Numerical Methods
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Lecture 39
R-K Methods for solving ODEs

Hello everyone. So in this lecture I am going to introduce another class of methods for solving ordinary differential equation numerically. This particular class or methods is called Runge kutta method. So or in short RK method so in the previous couple of lectures we have seen Euler's method. So in Euler's method in simple Euler's method we were having right to ratio of order of h while the modified Euler's method we were having it of order h square.

And we have seen in Euler's method if you want more accurate solution or a better approximation of the solution what you need to do? You need to reduce the step size that is you have to use smaller steps value of h or if we talked about Taylor method if you want more accurate solution what you need to do? You have to go for higher order terms. Higher order terms means higher order derivatives of a function.

But whether you are decreasing your step size or you are calculating the higher order derivatives in both the cases you need to to more calculation computational complexity will increase.

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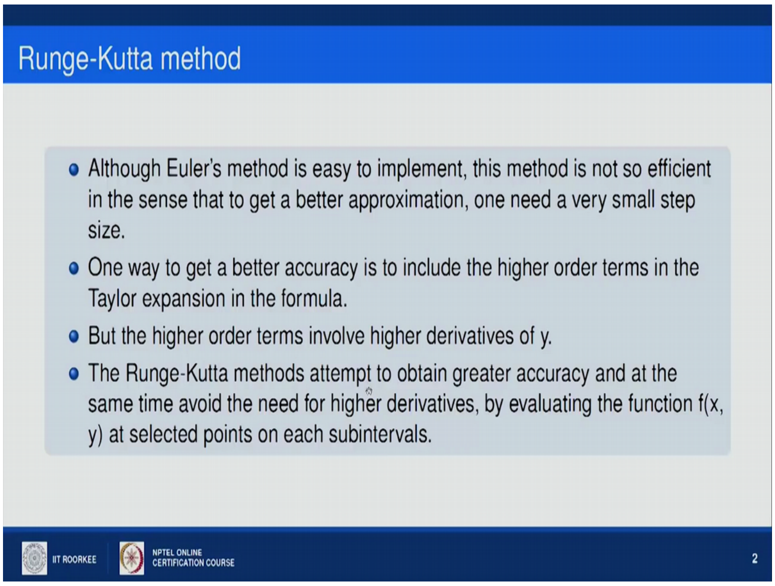

So in RK methods we attempt to obtain greater accuracy and at the same time avoid the need of calculation of higher derivatives or with a smaller step size. We don't want to reduce the step size. So how we can do it? We will evaluate the function f of x y.

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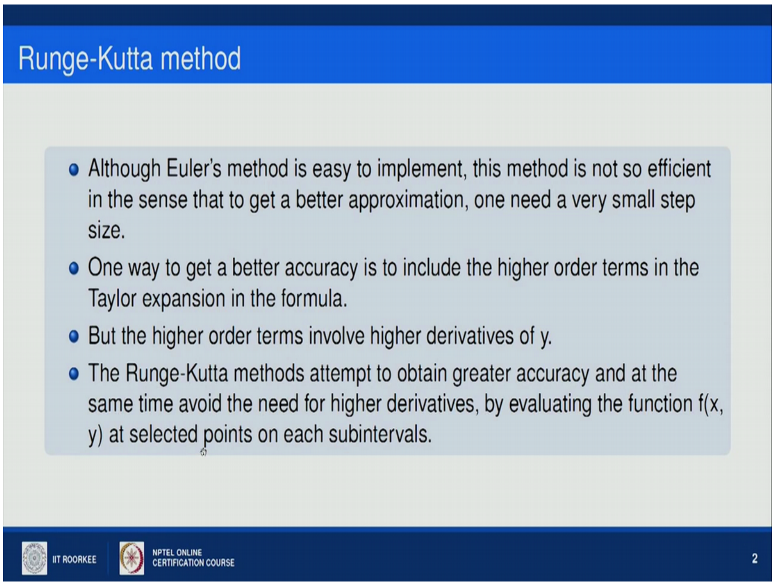

At some selected points on each subinterval like in simple Euler's method or Taylor method we are calculating at initial point and last point. Here what we will do? We will take some selected points on the subinterval. So consider this so first of all I will explain RK method of order 2 and then I will tell you how can we generalize it to in term in any order.

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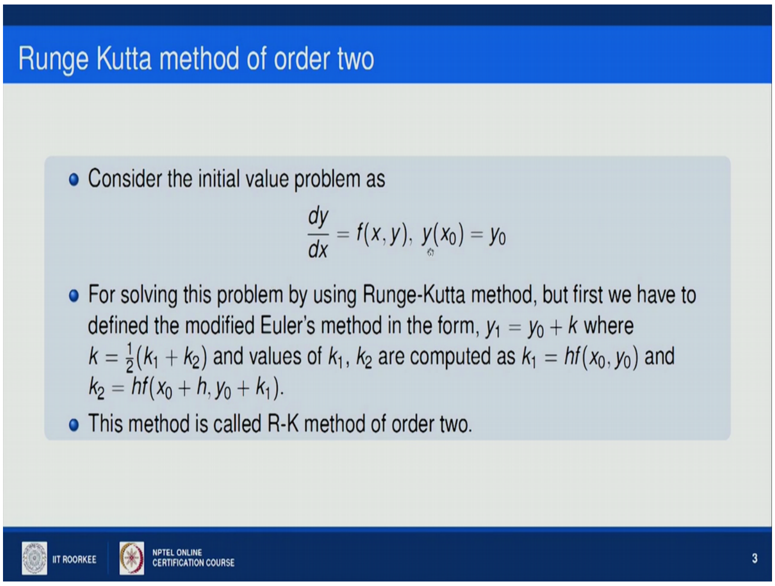

So consider the initial value problem is DY or DX equals to f of X Y with initial condition Y at X equals to X nought equals to Y nought. So for solving this problem using the runge kutta method of order to first we need to define the modified Euler's method in the form Y1 equals to Y0 plus K and here K will be half of K1 plus K2 where K1 is H times F of X nought Y nought and K2 is H times F X nought plus H Y0 plus K1 and this particular method is called RK method of order 2. So basically what we are doing?

