Numerical Methods: Finite Difference Approach
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Lecture - 12
ADI scheme for 2D paraboloc equations

Welcome to the lecture series on numerical methods finite difference approach, and in this lecture series, we have discussed in the last lecture that how we can just solve this two dimensional parabolic equations using explicit method and crank Nicolson method.

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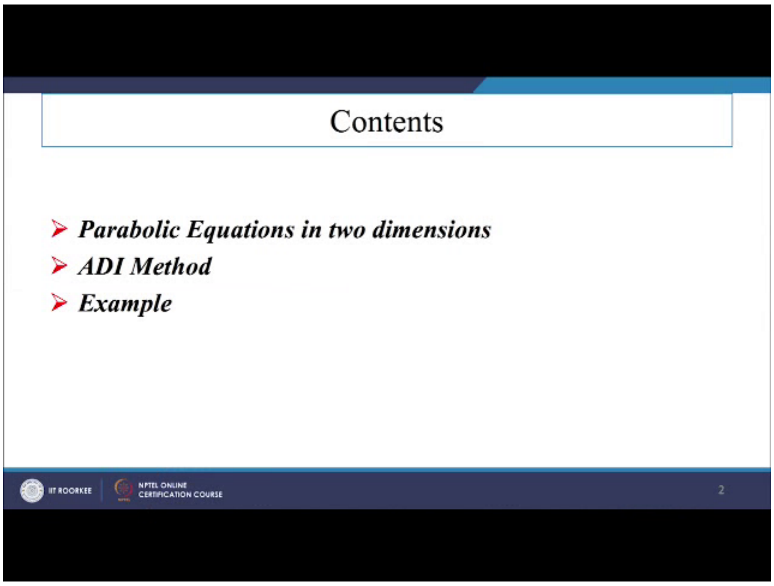

And in this lecture, we will start about these parabolic equations in two dimensions using alternating direction implicit method or ADI scheme.

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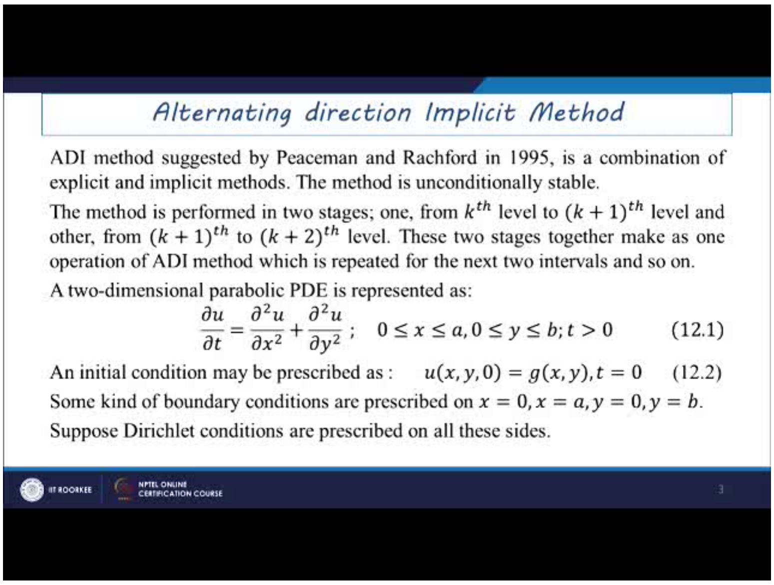

So, specifically if you will just go for this parabolic equation. So, it is written in the form of a like del u by del t equals to del square u by del x square plus del square u by del y square. Where we will have this prescribed boundary conditions at x equals to 0 and x equals to a or the last boundary of x and y equals to 0 and y equals to b; that is the first boundary along the x, sorry y axis and the last boundary along the y axis. And initially we should have to provide like this t equals to 0, all this initial guessed values at the first boundary level, and afterwards we can just use this boundary conditions in the x direction or a y direction keeping t changed, but this boundary conditions would be fixed there.

So, alternating direction implicit method suggested by Peaceman and a Rachford in a 1995. This is a combination of a explicit and implicit methods. So, that is why it is called a semi implicit method. This method is also unconditionally stable; since already we have discussed that this Crank Nicolson scheme is also unconditionally stable. And the method is performed in a two stages; in the first step we are just moving half time step in the x direction, and in the second step, we are just moving half direction in the y direction and afterwards we can just go like half time step in the x direction, again we will just go like a half time step in the y direction.

