Numerical Methods: Finite Difference Approach
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Lecture – 14
Solution of elliptic equations by SOR method

Welcome to the lecture series on numerical methods finite difference approach. In the last lecture we have discussed this elliptic equation, that is like classical elliptic equations as poisson equation and like Laplace equation, and we have adapted like a gauss iteration method or to find the solutions and in gauss iterated methods especially we are just going for like several iterations to get this corrected values there.

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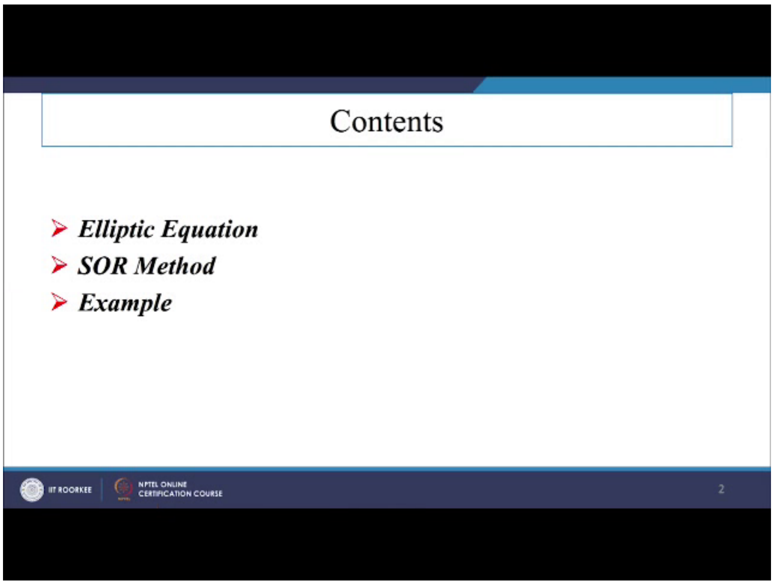

And in the present lecture we will discuss about this a SOR method, followed with some of the examples; and if you will just go for this like SOR method here.

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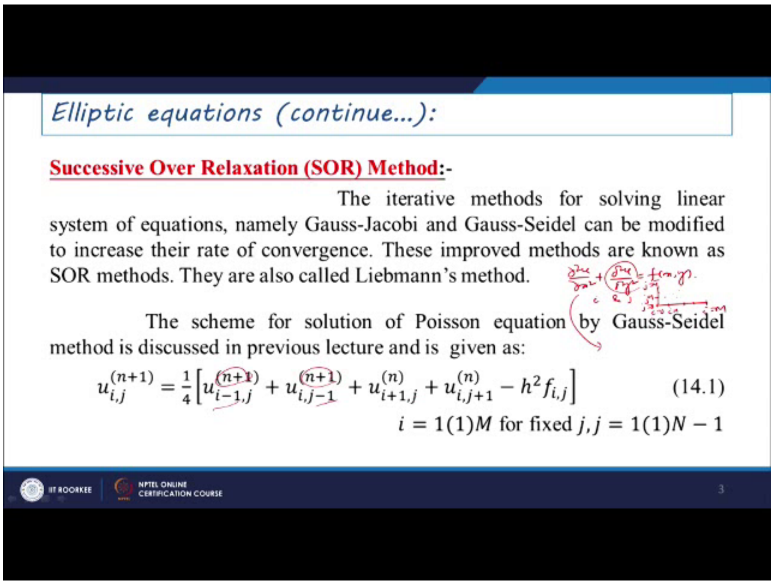

So, specifically if you will just say here the iterative methods for solving linear system of equations namely like Gauss-Jacobi and Gauss-Seidel method can be modified to increase their rate of convergence, and these improved methods are known as SOR methods. They are also called sometimes as Liebmann's method, and that they scheme for solution of poisson equation by Gauss-Siedel method is discussed in the previous lecture, where we have used like different iterative procedures to obtain the solutions and the different boundary condition treatments there itself.