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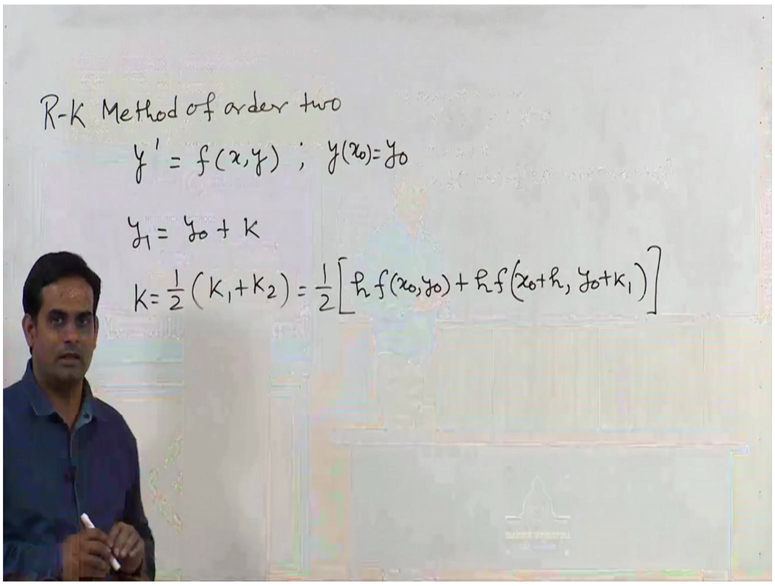

So RK method of order 2 can be given as suppose I am having initial value initial value problem is Y dash equals to F of X Y with an initial condition Y X nought equal to Y nought. So Y1 can be obtain as Y nought plus K. Where K will be half of K1 plus K2 and this half of K1 will be H times F X nought Y nought plus K2 will be H times F X nought plus H Y nought plus K1. So what we need to do first of all we need to calculate K1.

Then we need to calculate K2. We need to take the average of K1 and K2 that will be my K and Y can Y0 can be updated as Y1 equal to Y nought plus K. So this is the overall algorithm for Runge kutta method of order 2. Now graphically this method can be seen like this.

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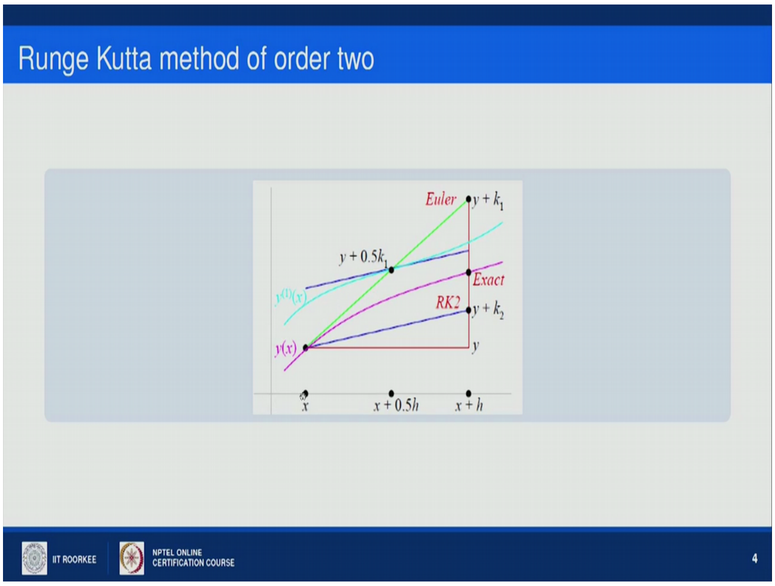

I am having initial point X at this point I am having the value of the function Y of X given by this particular point. I am having and this is my function this pink colour curve as X function. I am taking the slope at this particular point X which is given by this green line and I want to find out the value of the function Y at X plus H. So what will happen? I will take the midpoint of the slope that is somewhere X plus point 5 H.

So midpoint of the interval at this I will find out the point at the slope line. So this will this particular point will be the Y plus point 5 K1. Now at this particular point that is X plus point 5 H and Y plus point 5 K1 I will be having another solution curve given by this particular curve that is Y1 X. Now slope of this Y1 X at this particular point X plus point 5 H will give me the value of function Y at X plus H that is Y plus K2.

And that will be the half of the Euler’s step. You can see here this is the difference in Euler’s step and this is particularly Y plus K2 that is Runge kutta method of order 2.

