Numerical Methods: Finite Difference Approach
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Lecture – 19
Lax-Wendroff's method

Welcome to the lecture series on Numerical Methods Finite Difference Approach. In the present lecture we will discuss about hyperbolic equations of first order and for this we will just consider like Lax-Wendroff's method here and then we will just consider one example followed by this method.

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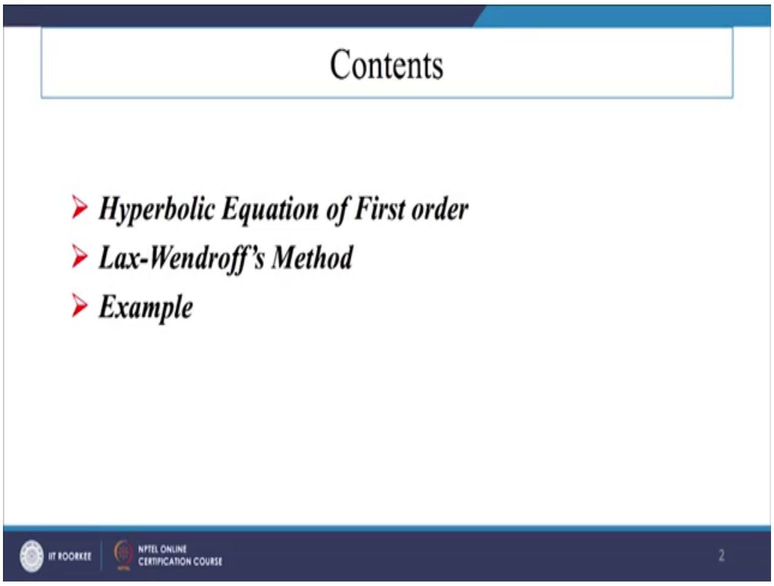

So, if you will just go for this hyperbolic equations of first order.

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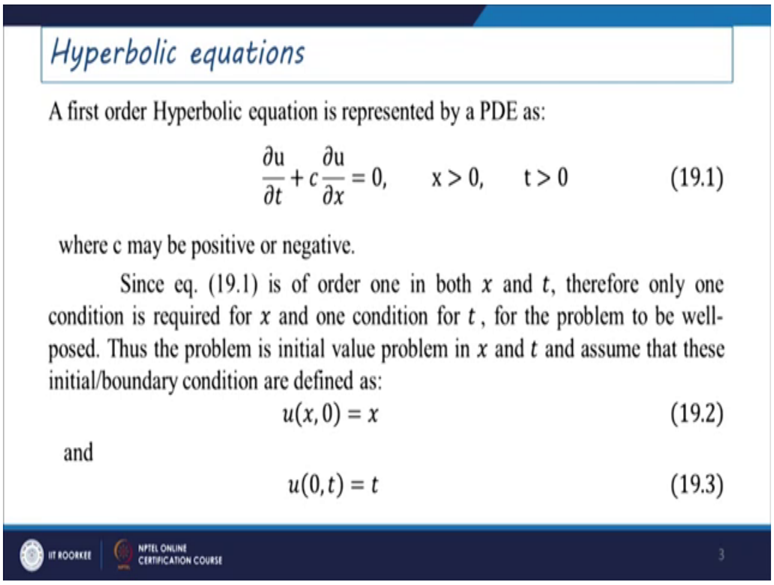

So, this equation can be written as like del u by del t plus c del u by del x this equals to 0 will have like x greater than 0 and t greater than 0, where c may be positive or negative since a always it is just to depends on this physical phenomenon that how c can behave.

And if you will just see this equation del u by del t plus c del u by del x here. So, if you will just see both the terms here they are of first order here in both x and t, therefore one condition is required for x and one condition it is required for t there for the problem to be well posed. And if you will just consider this one as a initial value problem in x and t we have to assume that we will have a initial condition and we will have a boundary condition there. So, this initial condition is prescribed as like u of x 0 equals to x here and this boundary condition that is prescribed as u of 0 t equals to t here.

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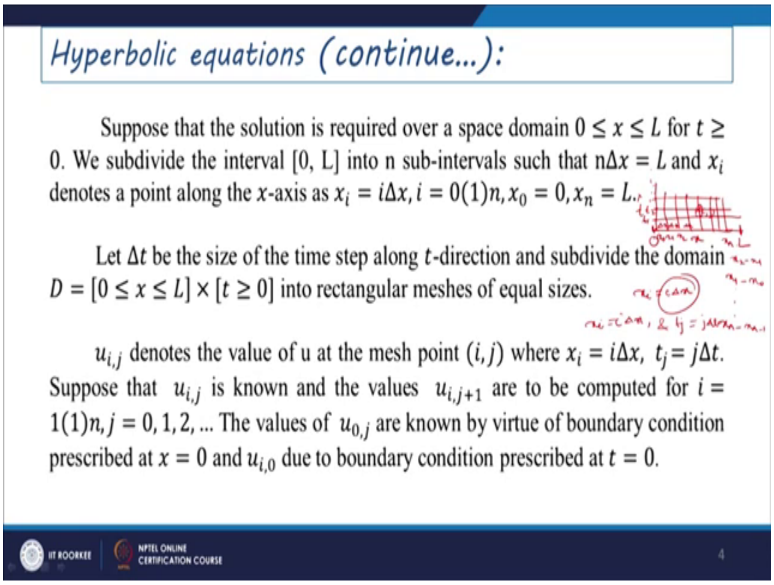

Suppose the solution is required over a space domain where the space is varying like in the x axis if you will just see here, this coordinate system if you will just consider here. So, x is the domain, where x is a just lying between 0 to L here and t is just increasing in y axis here.