And in a combined form like a zigzag form we are just obtaining the solution. And finally, in a coupled form we are just obtaining a converge solution. So, these two stages we can just consider for this ADI method in a combined form, if we are just considering two intervals at a time. So, the boundary conditions may be prescribed in the form of like u of x y 0, this equals to z of x y at t equals to 0, and some kind of a boundary condition that can be prescribed at a x equals to 0 and x equals to a, y equals to 0 and y equals to b.

Especially we have discussed that three types of boundary condition, especially we are just applying for this partial differential equations. First one it is the Dirichlet boundary condition, second one is a Neumann boundary condition and a third one is the mixed derivative conditions. Suppose here like Dirichlet conditions are prescribed at all these boundaries. This means that u equals to some u specified values it is just a given along the boundaries. So, alternating direction implicit method or ADI method can be described in the following steps.

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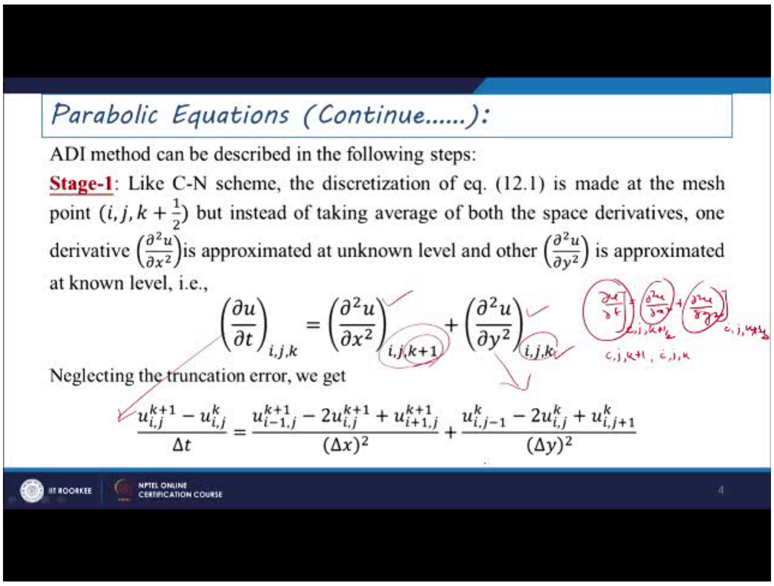

In the first step like Crank Nicolson scheme, the discretization of this equation; that is in the form of like del u by del t, this equals to del square u by del x square plus del square u by del y square. At the mesh point if you will just consider like I say k plus half, and suppose we are just considering like i j k plus half. If we are just using Crank Nicolson's scheme, especially we are just considering this term in the central difference form and this is just takes the coordinates as i j k, and it is just next, sorry k plus 1 and next it is just taking the coordinate as a i j k there, but if we are just considering the space derivative see here. So, then we are just taking the average of this space sizes in the x direction in the k th level, and x direction and y direction derivatives in the k plus oneth level, and average we are just considering to get it in a compiled form.

So, one derivative that is del square u by del x square is approximated at unknown level for this scheme, and other del square u by del y square is approximated at the known level. So, especially if you will just go like Crank Nicolson scheme, there itself you can just find that we are just considering average of this one and this one at a particular time level, and plus this one plus this one at the next time level, but especially here in this case, we are just marching if you will just see here, this x space coordinate in the i j k plus 1 level and y space coordinates at the i j k plus i j k th level here.

This means that half of the time step we are just moving in the x direction; that is at the unknown level and a half time step we are just moving in the y direction; that is a at the known level. So, if you will just a neglect this truncation error for this series here, then we can just obtain this series s, if you will just take this forward difference approximation for a time derivative here, this can be written as a i u i j k plus 1 minus u i j k by del t here, and this space coordinate x that is at k plus oneth level. So, it can be written as u i minus 1 j k plus 1 minus 2 u i j k plus 1 plus u i plus 1 j k plus 1 by del x square plus this space coordinate in the y direction at k th level here, that can be written as u i j minus 1 k minus 2 u i j k plus u i j plus 1 k divided by del y square here.