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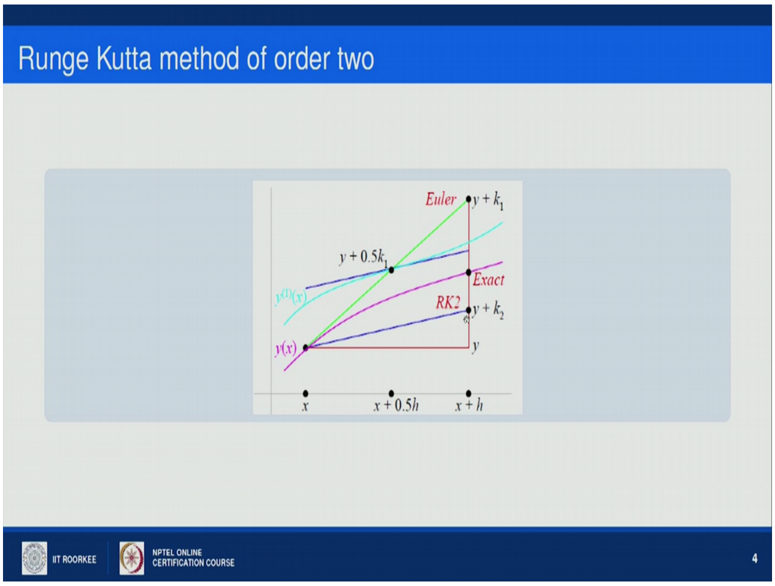

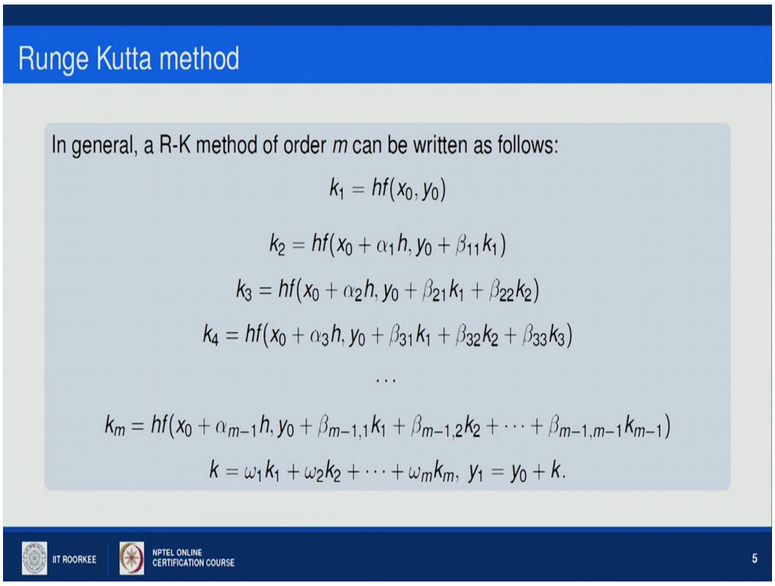

So in general RK method of order M can be written as follows like in order 2 we are having 2 terms K1 and K2. Like that in order M we need to calculate K1, K2 up to KM and where K1 is given just like as in order 1 method. K2 will be just like at order 2 method. K3 will be H time X0 plus alpha 2 H plus Y0 plus beta 21 K1 plus beta22 K2. Similarly K4 can be given by this particular expression and KM will be finally will this one.

Where K will be the weighted some of K1, K2 up to KM that is omega1 K1 plus omega2 K2 plus up to omega M KM where omega1, omega2, omega M are weights that is between 0 to 1. And finally once we will be having this K, I can write Y1 equals to Y0 plus K.

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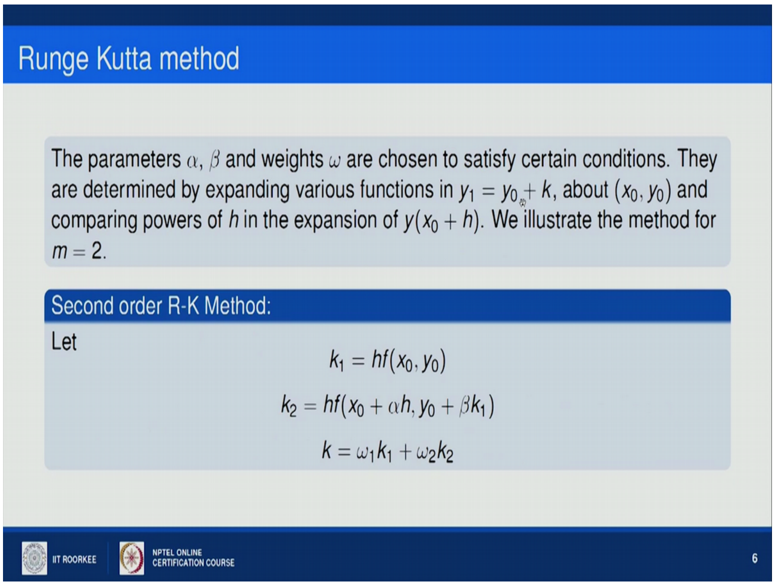

The parameters alpha, beta and weights w are chosen to satisfy certain conditions. They are determined by expanding various functions in Y1 equals to Y0 plus k about X nought Y nought and comparing powers of H in the expansion of Y X0 plus H. That can be seen means in case of order 2. I will derive it we are getting this particular alpha, beta and w for a RK method of order 2.

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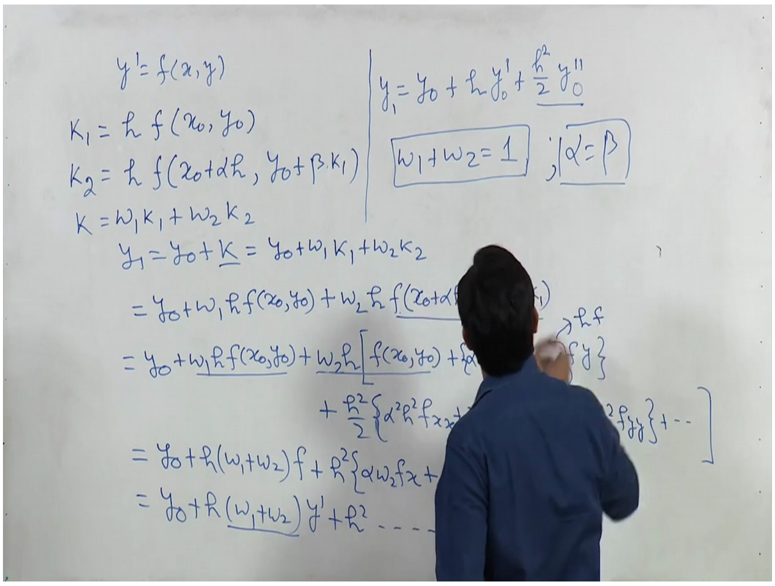

