this operator L is linear we call the differential equation a linear differential equation otherwise it is a nonlinear differential equation. So how to check that this operator is a linear operator, so look at this here the notation L y suggest that the operator L operates on a function y to give this y double dash+ py dash+qy as its value.

So it is a kind of an operator defined on the set of all functions where this y is coming and it gives you another function of t. So here L y give rise to y double dash+p t y dash+ q t y. So we can consider that L is an operator from V K to V K and here V is a vector space V defined on this scalar field K. So let us say that an operator L defined from V K to V K is said to be a linear operator on a vector space V defined on a scalar field K if it satisfies the following equality.

So what is this equality that if L operating on alpha x+beta y=alpha times L x+beta times L y and this is true for all x, y coming from this vector space V and all alpha, beta coming from this scalar field K. So if it is true for every x, y in vector space V and every constant in K then we say that this L is a linear operator otherwise it is called as a nonlinear operator.

So once we know how to define a linear operator, now with the help of this let us see what is linear differential equation or nonlinear differential equation.

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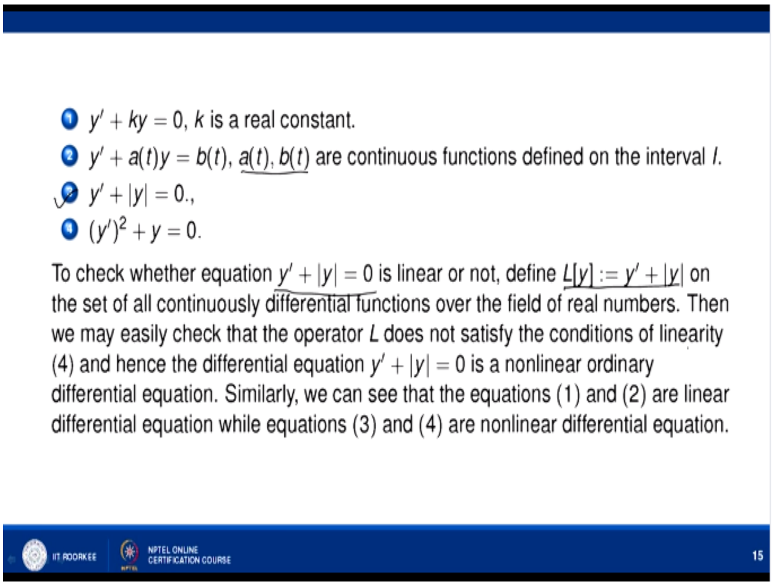

So let us look at the following problems, y dash+ky=0 where k is a real constant. Second problem is y dash+a t y=b t, here a t and b t are continuous function defined on some interval I. Third one is y dash+mod of y=0 and 4th one is y dash square+y=0. So we will do this classification or checking whether it is linear or nonlinear, we take one example. Let us take this example y dash+mod of y=0.

And you look at the working of this example and similarly you can check the remaining problems. So to check whether equation this y dash+mod of y=0 is linear or not we define L y as y dash+mod of y. So here we consider only that part which is depending on y, y dash and all. So here we will consider only that part which involves the variable y and just look at L y is defined as y dash+mod of y, now we want to check whether it is linear or not.

And it is defined on the set of all continuously differential functions over the field of real numbers. Then, we may easily check that the operator L does not satisfy the condition of linearity and how we can check.

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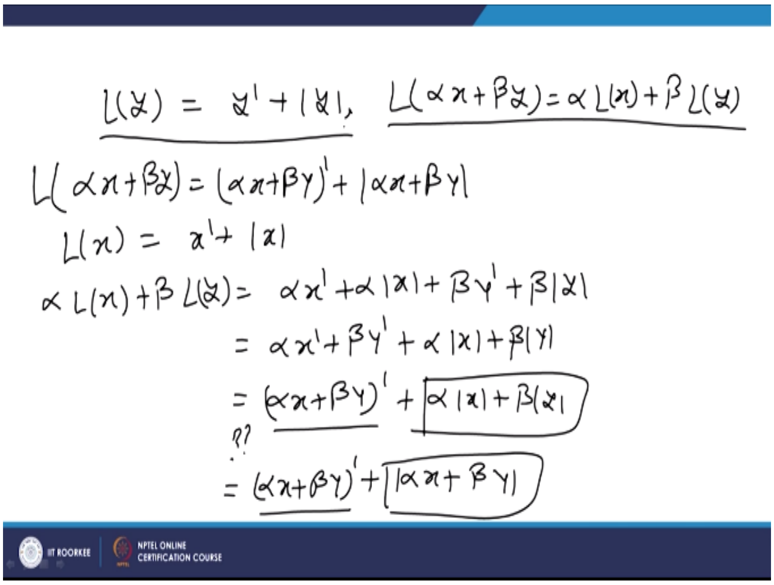

Let us look at here so here we have this L y is defined as y dash+ mod of y right. So the condition which we want to check is this that L alpha x+beta y=alpha L x+ beta L of y that we want to check. So operator is defined like this, so it means that L, what is alpha x+beta y so this value is given as alpha x+beta y derivative+ modulus of alpha x+beta y right and we can write L x as what L x is x dash+ modulus of x.

Now we want to show that this holds, so we need to see look at alpha times L x+beta times L y and if you look at it is what it is alpha times x dash+alpha times mod of x+ beta times y dash+beta times mod of y and if you simplify it is alpha x dash+beta y dash+alpha times mod of x+beta times of mod of y. So this I can write as alpha x+beta y derivative+alpha times mod of x+beta times mod of y.