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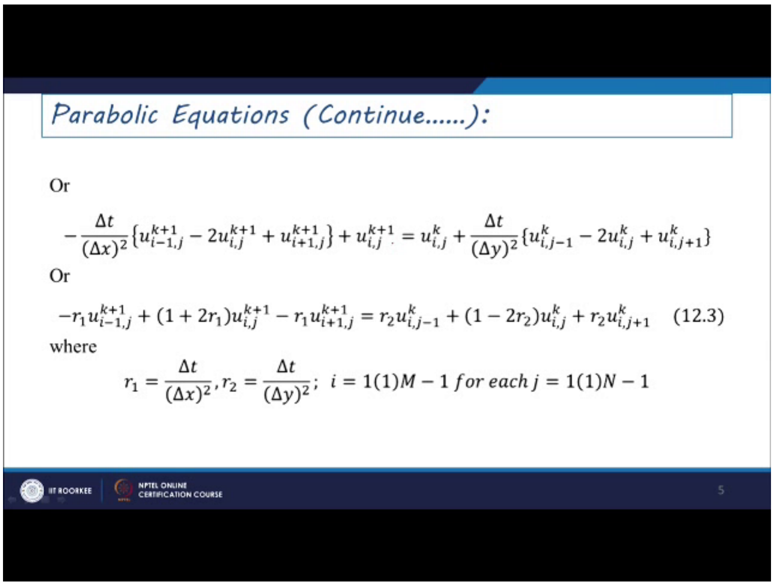

And if you will just combine all these terms and if you will just keep these unknown parameters in the left hand side and the known parameters on the right hand side. Then this system of equations can be written as minus del t by del x square u i minus 1 j k plus 1 minus 2 u i j k plus 1 plus u i plus 1 j k plus 1 plus, since a one coordinate it is a just associated with this time derivative there. So, that is why it can be taken this term as in the form of a u i j to the power k plus 1, and one term here u i j to the power k here.

And especially we are just specifying del t by del x square; that is as a r 1 here and r 2 as del t by del y square. For a i equals to 1 to m minus 1 and j equals to 1 to n minus 1. And if we will just write these coefficients r 1 and r 2 as a del t by del x square and del t by del y square, then this equation will be reduced in the form as minus r 1 u i minus 1 j k plus 1. So, 1 plus 2 r 1 u i j k plus 1 here. Since we are just minus sign is a multiplied here, so that is why it is just taking the plus sign there itself and one more u i j is a present there. So, that is why it is just a taken as 1 here.

So, last term it can be written as a minus is multiplied. So, minus r 1 u i plus 1 j k plus 1, and the right hand side if you will just look at, so we are just obtaining u i j minus 1 here and one term here itself. So, that is why it is just giving as a 1 here. So, 1 minus 2 r 2 u i j to the power k plus r 2 u i j plus 1 here.

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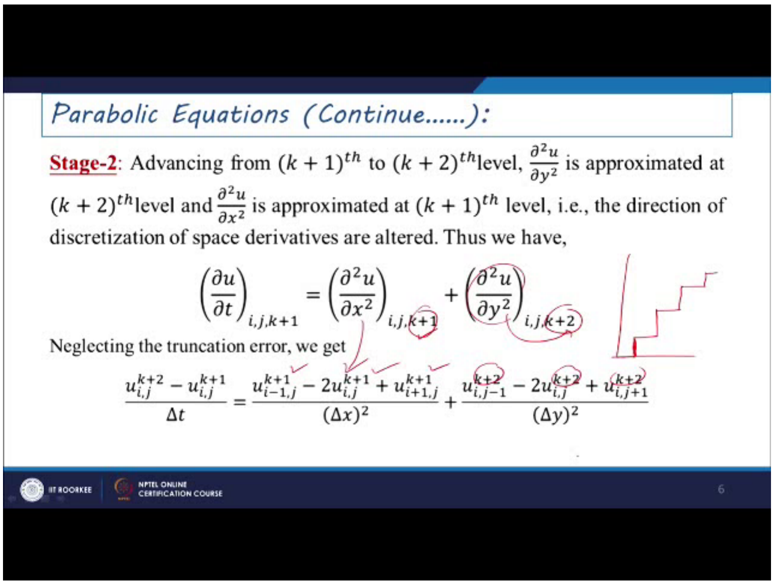

So, if you will just move further like stage 2. So, r ones in like a k plus oneth to k plus twoth level. We have to consider here del square u by del y square is approximated at a k plus twoth level and del square u by del x square is approximated at k plus oneth level.

If you will just see here, this marching step here. So, first we are just moving this x coordinate at a kth level here and we are just moving this one at k plus oneth level. Again x coordinate in the k plus oneth level, again this is in k plus 2 level. So, likewise we just we are moving here. So, the direction of this discretization of space coordinates are altered now. Thus we can just write here as a del u by del t at i j k plus oneth level here, and del square u by del x square at k plus oneth level and a del square u by del y square at k plus twoth level here.

So, if you will just neglect the truncation error, then we can just write this one as like u i j k plus 2. Since we are just considering here this a forward difference approximation. So, that is why it is just written as a u i j k plus 2 minus u i j k plus 1 by del t and the space derivative in the x direction this is a discretized at a i j k plus oneth level. So, all are k plus oneth level we are just writing, and this space coordinate in the y direction it is just described as at a k plus twoth level. So, that is why all terms are associated with k plus twoth level here.