So basically in RK method of order 2 I will be having K1 as H F X nought Y nought. K2 will be H time F X nought plus alpha H Y nought plus beta 1 or simply beta K1 and then I am having omega 1 K1 plus omega 2 K2. This is equals to K and finally Y1 will be Y nought plus K. So this is the general scheme for RK method of order 2. So Y1 is Y0 plus K that I can write Y0 plus omega1 K1 plus omega2 K2.

So Y0 plus omega1 the value of K1 I can substitute from here. H time F X nought Y nought plus omega 2 the value of K2 I can substitute from here. H F time X nought plus alpha H Y nought plus beta K. Or this can be written as Y nought plus omega1 H F of X nought Y nought plus let us explain this term by the Taylor series expansion about X nought Y nought.

So this will be omega2 H into F X nought Y nought plus alpha H F X plus beta K F Y. That is the first order term so F X at X nought Y nought F Y at X nought Y nought plus I will be having the second order term that is will be H square by 2 into alpha square H square the second order derivative of F with respect to X that is the partial derivative plus 2 alpha H beta K FYY sorry it will be K1. Because K we have taken here.

So K1 and then K1 plus beta square K1 square FYY plus higher order term. So this this can be expanded like this. Finally we can collect the coefficient Y0 plus H time omega1 plus omega2. So I have taken this into F so please note that now I am writing F X nought Y nought as F plus so I have taken this terms then I will be having this particular term H square.

So H into H H square alpha into omega2 into FX plus as you can note down this K1 can be written as from the this formula the H of F. So this term will be beta H F FY. So H square I have take out. So it will be beta into omega2 F into FY plus higher order term. This will be Y0 plus H omega1 plus omega 2 as you know for the initial value problem Y dash will be F of XY so this F I can replace with Y dash plus H square terms.

Now the simple Taylor series expansion of Y can be given as Y0 plus H time Y dash plus H square upon 2 Y double prime. Compare the various powers of H from here omega1 plus omega 2 will become 1. This is the first expression I am getting by comparing the power one of H and then from the second what I am getting? Alpha equals to beta.

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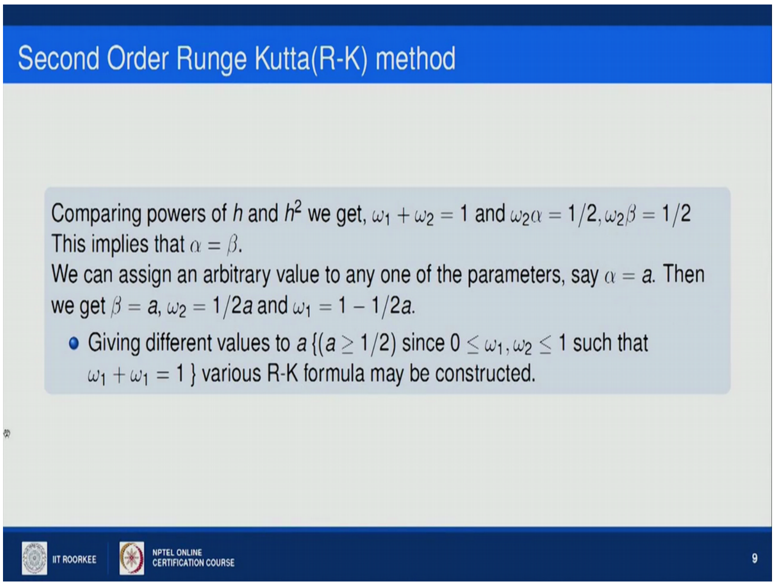

Hence I am getting if I choose alpha equal to A, I can get beta equals to A. Because alpha equals to beta and omega 2 will become 1 upon 2A and omega1 will become 1 minus 1 upon 2A. So giving different values to A in general we choose A greater than half or equals to half such that we can we can generalize the various RK methods of order 2.

For example the classical method of order 2 can be get can be obtained just by taking alpha equals to beta equals to 1. So here A is taken as 1. So omega1 will become half omega2 will become half and a corresponding this equation will become X nought plus H Y nought plus K1 which is the standard Runga Kutta method of order 2. So let us take again an example and solve it using method Runga Kutta method of order 2.

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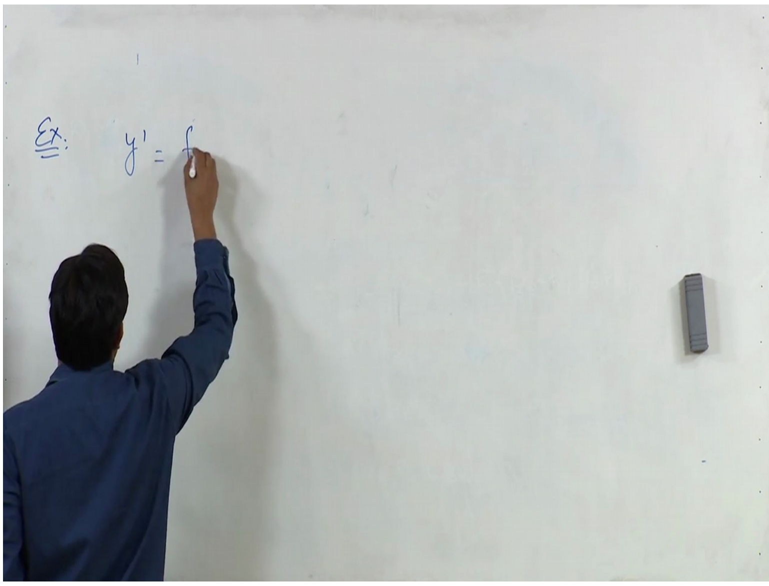