And if you will just subdivide this domain in the x range here that can be divided in like a n number of points suppose here. So, then we can just write each of these grid points as like x 0 x 1 x 2 up to x n here and the distance between two grid points we can just write this one as x 2 minus x 1 or we can just write as x 1 minus x 0 or the last point we can just write as x n minus x n minus 1 here. Hence at a particular point x i we can just define this point as existing at a distance of i del x from the initial position.

Since we are just considering here i is varying from 0 one two up to n there and x 0 is the starting point 0 itself there hence if you will just consider here as the small grid spaces has a del x del x del x then after like I grid points we can just consider the total distance as I del x there. Suppose in the time direction if you will just consider here del t be the small time steps we can just follow up in the time direction then we can just a subdivide the domain in the y direction into the meshes of equal size as del t there.

So, then we will have t greater than 0 we can just consider as this points as like t 1 t 2, likewise we can just move it about there. So, then we will have a variation in the x direction as a del x and we will have a variation del t in the t direction there and at a particular level suppose at a grid point like i j here where I denotes here as the x coordinate and z denotes the time level here we can just write this coordinate as x i as I del x and j as here as t j equals to j del t.

Since in each direction we can just say that we will have a small increment in x direction as a del x and we will have a small increment del t in the t direction here suppose uj denotes the value of u at the mesh point i j suppose here if you will just consider where you will have like x i and t j are the coordinates and which are existing at a distance like in the x position as I del x and from the j direction as the distance as j del t there.

And if you will just go for this computation at a level that is suppose u i j is known to us and we will just compute this value at the next time step level like u i j plus 1 there. So, then we will just vary for that particular level like variation of i from 0 1 to up to n points. And obviously, we can just say that since the boundary value it is known to us like u of 0 j are known by virtue of boundary condition at x equals to 0 and u i 0 is the boundary condition for prescribed t equals to 0 initially we can just consider that level.

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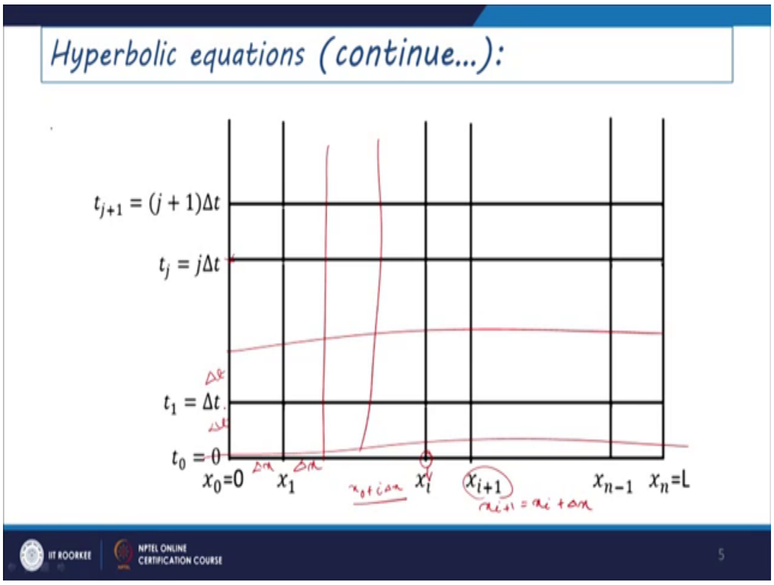

So, then we can just define this domain as in the form of like since you will have this total length that is a x is varying from 0 to L here and this small grid basis that is existing at a distance of for del x del x and at ith coordinate we are just signified as the point as x i here which is existing at a distance of like x 0 plus i del x distance. And we can just consider as this point as like x i plus 1 that is the immediate next point of x i, so that is why you can just write this one as x i plus 1 equals to x i plus del x here.

Similarly, in the j direction if you will just see here. So, that initial increment is just in each of these space grids are considered as del t here. So, del t. So, likewise if you will just consider at the jth position we can just get the total distance as j del t since initial time step we are just considering as like t 0 goes to 0. So, the total time it is just continued at jth step is j del t there itself.