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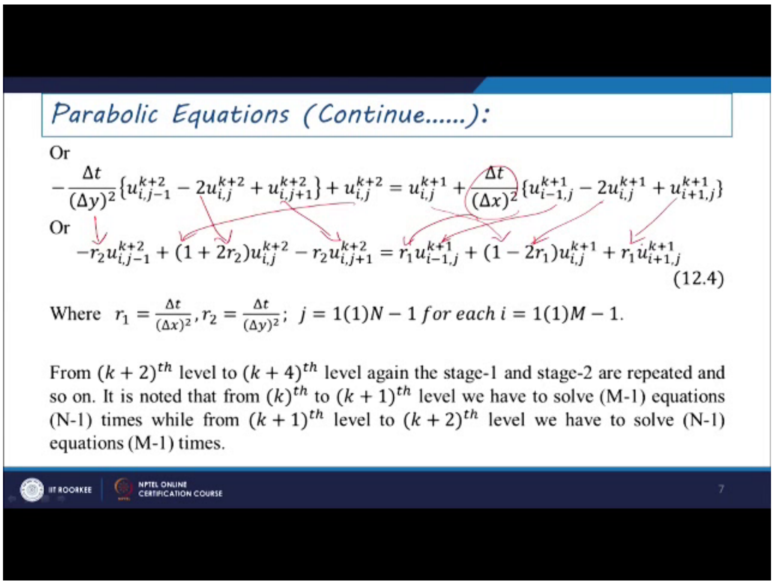

So, if we will just combine all these terms in k plus 1 and the k plus twoth level for a x direction and y direction values, then we can just obtained this approximated equation as minus del t by del y square this into u i j minus 1 k plus 2 minus 2 u i j k plus 2 plus u i j k plus j plus 1 k plus 2 plus u i j k plus 2 here.

Since this k plus 2 is a unknown variables we have now. So, that is why this k plus 2 values we are just keeping at the left hand side and k plus oneth value. So, that has been computed from the previous cycle of calculation. So, it can be put in the right hand side here. So, that is why this k plus oneth level values that are associated that can be, that is just kept in the right hand side here. So, which is in the form of u i j k plus 1 plus del t by del x square u i minus 1 j to the power k plus 1 minus 2 u i j k plus 1 plus u i plus 1 j k plus 1 here.

And if you will just put here r 1 as del t by del x square and r 2 is del t by del y square for a variation of a j from 1 to n minus 1 and i is varying from 1 to m minus 1 here, then we can just write this coefficient that is in the form of like minus r 2, then u i is a minus 1, then again this minus sign is multiplied with this minus 2. So, that is why this just give you 2 r 2 here and a one term it is just coming from here. So, that is why 1 plus 2 r 2 u i j k plus 2 here. And second term if you will just see here, so that is nothing, but minus r 2 and this term is written here.

Similarly if you will just go for right hand side here; so u i minus 1 j k plus 1 here associated with this term. So, that is why r 1 into u i minus 1 j k plus 1 here, and second term if you will just see here. So, one term it is just coming from this one and minus 2 r 1 and last term is a as usual there. So, from k plus twoth level to k plus fourth level. Again this stage 1 and stage 2 are repeated and so on we can just go. Like in each of these iterations we are just considering like first step as like k to k plus 1 then k plus 1 to k plus 2.

So, first level is complete up to k plus 2, then second level we can just move like a k plus 2 to k plus 4. Since the next iteration we can just consider like space derivative x in the k plus twoth level and like a space coordinate in the y direction it can be considered in the like k plus threeth level. So, whenever we will just go, like further o half time step, then we can just write space coordinate of x at k plus 3 level, and space coordinate in the y direction at k plus fourth level. So, likewise we can just proceed for a different time steps.