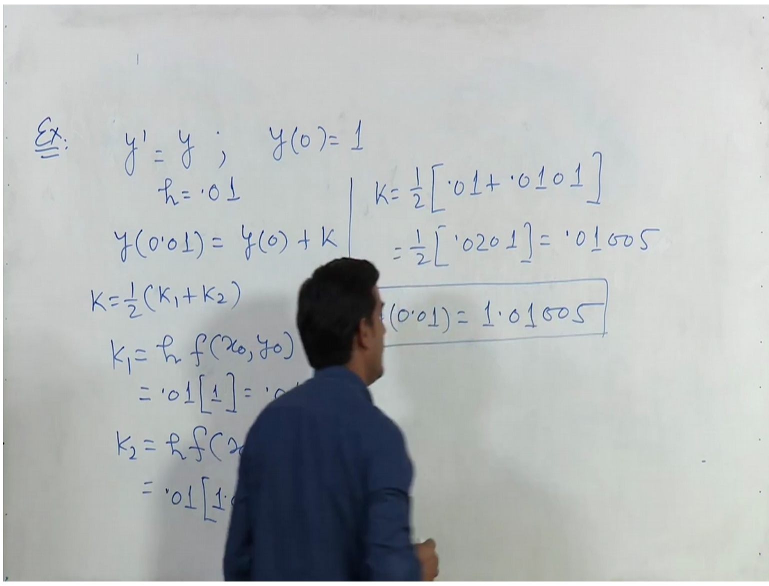

So I am taking Y dash equals to F of XY. So here let us take F of X Y as Y. So my differential equation is DY by DX equals to Y together with initial condition as Y0 equals to 1. So let us take H equals to point 1or let us find out more smaller value H equal to 0 point 1. So Y at 0 point 01 is given as Y at 0 plus K. Where K is half of K1 plus K2. So K is half of K1 plus K2.

So let us calculate K1 for this particular example. So K1 will be H times F X nought Y nought. So H is point 01 into F of X nought Y nought will be Y nought which is 1. Similarly I can calculate K2. K2 will be H time F X nought plus Y nought sorry H time F X nought plus H Y nought plus K1. So it will be point 01 into so this will become Y nought plus K1, Y nought is 1, K1 is point 01.

So 1 point sorry yeah 1 point 01 so it will become point 0101. Now K will become half of point 01 plus point 0101. This is half of point 0201 and it will be point 01050 hence Y at 0 point 01 can be given as 1 point 01005.

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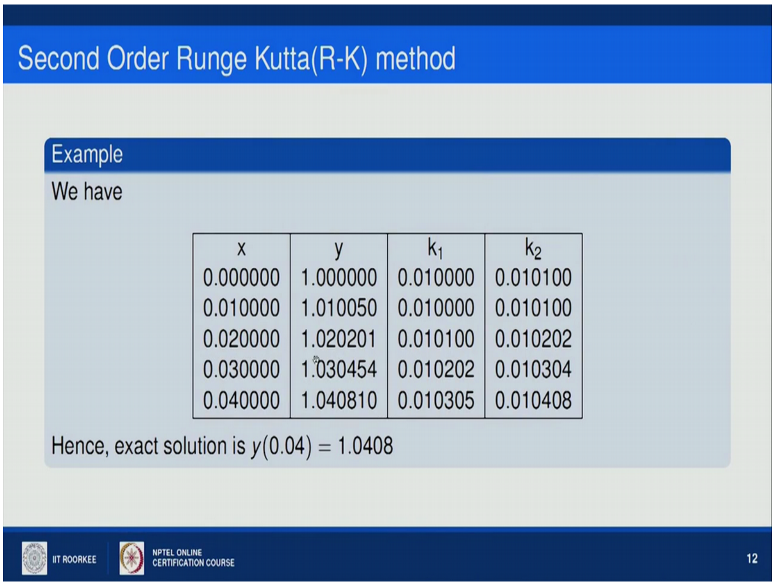

x	y	k_1	k_2
0.000000	1.000000	0.010000	0.010100
0.010000	1.010050	0.010000	0.010100
0.020000	1.020201	0.010100	0.010202
0.030000	1.030454	0.010202	0.010304
0.040000	1.040810	0.010305	0.010408

And so value from this table at X equals to point 02 Y will come out 1 point 020201 at point 03 Y will come out 1 point 0305454 at point 04 it will be 1 point 040810.