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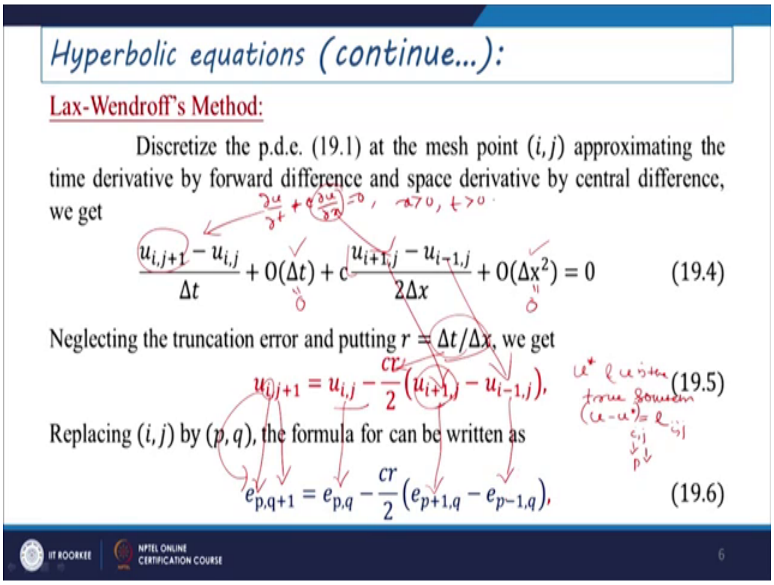

So, if you will just to discretize this first order partial differential equation which is defined as like our first order hyperbolic equation as given in a equation 19.1 here if you will just approximate this time derivative by forward difference and the space derivative by central difference scheme here. So, that is especially it is just written as like del u by del t this is nothing, but plus c del u by del x this equals to 0 here for x greater than 0 and t greater than 0 here.

So, if you will just use like this forward difference scheme for time approach here or the time dependent term here we can just write this one as like u i j plus 1 minus u i j by del t plus order of a del t there plus c into if you will just use the central difference scheme for space derivatives here it can be represented as c u i plus 1 j minus u i minus 1 j by 2 del x plus order of a del x square this equals to 0 here. And if you will just neglect this like truncation errors so, for the time like it is a for first order term here for the space we are just neglecting the second order terms here then we can just write this equation in the form u i j plus 1 this equals to u i j minus cr by 2 into u i plus 1 j minus u i minus 1 j where r is defined as del t by del x. Obviously, if you will just see if we are just considering this term as 0 here, this term equals to 0 here, then we will have these coefficients like u i j plus 1 it can be written as like del t can be taken to here itself. So, we can just write c del t by 2 del x and the coefficient del t will be like u i j; u i plus 1 j minus u i minus 1 j plus u i j there itself.

So, that is why it is just written as like u i j minus cr by 2 since r is defined as del t by del x here, and u i plus 1 j minus u i minus 1 j. And if you will just replace this i j coordinate by p q to obtain this error terms as we have defined in our earlier lectures that is like for i we are just replacing as p here, and j is replaced by q here and if u star is the approximated solution and u is the true solution here then the difference between u and u star it is just considered as the e term here and at each of this grid points if you are just defining u minus u star at the point i j as e i j here then especially we are just replacing here i by p and j by q here. So, that is why your u minus u star that is the true solution minus the approximate solution which can be defined as e at the point like p and q plus 1 where i is replaced by p and j is replaced by q here.

Similarly, u i j minus u star i j which can be replaced as e of p q here minus cr by 2 and u i plus 1 j minus u star i plus 1 j which can be written as e of p plus 1 q term here and similarly if you will just replace here u of i minus 1 j minus u star of i minus 1 j and we can just denote this difference as e of p minus 1 q here.

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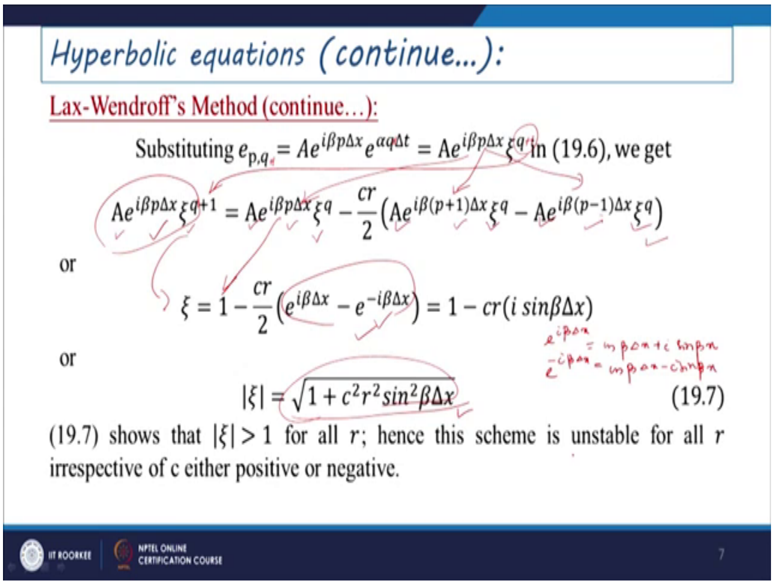

So, if you will just substitute our earlier formation that is just defined in a exponential form that is e of p q equals to Ae to the power i beta p delta x into e to the power alpha q del t this equals to e to the power i beta p delta x and zeta to the power q then we can just obtain this equation is Ae to the power if you will just see here this equation here that is nothing, but e p q plus 1 here. So, that is why we are just writing e p q plus 1 here.