So, if you will just use the same Gauss-Seidel method here. So, then this equation which is represented in the form of like del square u by del x square plus del square u by del y square, which is written as a f of xy and if we are just denoting this grid points are in the form of like x coordinate as the I grid points and the y coordinates as the j coordinates there.

So, then we are just signifying all this grid points are in the form of like here as i is varying in the x axis which is just a denoted as i equals to 0, i equals to 1to opt i equals to m here and j is varying from j equals to 0 to j equals to 1 to up to j equals to n here, and especially in each of these coordinate points if you will just write that are in the form of like ij there. So, that is why we are just signifying, this central difference scheme as in the form of like ui minus 1 j minus to uij plus ui plus 1 j by del x square plus if you will just go for this discretization, here it can be written in the form of like uij minus 1 minus

2 uij, plus uij plus 1 by del y square this equals to fij and which in turn can be written in the form of like uij to the power n plus 1, this equals to 1 by 4 ui minus 1 j whole to the power n plus 1, plus u ij minus 1 and plus 1 ui plus 1 j whole power n plus uij plus 1 n minus h square fij.

And already in the last lecture I have discussed that why we are just considering this i minus 1 and j minus 1 in the powers of n plus 1 and this I plus 1 and j plus 1 as power of m there itself since this values like i minus 1 and j minus 1 it can be computed from this previous steps which can be updated in the next iterated level there. Where itself we can just consider i is varying from one to m here and like j is varying from 1 to n minus 1 if we will have this a boundary condition like a derivative boundary condition along this a boundary i equals to like last boundary m there itself.

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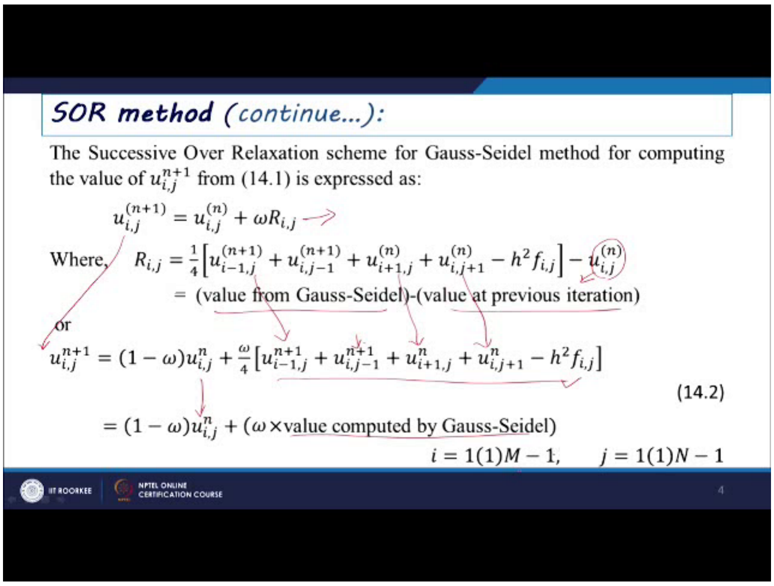

And if you will just go for this a successive over relaxation scheme for a Gauss-Seidel method for computing the value of uij to the power n plus 1 from this previous equation which is expressed in the form of equation 14.1 here that can be written in the form of uij to the power n plus 1 this equals to uij to the power n plus omega rij here.

Where rij is nothing, but if you will just see here this can be written as like a previous iterated scheme whatever we have just written that is in the form uij to the power n plus 1 minus uij to the power n here. So, this means that the value obtained from this Gauss-Seidel iterated method, minus value from this previous iteration. So, which can be written as the residual here or rij especially we can just say and if you will just put this rij value in the first expression here, then we can just write this modified expression as uij to the power n plus 1 yes since a ij is present there itself also here. So, that is why uij can be taken common. So, 1 minus omega into uij to the power m plus omega y 4, ui minus 1 j to the power n plus 1 uij minus 1 over n plus 1; so ui plus 1 j 2 the power n here then uij plus 1 whole to the power n minus h square ij here.