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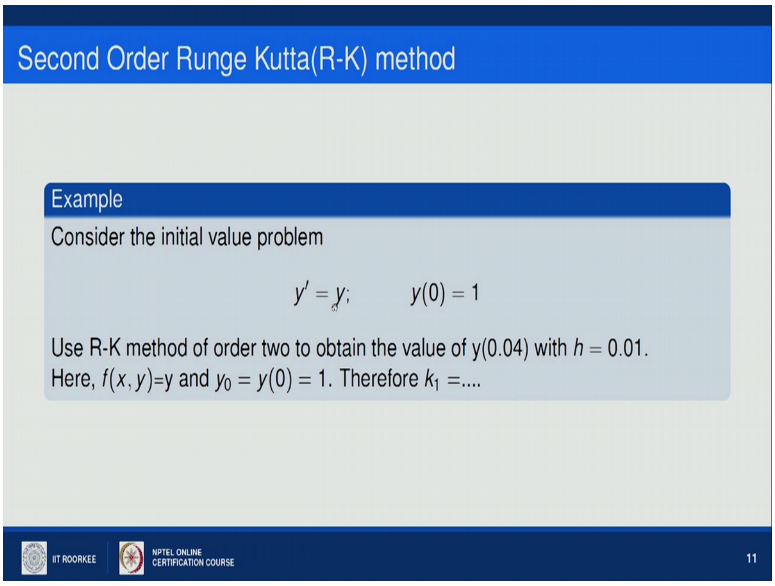

The exact solution of this particular differential equation is Y equals to E rest to power X that can be obtained separating the variables of Y index in different size.

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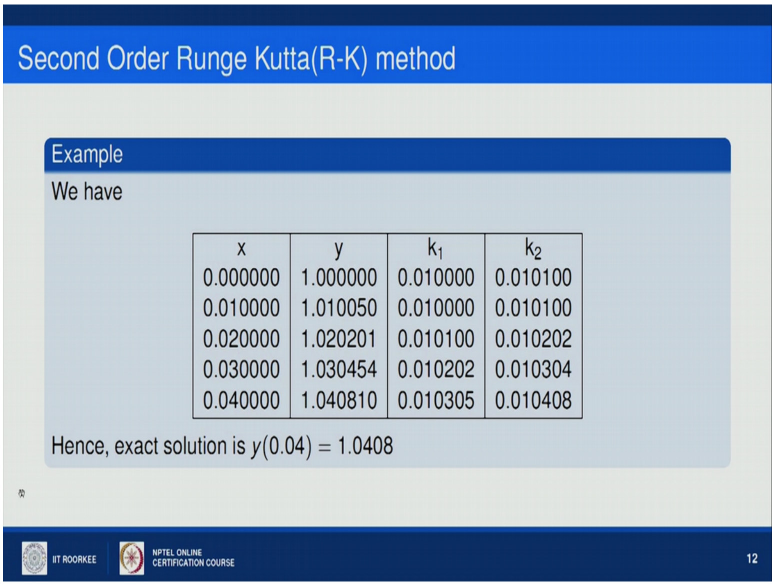

And integrating and exact solution at Y of Y at X equals to point 04 is given by 1 point 0408 which is same as we are getting using the Runga kutta method of order 2 for this particular example.

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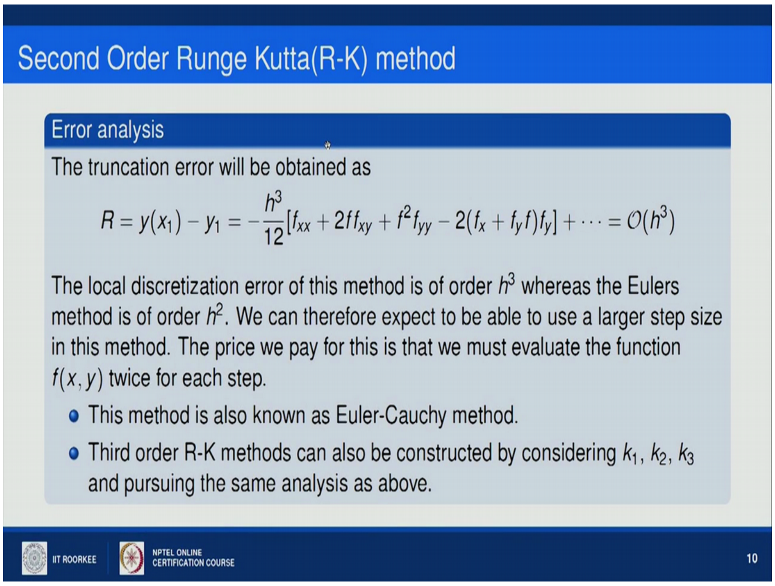

Then if we talk about error in this particular method RK method of order 2 then the truncation error will be obtained as R equals to Y X1 minus Y1 and that will be the term which we have left out for third order of H cube. So hence error will be accuracy will be order of H cube so local discretization error of this method is of order H cube whereas the Euler's and quadratic Taylor methods were having order of H square.

So there therefore we can expect to be able to use a larger step size in this method when compare to the Taylor Euler's method. The price we pay for this is what we must evaluate the function F XY twice that is 1 for K1 another 1 for K2. This method is also known as Euler quasi method. Third order RK method can also be constructed by taking K1, K2, K3 and carrying out the same analysis as we have done for order 2. Here we are talking the RK method of order 4.

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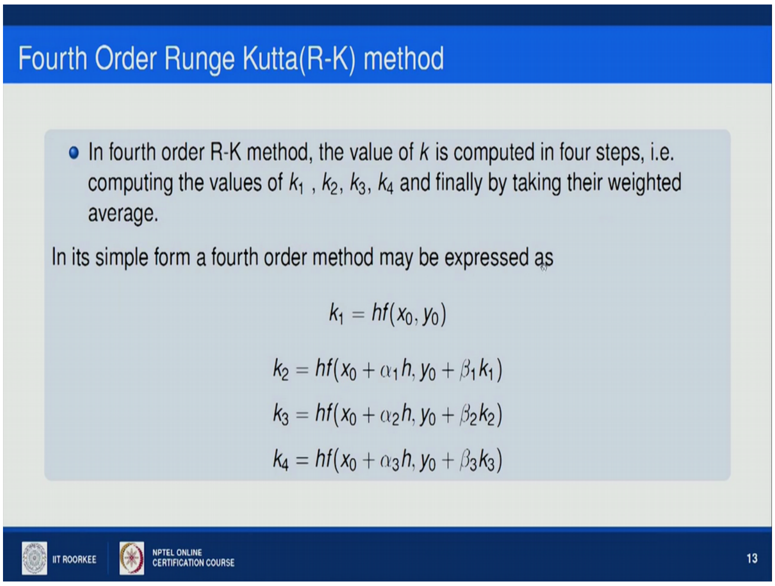