So, if q is replaced by q plus 1 we can just replace here q plus 1 we can just replace here as q plus 1. So, that is why we are just writing this one as like Ae to the power i beta p delta x zeta to the power q plus 1 which is nothing, but this term here. Similarly we are just writing our term like e p q term e p q directly we can just write this term here only similarly we will have like one term which is as p plus 1 q term and p minus 1 q term there. So, directly we can just replace here that is Ae i to the power beta p plus 1 delta x zeta to the power q term and similarly for a p minus 1 we can just write Ae to the power i beta p minus 1 delta x into zeta to the power q here.

And if you will just cancel this terms that as just taken as like if you will just see here A is common at each of these terms here and e to the power i beta p delta x it is also common in each of these terms here and then if you will just see here that zeta to the power q it is also common to each of these terms here. Hence if you will just neglect or we can just cancel it out both the sides here the terms containing like Ae to the power i beta p delta x zeta to the power q then we can just obtain this reduced form as since we are just taking common at both the sides and we are just cancelling this total term then we will have zeta here then we will have one here then minus cr by 2 and 1 surplus term it is just existing as e to the power i beta into delta x term since a p delta x it has been taken common minus e to the power minus I beta delta x it is just present it over there.

And if you will just write this terms like e to the power i beta delta x plus e to the power minus i beta delta x by 2 i it can be represented as a sin beta delta x there. So, that is why 2 is there itself present. So, we can just write it as 1 minus c or i sin beta delta x here. Since a especially if you will just express like e to the power i beta delta x it can be expressed as like cos beta delta x plus i sin beta delta x. Similarly if you will just write e to the power minus i beta delta x it can be written as a cos beta delta x minus i sin beta delta x you can just subtract these two terms then you can just find this expression there.

So, if you will just write absolute value of zeta which can be written as like a square root of one plus c square r square sin square beta delta x here. And this equation like 19.7 shows that if you will just consider like absolute value of zeta which is a greater than one if you will just see this square root on here. So, especially always it will be greater than one for all r hence this scheme is unstable for all r irrespective of c is whether it is positive or negative there itself.

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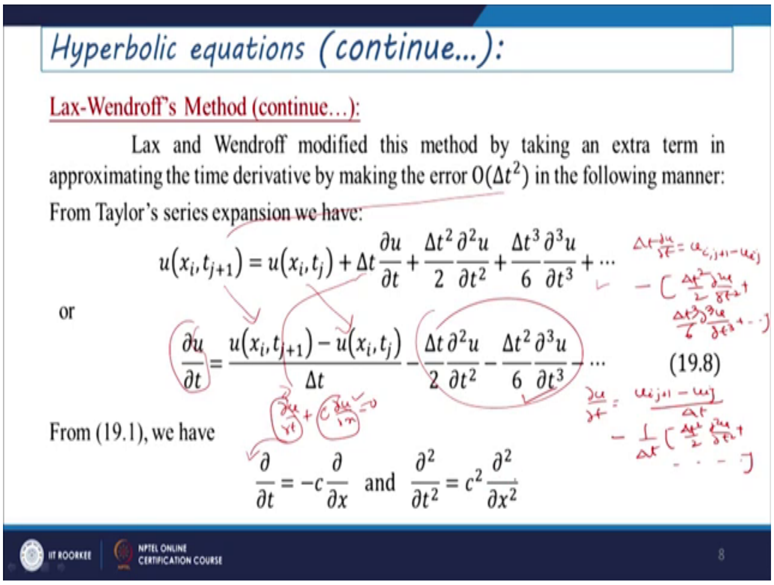

Similarly, if you will just go for like a Lax-Wendroff's method here Lax and Wendroff modified this method by taking an extra term in approximating the time derivative during this Taylor series expansion. And in this Taylor series expansion if you will just consider this higher order terms containing order of del t square is like the expressions u of x i t j plus 1 as u of x i t j plus del t del u by del t plus del t square by 2 del square u by del t square plus delta t q by 6 del q u by delta t q terms here, but we will just consider these terms up to like second order terms for the time derivative here that is in the form of as del u by del t as u of x i t j plus 1 minus u of x i t j by del t it has been divided it off here.

Since directly we can just write this one as del t del u by del t this is nothing, but u of i j plus 1 minus u i j minus all other terms containing like del t square by 2 del square u by del t square plus del t cube by 6 del q u by del t q plus all other terms there. And directly we are just writing here del u by del t is nothing but just to the division here, that is nothing but del u by del t this can be written as u i j plus 1 minus u i j divided by del t minus 1 by del t into del t square by 2 del square u by del t square plus all other higher order terms containing del t.

And if you will just divide del t term with these terms here then we can just get the rest of the terms here, that is as minus del t by 2 del square u by del t square again if you will just consider the immediate next term here that is in the form of like del t q. So, it will be del t square by 6 del q u by del t q. And if you will just use this approach since del by del t is the operator which is just operating on this variable u here we can just write it in a separate form as a del by del t equals to minus c del by del x from the original equation especially the original equation it is just written in the form of a del u by del t plus c del u by del x this equals to 0.

Since we if you will just see here del by del t is the operator and c del u by del x c is the constant here. So, del by del x is the operator which is operating on the variable u there itself. So, you can just write these terms as the operator is del by del t equals to minus c del y del x that is nothing but the operators or the functions we can just say. So, once more if the operator will be operated there itself we can just write that is like del square by del t square this equals to c square del square by del x square.