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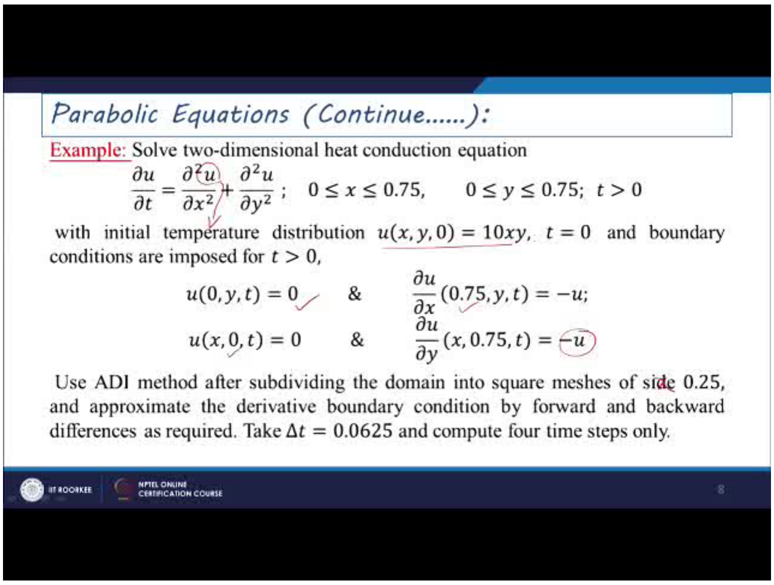

And, if we will just consider this example for 2 dimensional heat conduction equation here; that is in the form of a del u by del t equals to del square u by del x square plus del square u by del y square, where x is specified as a lying between 0 to 0.75 and y is lying between 0 to 0.75 for each t greater than 0 with initial temperature.

Since we are just considering, u as the temperature here and u of x y 0; that is at initial level t equals to 0, this value is given as a 10 x y here, and the boundary conditions are imposed as u of 0 y t equals to 0 and del u by del y x 0.75 y t there is equals to minus u, u of a x 0 t is 0 and a del u by del y x at 0.75 t is equals to minus u, it is just given to us. And the question is asked use alternating direction implicit method after subdividing the domain into square messes of size 0.25 and approximates this derivative boundary condition by forward and backward differences as required.

So, if you will just consider this a time step del t equals to 0.0625 make the computation up to 4 time steps. So, if you will just consider this problem with the specified boundary conditions.

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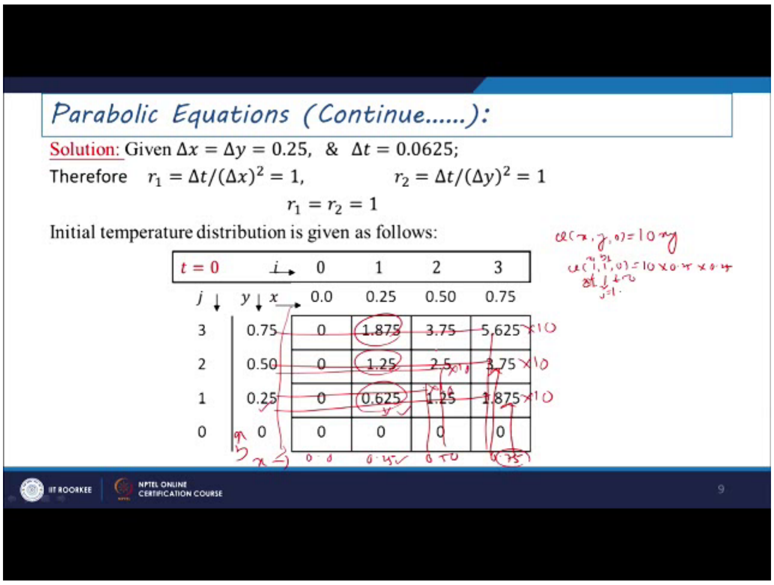

So, then we can just write this domain in the form like del x equals to del y, it is just given as 0.25. So, that is why we can just formulate this domain as a x is varying from 0.0 0.25 0.50 and 0.75 here, y is also starting that is a 0, 0.25, 0.50, 0.70. Especially you can just move this one to here and you can just write this as like x is varying from 0.0 0.25 0.50 0.75 here also.

So, y is just varying in this direction and x is varying in this direction here. So, for like your compatibility you can use in this direction or the specified direction it is just given in the slides. And the boundary condition it is just specified as if you will just see here that as x equals to 0. So, it is just a given as a 0 here, and at a last boundary x equals to 0.75, this normal derivative equals to minus u there or del u by del x equals to minus u, and especially for a boundary condition at y equals to 0, it is just given as a u equals to 0 the Dirichlet condition.

And this derivative condition that is del u by del y at the last boundary; that is just taken as minus u there. So, if you will just a first consider this u of x y 0 equals to 10 x y here suppose. Then first coordinate if you will just see here this is 0, this is 0. So, it can take a 0 value here. And if you will just consider this first coordinate here, suppose we will just consider here like u of x y 0 is a 10 x y. So, if you will just consider like a x equals to 0.525 here and y equals to 0.25 here suppose, then we can just write this one at this point as, since this is the starting point if you will just see here 1 1 point.