And in a combined form if you will just write here then it can be written as 1 minus omega uij to the power n plus omega into value, if you will just see this value which is computed by only this Gauss-Seidel iteration method. So, that is why it is just written value computed by Gauss-Siedel method there itself and if this condition is not prescribed as the normal derivative condition or the derivative condition along the boundary. So, you can just vary I this values i from 1 to m minus 1 and j is varying from 1 to n minus 1 there itself. And specifically if the derivative bounder condition if it is there then we have to vary this value i is from 1 to m there itself.

So, how we can just choose this omega value, that specifically depends on like a some scheme here.

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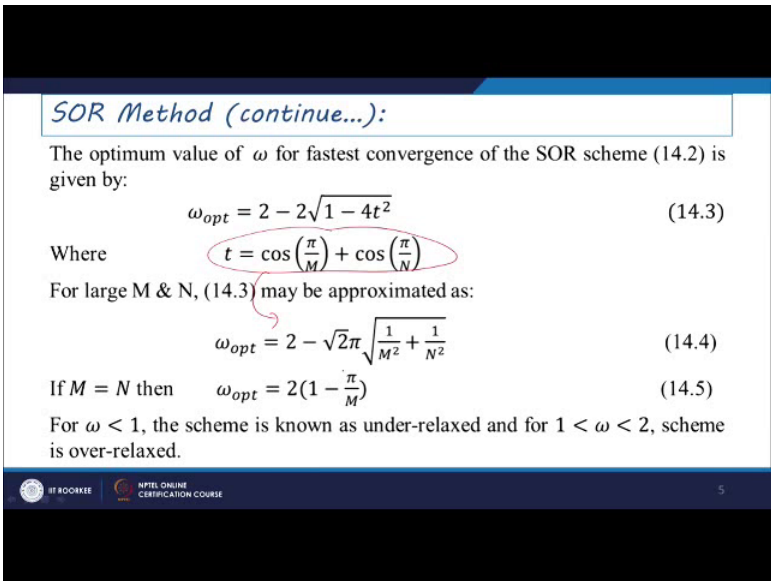

So, especially this scheme I have just written it in the direct form, since there are some specified theorems for this determination of omega optional values. So, this opted value if you can just see here that omega opted as like 2 minus 2square root of 1 minus 4 t square, and which can be obtained from this a Eigen value calculation especially this t value, and we have just written it in a direct form as t goes to cos of pi by M plus cos of pi by M and for large m and n. So, especially this term will be like variety as here 2 minus square root of 2 pi square root of 1 by m square plus 1 by n square and if we will just consider this grid spaces or they say grid sizes are equal and this total length is also equal m equals to n is the total grid a sizes.

So, then we can just write this one as one 2 y m square here. So, 2 by m square means square root 2 by m here. So, square root 2 into 2 so, that is why it is just giving you two pi by M and if I will you just a common here 2. So, I can just write 1 minus pi by M here. So, we are just opting here two different conditions that is a successive under relaxation and successive over relaxation sometimes whenever we will just go for this higher computations . So, some schemes maybe it will converge within certain ranges like if omega is mentioned like 0 to 1 there itself, and sometimes we can just find this system will converge when omega is lying between 1 to 2 there itself. And if we are just choosing omega less than one here this game is known as under relaxed, and if we will just consider this omega lying between 1 to 2 here scheme is said to be over relaxed.

So, by iterated manner we can just test and we can just say that whether we can just apply this say over relaxation method or the under relaxation method.