So in its simple form a fourth order RK method may be expressed as so there we will be having four terms K1, K2, K3, K4. K1 is given as H times F X nought Y nought. Then we will make use of this k1for calculating K2. So K2 will become H times F X nought plus alpha 1 H Y nought plus beta 1 K1. K3 will be become H times F X nought plus alpha 2 H plus into what comma Y2 Y nought plus beta 2 K2.

And finally K4 will be become H times F of X nought plus alpha 3 H comma Y nought plus beta 3 K3 where K is omega1 K1 plus omega2 K2 plus omega3 K3 plus omega4 K4. And finally Y1 can be obtained just like in the method of order 2 that is Y nought plus K.

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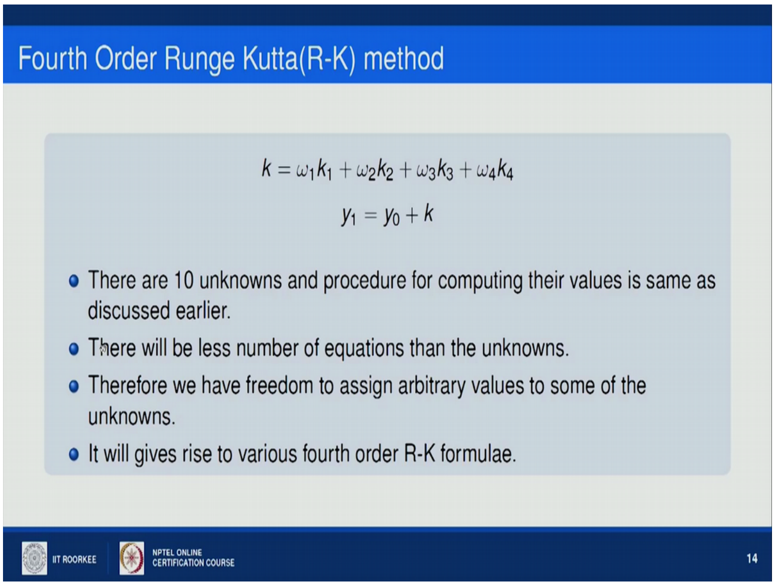

Basically we are having 10 unknowns and procedure for computing their values is same as we have discussed in order 2 method. However we will be having less number of equations than the unknown therefore we have freedom to assign arbitrary values to some of the unknowns and in this way we will get a class of RK method of order 4.

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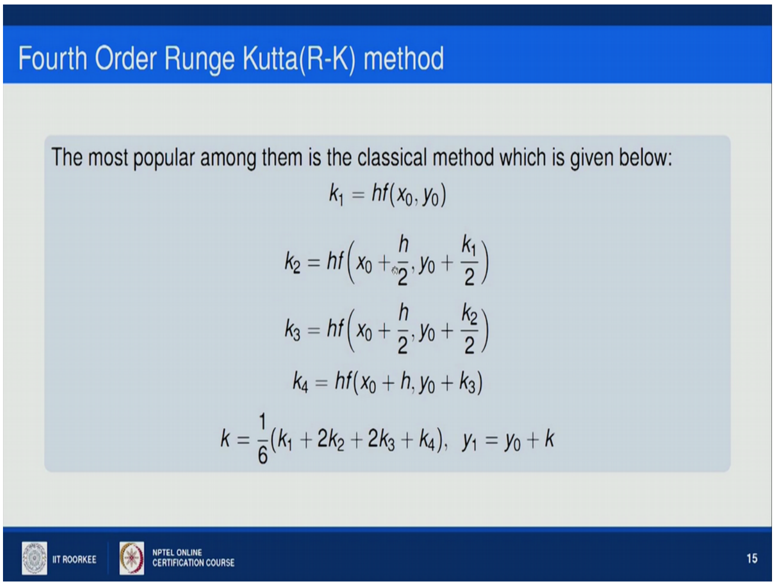

The most classical method which we take is something like this. We take alpha 1 as 1 by 2 beta 1 as 1 by 2 then alpha2 as 1 by 2 beta2 as 1 by 2 and finally alpha3 beta3 as 1 and 1. And we take the weighted average as omega1 as yeah omega1 K1 so 1 by 6 omega2 as 1 by 3 omega3 as 1 by 3 and omega4 as 1 by 6.

So scheme will become like this. Let us take an example and as we have seen in the method the order 2 method the truncation error is of order H cube. So here order error will become of order H5 H rest to power 5 one less than the polynomial degree of the polynomial which we are taking.