So, u at a 1 1 0; that is t equals to 0 here, x is the first coordinate we can just write i equals to 1 there and j equals to 1 there so; that means, we can just write this one as a x 1 this is as y 1 there, and we can just write 10 into 0.25 into 0.25 there. So, that is nothing, but 0.625 here. So, likewise if you will just consider here; like 0.25 into 0.50 into 10 this is just giving you 1.25 here, and if you will just consider 0.25 and 0.75 this is just giving you into 10, this is just giving you 1.875.

So, similar way all other coordinates it has been calculated; like 0.75 this is just a reflects this point here into 10. This just takes this into this one into multiplied by 10 here. This just takes this one into this one into multiplied by 10 here. So, likewise all these coordinates here; so this one and this one here into 10. So, this value into this one into 10 here. So, likewise all these values has been computed at the initial time level and it has been fixed in the first plane there.

So, especially if you will just see our previous slide we have given a clear contradiction that how this x coordinate is varying, or how this y coordinate is varying and t is just in the perpendicular direction it is just varying. So, each of these planes; since in the first plane if you will just see we are just fixing like t equals to 0 there, and at each of these grid points we are now have the values that is just calculated here. And for the next immediate step of calculation for a del t; that is a t equals to del t we have to move to the next immediate step there. So, for this, if you will just go for this next slide here.

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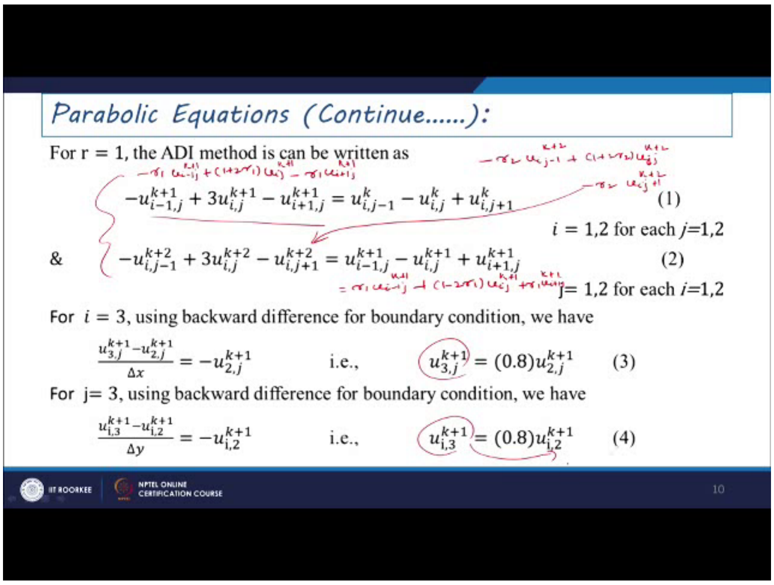

So, for r equals to 1; the ADI method can be written as like minus u i minus 1 j k plus 1 plus 3 u i j k plus 1 minus u i plus 1 j k plus 1. Since r is a 1 as considered here. So, especially if you will just see here. So, that r is a considered as a like, sorry this is a we have considered here if you will just see here are r 2 equals 2 r 1 equals to 1 here; that is a r 1 is a del t by del x square and r 2 is a del t by del y square. So, both these values are considered as one here. So, that is why r 2 r 1 equals to r 2 equals to 1 we have consider. So, if you will just put here r 1 equals to 1 and r 2 equals to 1 here. So, first coordinate if you will just see, this equation is written in the form like minus r 2 u i j minus 1 k plus 2 and plus 1 plus 2 r 2 u i z k plus 2 minus r 2 u i j plus 1 k plus 2 here.

So, if you will just see this equation here you can just find that minus u i j minus 1 k plus 2 plus 2 plus 1. This is nothing, but 3 here u i j k plus 2 minus u i j plus 1 k plus 2. So, right hand side if you will just see here. So, this is nothing, but r 1 u i minus 1 j k plus 1 minus, sorry plus 1 minus 2 r 1 u i j k plus 1 plus r 1 u i plus 1 j k plus 1 here. So, and of course, if you will just put all these values then you will have this two set of equations, whatever it is just written here. Since the first time step we are just a calculating here that as the values as like a minus r 1.

So, first time step I can just write in here also minus r 1 u i minus 1 j k plus 1 plus 1 plus 2 r 1 u i j k plus 1 minus r 1 u i plus 1 j k plus 1 here. So, that is why r 1 equals to 1. So, that is why it is just take this values in this form here. Similarly this right hand side you can just get it from using this equation number 12.3. So, once we are just using i equals to 3 here, using backward difference for this boundary conditions, then we will have this conditions like u 3 j k plus 1 minus u 2 j k plus 1 by del x as minus u 2 j k plus 1.