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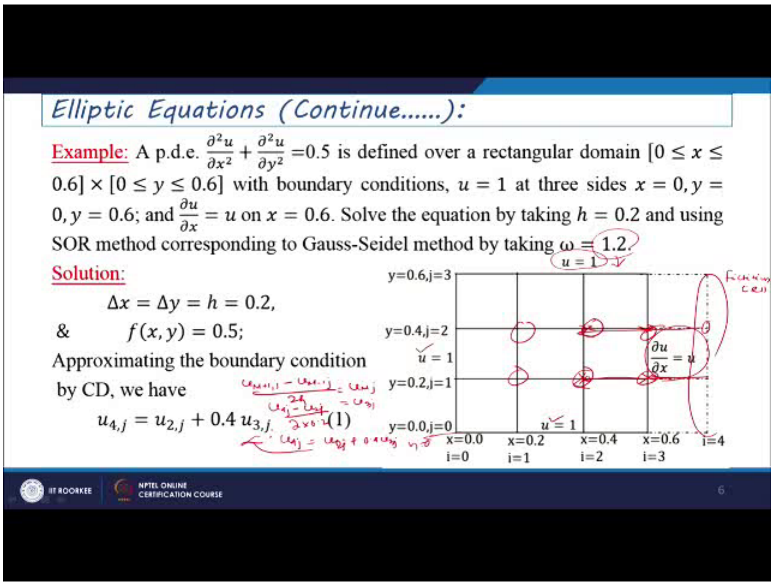

So, even just to go for this a practical example here that we have just considered for like earlier method that is Gauss-Siedel method. So, how these values are improved, we can just check by considering the same example here also. So, if you will just consider the same example here that is in the form of a del square u by del x square plus del square u by del y square this equals to 0.5, which is defined over a rectangular domain x lies between 0 to 0.6 and y is lying between 0 to 0.6 with boundary conditions u equals to 1 at 3 sides, especially that is x equals to 0 y equals to 0 and y equals to 0.6 and del u by del x equals to u on x equals to 0.6 here by considering a is equals to 0.2.

And specifically we will just use here SOR method. So, that is why we need a like relaxation parameter especially this is just provided as over relaxation parameter, which is given as 1.2 here. And if you will just specify this boundary conditions in the grid domain here that can be written as since we have just defined at y equals to 0, u is a specified as 1 here and at x equals to 0, this is also specified u equals to 1 here, and at y equals to n it is just as specified as u equals to1 there, and at the last boundary x equals to like m here or x equals two we can just say that the last boundary i equals to 3 here we will have del u by del x equals to u there .

So, that is why we are just considering this cell as the fictitious cell, where these values will be updated from these two values there itself. When they say j will vary from one to two there itself and i will be vary from like 1 to 3 here, since these are this unknown a nodes there itself. And if you will just go for this like central difference approximation along the last boundary for x equals to 0.6, we can just find they say you for j this equals to u 2 j plus 0.4 you 3 j since a from this point we can just say that that is as a plus 1 j minus 1 j by 2 h this is defined as in the form of umj there.

So, that is why if you will just keep fixed j there itself and m is considered here as a 3. So, that is why we can just write this one as u 3 j, this equals to u 4 j minus u 2 j by 2 h. So, 2 h a specially 2 into h is considered as a 0.2 here. So, that is why this you will just give you 0.4 and u 4 j especially it can be represented as like you 2 j plus 0.4 into u 3 j which is a this statement here.

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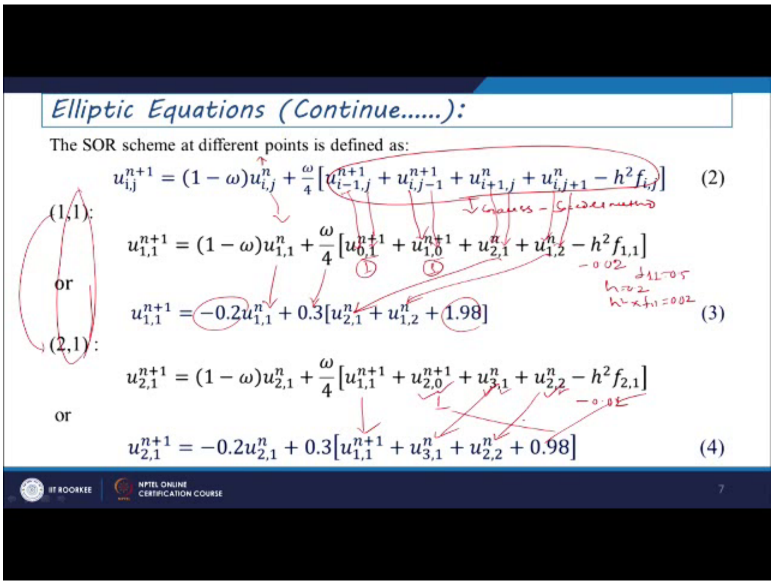

And if you will just go for the further process here the successive over relaxation scheme at different points is defined in the form of this total scheme which is a obtained from this a like a Gauss-Siedel method, and this will be just computed from this previous type of calculation there.