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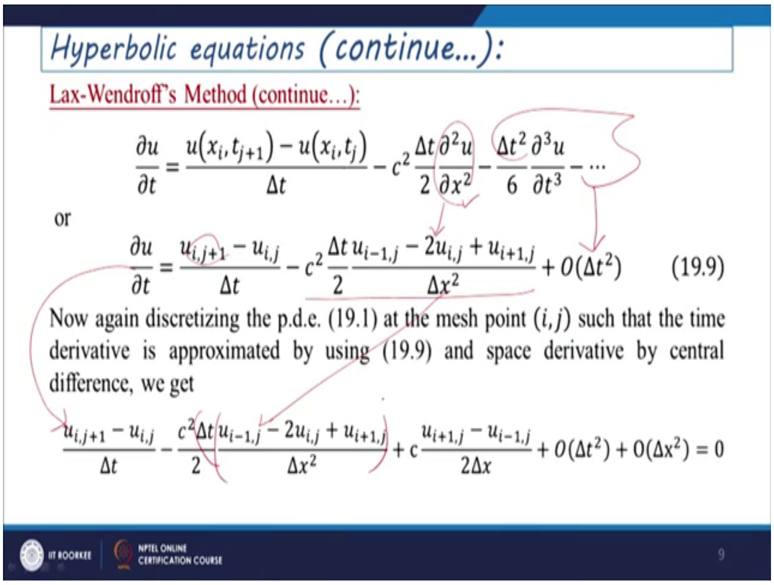

And if you will just put these terms then we can just often the series expansion as a del u by del t equals to u x i t j plus 1 minus u of x i t j by del t minus c square del t by 2 del square u by del x square minus del t square by 6 del q u by del t q minus the all other terms. And if you will just see here that terms containing like del e del by del t here or del square by del t square which can be replaced by the terms containing x here. So, that is why we can just have this expansion as del u by del t equals to u i j plus 1 since you in the point or the position vector form we are just writing here. So, u i j plus 1 u i j by del t minus c square del t by 2.

So, if you will just go for this space derivatives here in a central difference approximation we can just write it as u i minus 1 j minus 2 u i j plus u i plus 1 j by del x square plus the terms containing like delta t square and higher order terms are neglected here or we can just consider these terms are as the reminder term.

Now, again if you will just to discretize these partial differential equations, so we can just obtain this series expansion is u i j plus 1 since a del u by del t especially if you will just see here as u i j plus 1 minus u i j by del t minus if you will just consider here this term here as minus c square by two del t u i minus 1 j. So, u i minus 1 j minus 2 u i j plus u i plus 1 j by del x square here plus c terms containing like u i plus 1 j minus u i minus 1 j by 2 del x here plus order of a del t square plus order of a del x square this equals to 0. So, that is why we are just obtaining this modified formulation for this method.

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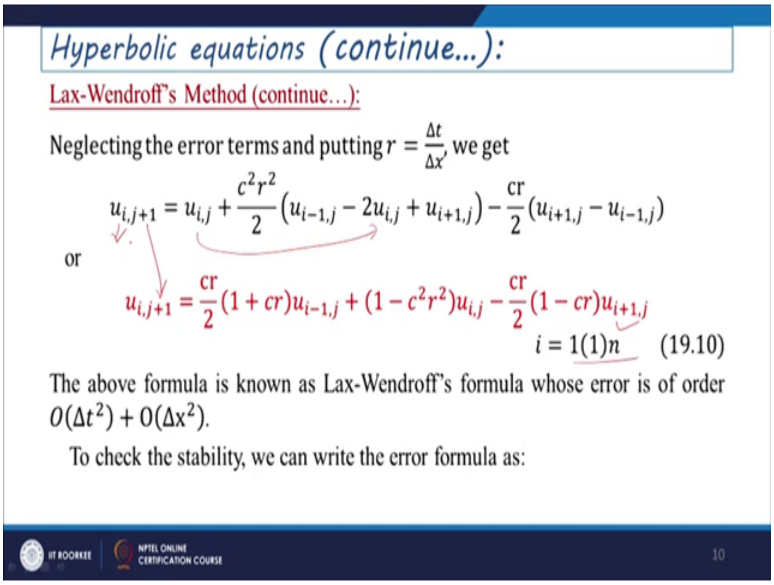

And if you will just neglect this error terms by putting r equals to del t by del x here we can just obtain this expansion as u i j plus 1 this equals to u i j plus c square r square by 2 into u i minus 1 j minus 2 u i j plus u i plus 1 j minus cr by 2 u i plus 1 j minus u i minus 1 j. And since we want to calculate this terms in the next time step so that is why we just try to separate this term j plus 1 as in the left hand side and the remaining terms in the right hand side as in the form here cr by 2 since u i j is also appearing in the middle of this a terms here. So, that is why it can be written as cr by 2 one plus cr u i minus 1 j plus 1 minus c square r square u i j minus cr by 2 into 1 minus cr u i plus 1 j here, where i is especially since a I equals to 0 it can be taken as this like boundary condition. So, afterwards we will just start this computation. So, that is why i is varying from 1 to n point there.