So, from there itself this boundary point that is u 3 j k plus 1, this can be obtained as in the form of 0.8 u 2 j k plus 1 there. For j equals to 3 using backward difference for a boundary condition, if you will just see here, we can just write as u i 3 k plus 1 minus u i 2 k plus 1 by del y, this is as written as minus u i 2 k plus 1. So, that is this boundary point u i 3 k plus 1 it can be written in terms of the domain values or the domain variable u i 2 as 0.8 u i 2 k plus 1 here.

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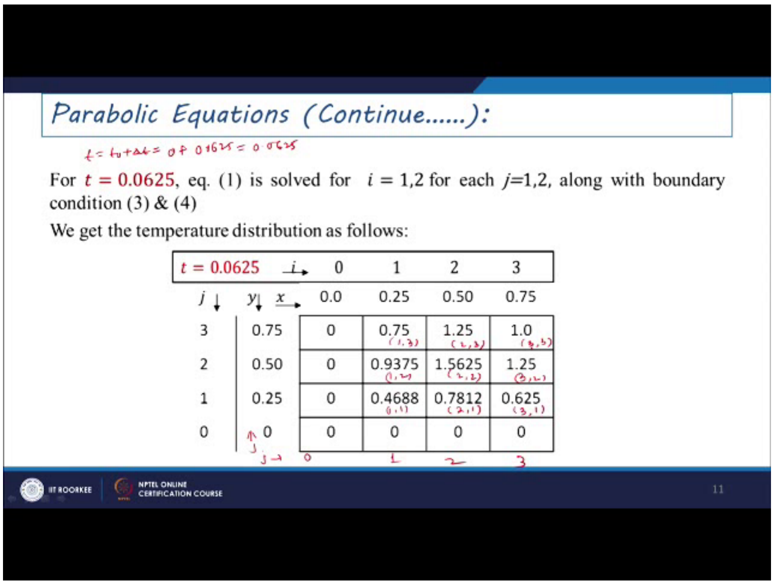

So, for t equals 2 like a first increment if you will just consider here. So, t can be written as like a t 0 plus del t that is nothing, but a 0 plus 0.0625. So, that is why it is just written as 0.0625 here and if you will just consider like i equals to 1 to 2 here, for each j equals to 1 to 2 along with this boundary condition. So, we can just write these values of temperature at a different node points here. Here also you can just shift this one j is varying from like a 0 0.25 0.50 0.75 and i can be written as like 0 1 2 3 here and if you will just compute all these values from this set of equations. So, that has been formulated here.

Then we can just find the temperature values at a different grid points; that is 1 1,this is just like 1 2 1 3, this is like you will have 2 1 2 2 2 3, then it will have 3 1 3 2 3 3 coordinates. So, this is the computed value. So, that has been computed by using this previous equations.

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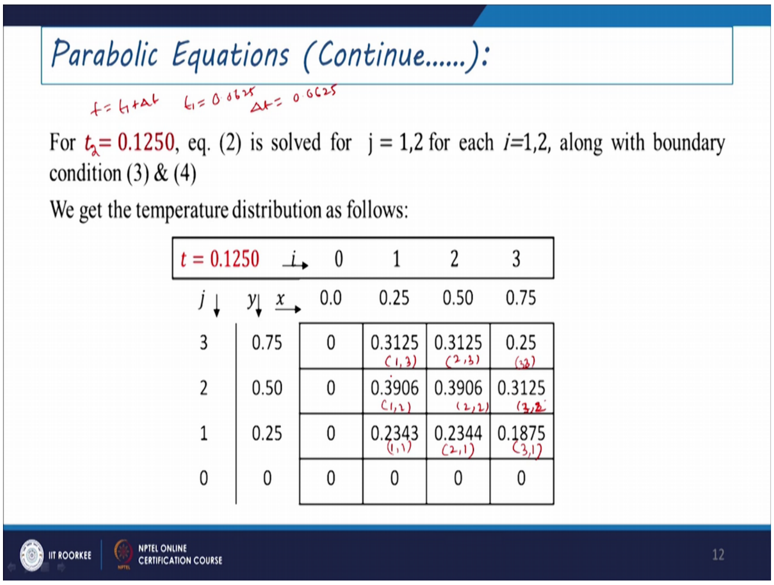

So, if you will just go for the next iteration step, we can just write t can be written as like t 1 plus del t; since t 1 it is already calculated as it is in the previous slide as 0.0625 and incremented value del t is equals to 0.0625 there itself.