So, that is why uij to the power n plus 1 it can be written as 1 minus omega uij to the power n plus omega by 4 into Gauss-Seidel iterated method value. That is nothing, but ui minus 1 j power n plus 1 plus uij minus 1 n plus 1 plus ui plus 1 j n plus uij plus 1 n minus h square fij. So, if you will just put here i equals to 1 and j equals to 1, then we can just write this one has like 1 minus omega and this is nothing, but ur 1 to the power n here plus t2 omega by 4. So, if I will just put here i equals to 1, then it the value will be reduced to like 0 here. If I will put j equals to 1 here, this can be reduced to as a one there and i remain same here. So, i equals to 1, then j minus 1 this is nothing, but 0 here then i plus 1 that is a 1 plus 1 is 2 here and j equals to 1 then i equals to 1 here then j plus 1, 1 plus 1 this is nothing, but two here. So, minus h square f 11 there.

If you will just see here. So, this 0 1 condition it is known to us 1 0 condition it is known to us. So, that is why this is has just taken the value as 1 here, this has taken the value is 1 here and this can be considered as a minus 0.02 here. Since af 11, it is just a prescribed as a 0.5 and h is given as a 0.2. So, that is why h square equals to you can just write as multiplication of this f 11, which will be nothing, but 0.02 here.

And if you will just put this say 1 minus omega in a compact form here this means that omega value is also specified here omega is given as like 1.2. So, 1.2 this means that 1 minus 1.2 you can just write this one as minus 0.2 here, and this is nothing, but multiplication of e 11 to the for n plus this omega is 1.2 by 4 here. So, that is nothing, but a 0.3 here and into this one u 2 1 and u 1 2, and this is the modified value that is as 1.98 here. And similarly if you will just go for u 2 1 value keeping fixed j if you will just see here j value is fixed there only i is varying here i is varying from 1 to 2.

So, that is why you will just consider i equals to 2 and j equals to 1 here. So, especially you can just find this 1 minus omega then this will be u to 1 and here plus omega by 4 and if I will just put I equal to 2 here this will just give you like 2 minus 1, that is nothing, but 1 here j is one then i is a 2 here then j minus 1 means this can be reduced as a 1 minus 1 as a 0 here.

So, similarly if you will have just put2 i equals to 2 here. So, this will just take you 3 here and j is 1 there then i is a 2, then j plus 1 means 1 plus 1 is two here minus f square h square f to 1 and if you will just put this 2 0 condition here that is nothing, but one. So, one this is nothing, but minus 0.02. So, this just takes the value 0.98 here and rest of this value e 11 is present u 3 1 is present u 22 is present there, and if you will just go oh by fixing like since i will be vary from 1 2 3 and j is fixing there. So, j equals to 1 there itself.