And this formula is known as Lax-Wendroff's formula and whose error is of in the order of like order of del t square plus order of del x square here. And if you will just go for this stability of this scheme we can just write this terms as e of p q plus 1, since we are just writing this u as the true solution and u star is the approximate solution if you will just take the differences of u minus u star it can be just denoted as e term there.

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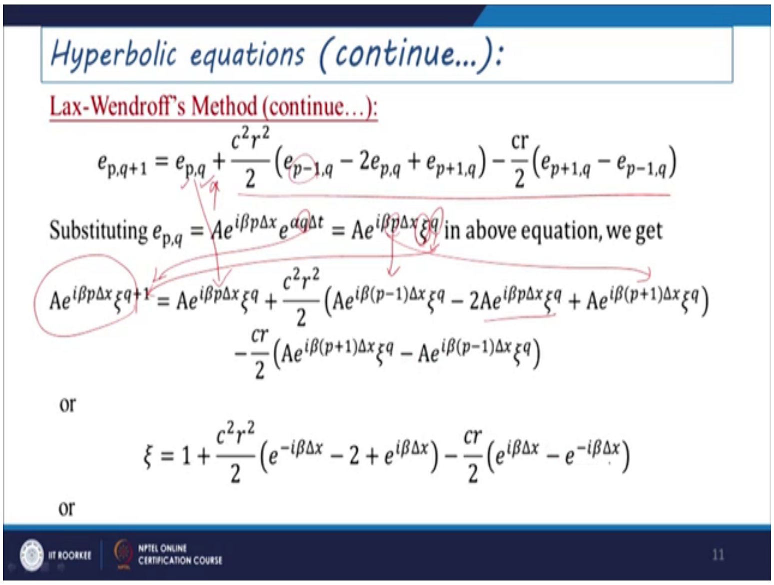

So, each of these coordinate points especially we are just replacing i as the point of p there and j in terms of q there. So, that is why it can be written as e of p q plus 1 here and rest of these terms it is just followed in the coordinate form that is i as p there and j as q there and rest of the terms accordingly it can be just placed. And if you will just put e of p q s in terms of Ae to the power i beta p delta x e to the power alpha q delta t and which is especially expressed in the form of like Ae to the power i beta p delta x and this is just written as e to the power alpha delta t as zeta there. So, that is why it is just written as zeta to the power q in the above equation.

And if you will just put these terms in the equation here then we can just obtain as Ae to the power i beta p delta x and in plus of like q plus 1 here we are just replacing this one as q plus 1 here. So, that is why it is just replaced as q plus 1 term here and this term remains there itself like Ae to the power i beta p delta x zeta to the power q plus c square r square by 2 and p minus 1 it is just replaced here as p minus 1 here. Similarly, p q itself it is the present. So, directly we can just write this term there itself and in terms of a p minus 1 sorry p plus 1 we are just substituting this term as e to the power i beta p plus 1 delta x zeta to the power q here.

So, if we will just go for like the final step of this calculation here since both the sides. So, this part can be cancelled it out. So, that is why zeta is remaining there. So, zeta equals to 1 plus c square r square by 2 into e to the power i beta delta x minus 2 plus e to the power i beta delta x minus cr by 2 and this is e to the power i beta delta x minus e to the power minus i beta delta x.

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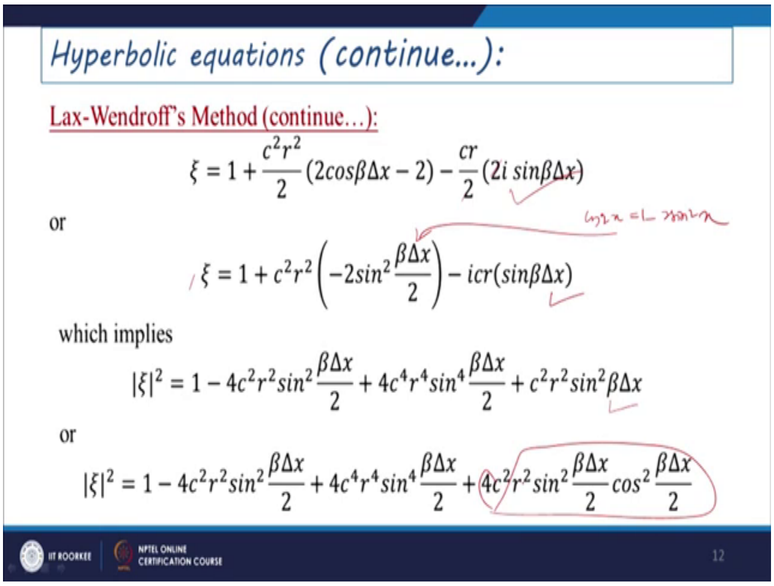

And if you will just go for this a further process of this scheme here, so zeta can be written as like 1 plus c square r square by 2 here as e to the power minus i beta delta x plus e to the power i beta delta x, so that is why this can be replaced as a 2 cos beta delta x minus 2 itself it is just present minus cr by 2 into. So, e to the power i beta delta x minus e to the power minus I beta delta x, so this can be just written as 2 i sin beta delta x here.