And we want to show that we need to show that is = alpha x+beta y dash +modulus of alpha x+beta y. So here if you look at this first term is matching no problem but if you look at this term, this term may not be equal. This can be equal only when this x and y both are positive but this may not be necessary because y is an unknown function, it may be positive, it may not be positive.

So it means that here we see that this is not always true that alpha L x+beta L y=L of alpha x+beta y. So we can say that these 2 are not matching right. So it means that here this LHS is != RHS, so we can say that this operator L which is defined like this is not a linear operator.

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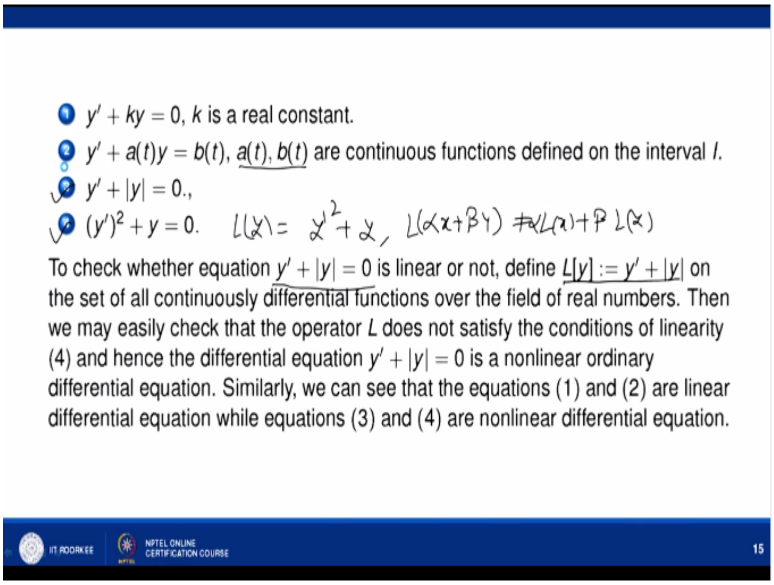

So we can say that this L does not satisfy the condition of linearity and hence the differential equation y dash+mod of y=0 is a nonlinear ordinary differential equation. So here we have done for y dash+mod of y=0. You can check others also. For example, here also you can check in a similar manner. Here in this case you can define L of y as y dash square+y, so you can easily check that alpha of alpha x+beta y is not coming to be alpha times L x+beta times L y right.

So from this observation we can say that this 3 and 4 are nonlinear ordinary differential equation but 1 and 2 are linear differential equation. So we have discussed the classification based on dependent variable and here we have checked that 1 and 2 are linear differential equation while this 3 and 4 are nonlinear ordinary differential equation. Why this classification is important and why we have discussed this classification.

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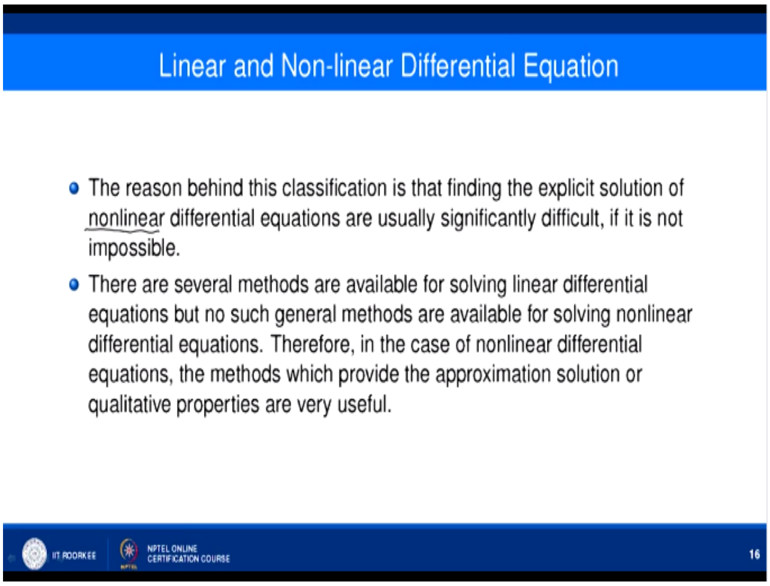

The reason behind this classification is that the finding the explicit solution of nonlinear differential equation. These nonlinear differential equations are usually very, very difficult if it is not impossible. Many times it is very, very difficult to find out the solution in explicit form. So we need to know before proceeding further whether we are dealing with linear problem or a nonlinear problem.

And there are several methods available for solving linear differential equation but no such general methods are available for solving nonlinear differential equation. So given a nonlinear differential equation we have to deal according to the nonlinearity present in the ordinary differential equation but in case of linear ordinary differential equation we have several methods available and we will try to discuss some of them.

So therefore in the case of nonlinear differential equation the method which provide the approximation solution are qualitative properties are very, very useful because we are not able to find out all the times the exact solution of nonlinear differential equation and nonlinear ordinary differential equation.

In that case, we try to discuss some other method for example approximation solution or the qualitative properties we want to know rather than finding the exact solution we try to find out the qualitative properties of the solution of the ordinary differential equation. So here we stop and in today's lecture what we have seen is that what is differential equation, how we can classify of differential equation in terms of number of independent variables and depending on the dependent function involved in the differential equation.

So thank you for listening us. Thank you.