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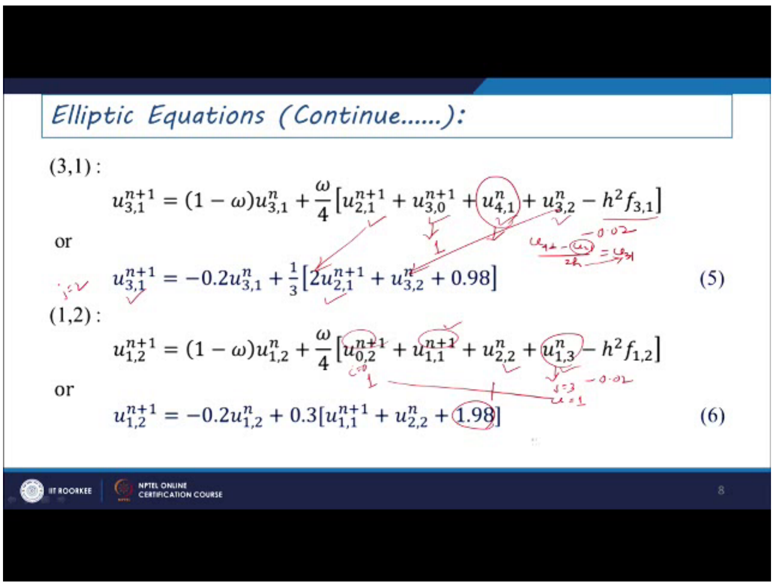

So, further this last equation for i equals to 3, it can be written in the form of u 3 1 to the power n plus 1 this is nothing, but 1 minus omega, u 3 1 to the power n plus omega by 4 u 2 1 to the power n plus 1 plus u 3 0, n plus 1 plus u 4 1 plus u 3 2 minus h square f 3 1 here; if you will just see here. So, these 3 0 value that is nothing, but one there itself and this is nothing, but minus 0.04, but this u 4 1 it can be updated from this values of a 3 1 and 2 1 there itself.

So; obviously, you can just write this one as a u 4 1 minus u 2 1 by 2 h that is nothing, but u 3 1 here. So, that is why you can just multiply the here two h into u 3 1 and with the multiplication of a wi 4 here. So, that can be taken to the left hand side and it can be this a fractional terms which will be multiplied by u 3 1 that can be taken out to the right hand side and it can be multiplied bt itself .

So, that is why this u 2 1 is repeated there. So, that is why they say 2 u 2 1 to the power n plus 1 is there u 3 2 is there and u 4 1 it can be replaced in terms of u 3 1 and u 2 1. Similarly if you will just very like j equals to 2, now here with fixing i equals to 1 2 3 again we can just obtain this sequence as a u 1 2 to the power n plus 1 this equals to 1 minus omega u 1 2 to the power n plus omega y 4 u 0 2 plus u 1 1, which is at n plus oneth level, u 2 2, u 1 3 minus h square f 1 2.

If you will just see here; so u 11 to the power n plus 1. So, this is already calculated here. So, that is why we have just written this one as a u 11 to the power n plus 1 here and a 0 to n plus 1 means i equals to 0 we are just considering. So, this is nothing, but the fixed boundary condition it is defined as a one there and this is nothing, but minus 0.02 there. Another condition if you will just see here 1 3 it is also known to us from the last boundary of j, j equals to 3 it is just to defined as u goes to 1. So, one then it is added with this one minus obstruction of this one. So, that is why it is just giving a 1.98 here and u 1 2 to the power n plus 1, it can be written in the form of minus 0.2, u 12 to the power n plus 0.3 1 to the power n plus 1 plus u 22 to the power n plus 11.98.

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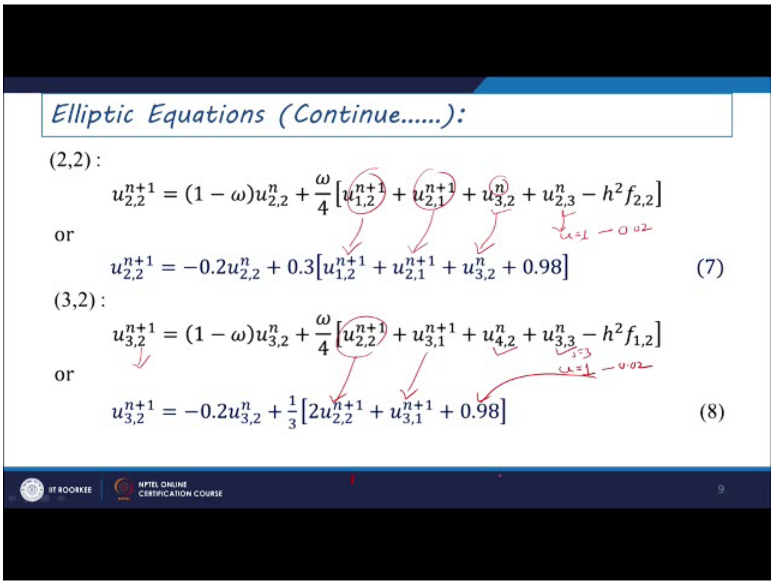