So, that is why this final t or you can just write this one as a t 2 here that is nothing, but a t 1 plus del t. So, it is written as a 0.1250 here, and if we will just go for this calculation of a temperature here. So, then this calculation, we can just get it as like for first coordinate, we can just get this one as a 0.2343 and for like 2 1 coordinate we can just get it as a 0.2344 and third coordinate we can just get it as a 0.1875.

So, likewise further if you just move here, we can just get it as 1 2 then 2 2 then 2, sorry this is like 3 2 and then this is 3 3 and this is like 2 3. So, this is like a 1 3. So, all of these coordinate values; like corresponding i and j coordinates if you will just put it and you can just obtain this values. So, for like further increment of this time step, if you will just use like a x coordinate as in this direction here and y coordinate as in this direction.

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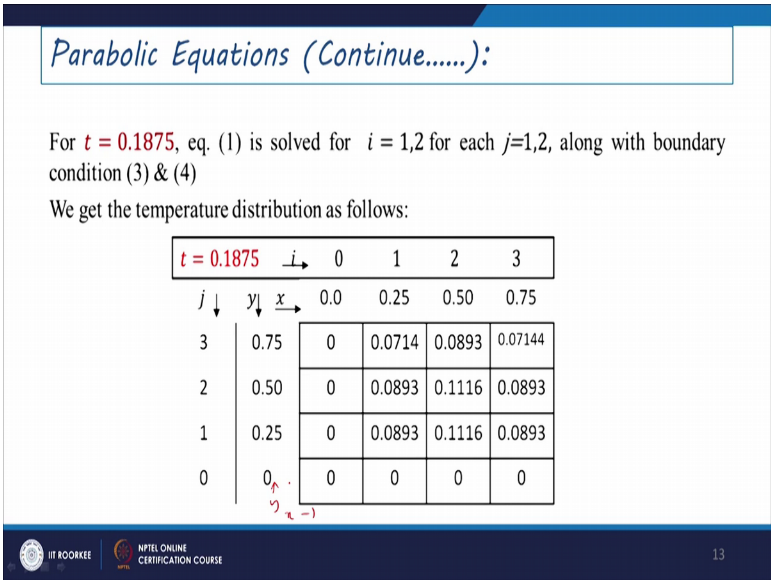

t = 0.1875	i →	0	1	2	3
j ↓	y ↓ x →	0.0	0.25	0.50	0.75
3	0.75	0	0.0714	0.0893	0.07144
2	0.50	0	0.0893	0.1116	0.0893
1	0.25	0	0.0893	0.1116	0.0893
0	0	0	0	0	0

And put all these values then we can just obtain these values for a temperature at a different grade points.

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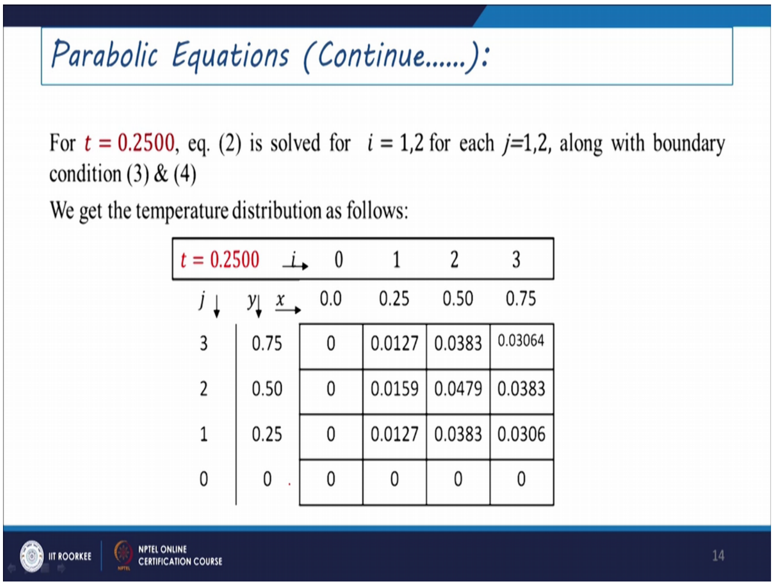

t = 0.2500	i →	0	1	2	3
j ↓	y ↓ x →	0.0	0.25	0.50	0.75
3	0.75	0	0.0127	0.0383	0.03064
2	0.50	0	0.0159	0.0479	0.0383
1	0.25	0	0.0127	0.0383	0.0306
0	0	0	0	0	0

And further moment of for t, we can also obtain these values are in this form, and this can be tested easily since all of these calculations this proceeds with 1 by 1 1 by 1. So, that is why you can just compute and you can just find these values here

Thank you for listen this lecture.