So, if you will just take this since we have already known that cos 2 x it can be written as like 1 minus 2 sin square x. So, that is why if you will just put this formulation here then we can just obtain this one as like 2 cos beta delta x minus 2 is minus 2 sin square beta delta x by 2 minus icr since 2, 2 it is just cancel it out. So, sin beta delta x here, which implies that we can just write absolute value of zeta square then we can just obtain this series expansion as 1 minus 4 c square r square sin square beta delta x by 2 plus 4 c 4th, r 4th, sin 4th beta delta x by 2 plus c square r square sin square beta delta x by beta delta x here.

And since we want to go for the stability of the schemes, so you have to find this zeta value so that is why if you will just go for further simplification of this equation here we are just obtaining as 1 minus 4 c square r square sin square beta delta x by 2 plus 4 c to the power 4 r to the power 4 sin to the power 4 beta delta x by 2 plus since we are just writing this one as sin square beta delta x. So, that is why we can just write this one as like 2 beta delta x. So, if you will just write sin 2 x is equal to 2 sin x into cos x. So, in that form we are just expressing this term here.

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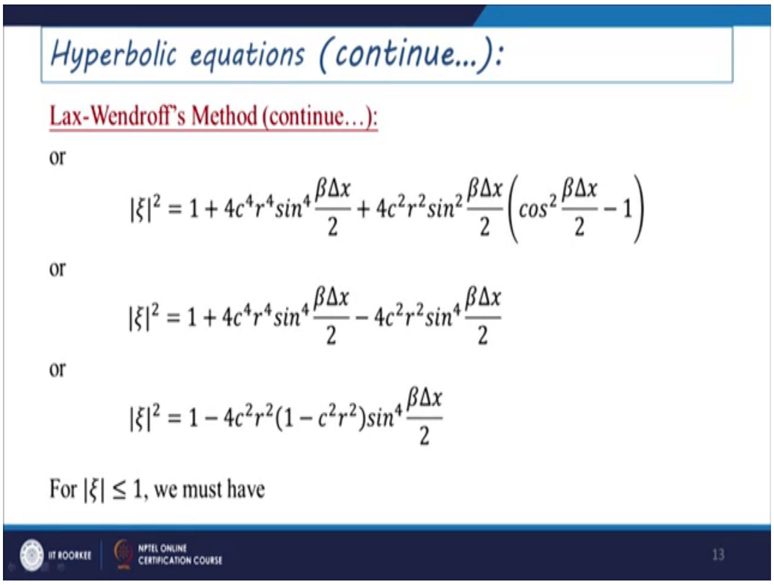

And in the final form this equation is written as like absolute value of zeta square this as 1 plus 4 c 4th, r 4th, sin 4th, beta delta x by 2 plus 4 c square r square sin square beta delta x by 2 into cos square beta delta x by 2 minus 1. Our aim is that if you will just consider like sin x function as the compact function in the right hand side, since sin x is a absolute value is always less or equal to mod x. So, we can just consider this scheme as this table whenever we will have this right hand side it should be less or equal to 1 there.

So, that is why if you will just go for this further process of this equation here we can just write it as 1 minus 4 c square r square into 1 minus c square r square sin to the power 4 beta delta x by 2. So, for like absolute value of zeta if you we will just assume that it should be less or equal to 1 this means that we should have like 4 c square r square into 1 minus c square r square it should be less than 1. And for this if you will just consider like left inequality here. So, you can just write this one as c square r square into 1 minus c square r square this should be greater or equal to 0 this implies that c square r square it should be always less or equal to 1.

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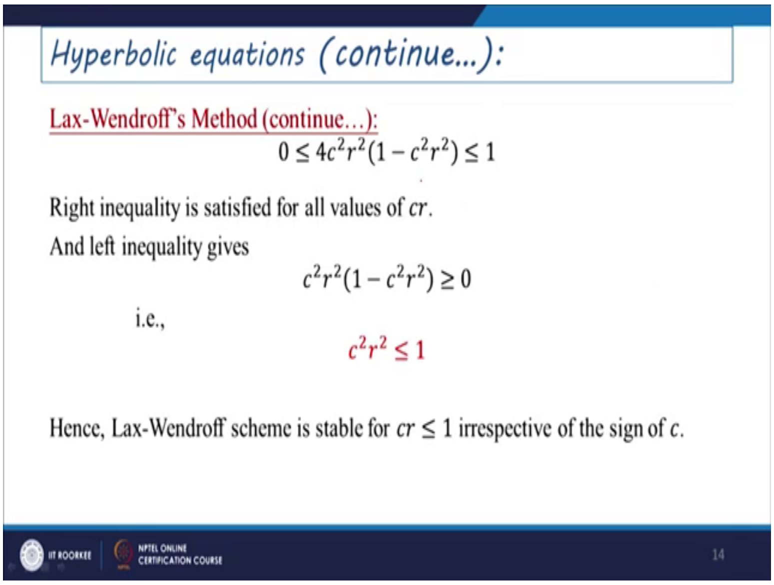

So, we will just depend the values of c and r to get the stability of this Lax-Wendroff's scheme here irrespective of this like sin of any c or r value there. So, to get the solution of using this Lax-Wendroff's schemes here we are just considering example as del u by del t plus del u by del x equals to 0 with the boundary conditions as u of x 0 equals to x and u of 0 t equals to t.