And fixing like j equals to 2. So, we will just go for 2 and 3 there itself. So, for this a 2 point if you will just see here. So, these 3 2 and 2 3 is there. So, 2 3 means this is nothing, but u equals to 1 it will just take, and this is nothing, but minus 0.02. So, 3 2 is drawn on here. So, which is a known to not known to us. So, this would be ah known from this previous cycle of calculations, and u 2 1 this is already computed and you want already it is computed. So, that is why we have just kept it as a u 1 2 here, then u 2 1 here, and u 3 2 here and it can be taken as 1 minus 0.2 02 there, and similarly if you will just go for u 3 2 n plus 1 here keeping a fixed to for i equals to 3 here.

So, we can just obtain this value as one boundary as here 4 2 here and another boundary we can just find it as a u 3 3 here. So, this is nothing, but j equals to 3, we will have u equals to 1 minus this is 0.02. So, it takes the value 0.98 here, but this u 4 2 it can be replaced by earlier condition that is the central difference scheme there. So, it can be written as like u 4 2 minus u of 22 this equals to like two h into u 3 to there.

So, that is why u 3 two can be taken out to the left hand side and it can be separated it.

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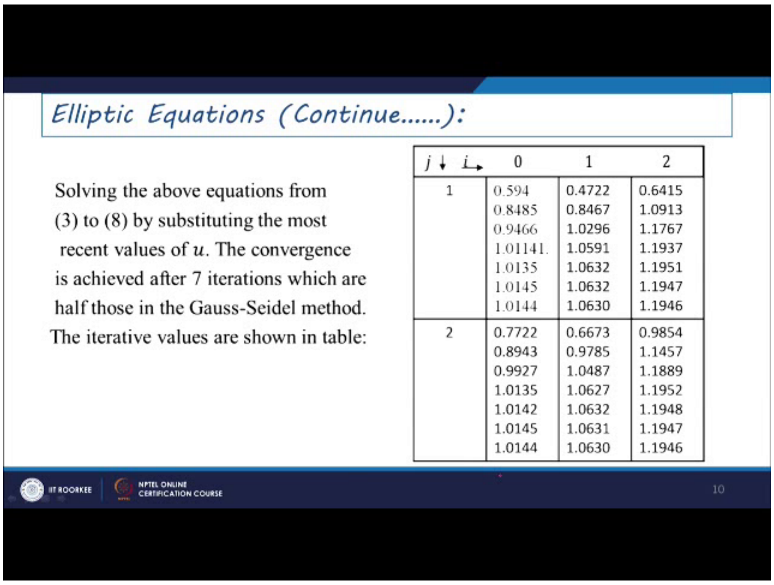

So, if you will just separated out. So, u 22 can be repeated there itself. So, that is why two u 22 it is here plus u 3 1 plus 0.98 here. So, if you will just solve this set of equations 3 to 8 by substituting these most recent values of u the convergence is achieved seven iterations, which are half than those of Gauss-Siedel methods. Since after fourteen steps we use we have got these convergence solutions in a Gauss-Siedel method, but this iterated steps, how to incise whenever we are just to go for this improved Gauss-Seidel method especially called your successive over relaxation method here.

So, the iterative values are shown in the table if you will just see, and we are just considering these differences in each step there and in the final step if you are just seeing the differences here. So, you can just find this order of convergence. So, this is going off to like fourth order here. Then similarly if you will just go for this last step here we can just often these differences are they are just going up to fourth order of there itself also; so in each of these variables here. So, with these say we have just ended up this lecture

Thank you for listen this lecture.