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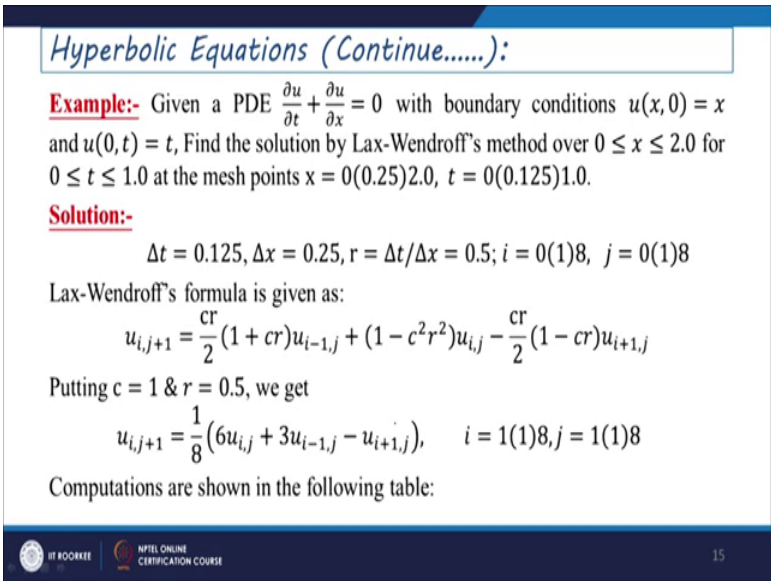

Here and if the question is you ask you to find the solution using this scheme over like x from 0 to 2.0 and t is incrementing from 0 to 1.0 with the mesh sizes as like x equals to 0

0.25, increments of 2 last point as like x n equals to 2.0 and time increments as like 0.125 starting from 0 to 1.0 then we can just proceed as like del t equals to 0.125 since it is just given in the question here and del x just it is given as 0.25 here. So, you can just define r as del t by del x which is nothing, but 0 point one two 5 by 0.25. So, that is why this value is 0.5 here and i is varying from like 0 to 8 and j is varying from 0 to 8 here, since if you will just see here our increments are of like 0.25 here. So, up to 1 we will have 4 steps and up to 2 we will have like another 4 steps, so total is 8 steps.

And if you will just use this Lax-Wendroff's formula here, then this formula especially it is just written as in the form of u i j plus 1 that is as cr by 2 into 1 plus cr u i minus 1 j plus 1 minus c square r square u i j minus cr by 2 into 1 minus cr u i plus 1 j here. So, first if you will just see here, c is taken as 1 here since in this equation if you will just see del u by del t plus c del u by del x this is 0 so obviously, c equals to 1 and r is considered as 0.5 here. So, directly we can just put c and r value. So, this equation will be reduced as in the form like 1 by 8 6 c y j plus 3 u i minus 1 j minus u i plus 1 j here.

Since a u i j it is just occurring there itself, so u i plus 1 j and u i minus 1 j here. So, if you will just proceed like i is varying from 1 to 8 and j is varying from 1 to 8, we can just get it as since our initial values it is just given as the time increments as 0, 0.125, 0.250 up to 1.0 here and x is just varying as incremented as 0.25, 0.25 plus 0.255 then 0.5 plus 0.25 this is 0.75. So, it is just moving up to 2.0 here.

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Hyperbolic Equations (Continue......):

j \ i =		0	1	2	3	4	5	6	7	8
	t \ x	0.0	0.25	0.50	0.75	1.0	1.25	1.5	1.75	2.0
0	0.00	0.000	0.250	0.500	0.7500	1.000	1.2500	1.5000	1.7500	2.000
1	0.125	0.125	0.1250	0.375	0.6250	0.8750	1.1250	1.3750	1.6250	1.8750
2	0.250	0.250	0.0938	0.2500	0.5000	0.7500	1.000	1.2500	1.5000	1.7500
3	0.375	0.375	0.1328	0.1602	0.3750	0.6250	0.8750	1.1250	1.3750	1.6250
4	0.500	0.500	0.2202	0.1231	0.2632	0.5000	0.7500	1.000	1.2500	1.500
5	0.625	0.625	0.3373	0.1420	0.1811	0.3800	0.6250	0.8750	1.1250	1.3750
6	0.750	0.750	0.4696	0.2104	0.1416	0.2748	0.5012	0.7500	1.000	1.2500
7	0.875	0.875	0.6072	0.3162	0.1508	0.1966	0.3852	0.6254	0.8750	1.1250
8	1.000	1.000	0.7440	0.4460	0.2071	0.1558	0.2844	0.5041	0.7502	1.002

And at the first step if you will just see here. So, the coordinates if you will see here that is as u of 0 t that is just a given as a t here and u of x 0 that is just given as x here. So, at t equals to 0 if you will just see here, t equals to 0 we will have these values here like 0.000, then 0.250, then 0.500, then 0.7500. So, all these value just it is taken from this values there and afterwards we can just follow up this method and we can just obtain the solutions as it is just described here.

Thank you for listen this lecture.