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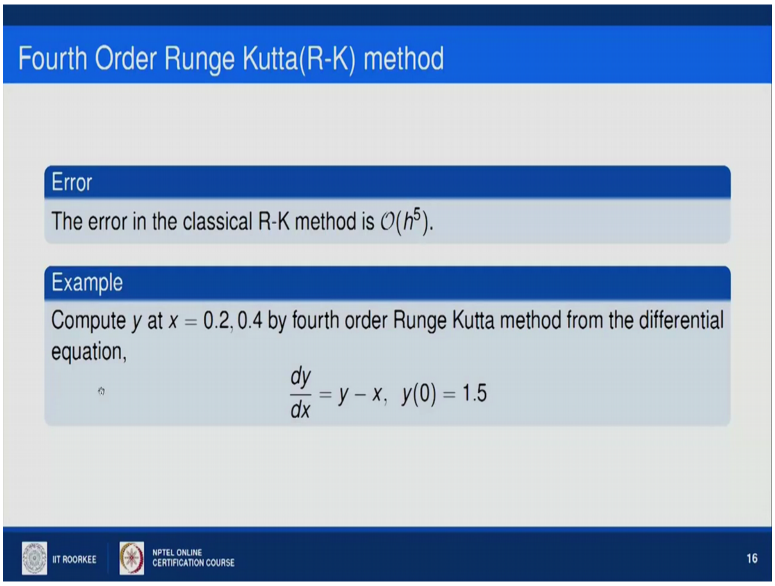

So compute Y at F equals to point 2 and point 4 by fourth order Runge Kutta method for the differential equation DY by DX equals to Y minus X where Y at X equals to 0 is given by 1 point 5.

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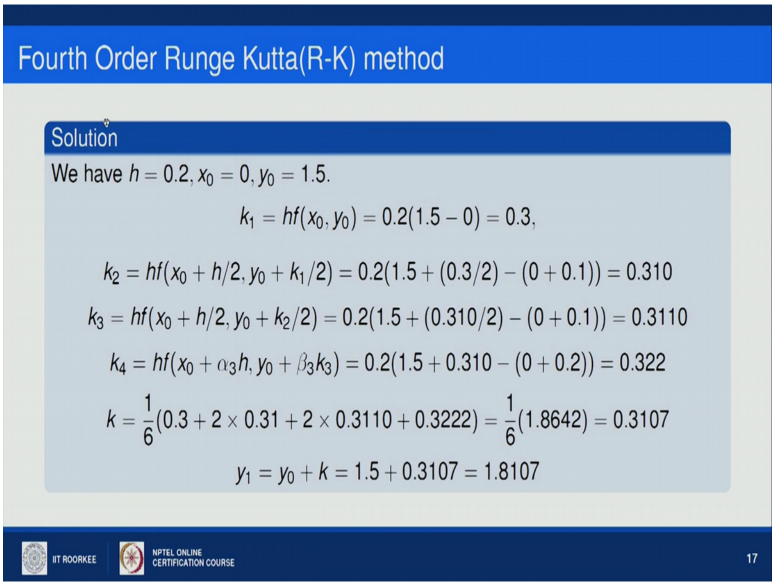

So first of all we will take H as point 2 X0 is 0 Y0 is 1 point 5. So K1 will become H times F X nought Y nought so it will be point 2 into 1 point 5 minus 0so point 3. The K2 will come out as point 310. K3 will come out as point 3110 and finally K4 will be point 322. By taking the weighted average of K1, K2, K3 and K4 I will calculate K and K will be my 1 by 6 point 3 plus 2 times point 31 plus 2 times point 3110 plus point 3222.

And it is coming out as point 3107. So finally Y1 that is the value of Y at X equals to 0 point 2 will be Y nought plus K and it will be 1 point 5 plus point 3107 that is 1 point 8107. Similarly taking the initial values at X1 point 2 and Y1 as point 8107 I will calculate the value of Y at X2 that is point 4 that is basically Y2. So similarly for this I will calculate K1, K2, K3, K4.

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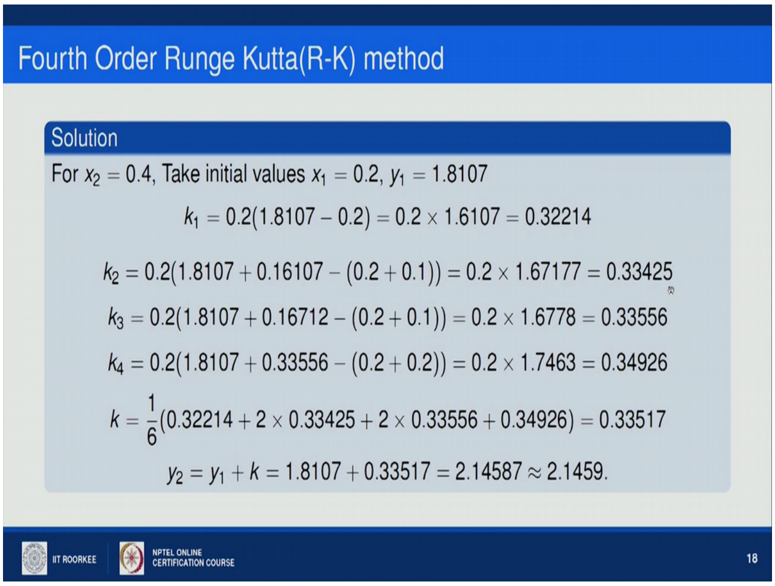

So these are the this is the value for K1, the value for K2, value for K3, value for K4. Finally weighted average will give the K which will be point 33517. So Y2 will come out finally Y1 plus K and it is 2 point 1459. So in this way I can implement the Runga Kutta method of order 4 which is having the accuracy of order H rest to power 5. I can generalize the method of order 3 also which will be having accuracy of H rest to power 4.

So in this lecture what we have learned? We have learned another class of numerical methods that is Runga Kutta methods and we have done the method of order 2 in detailed for this we have done the derivation, we have done the error analysis, we have also seen the RK method of order 4 in detail. We have taken an example. We have found the value of Y at 2 points that is X equals to point 2 and point 4.

If the value of X at 0 value value of Y at X equals to 0 is given. So all these methods staring from the Euler's then Taylor then RK methods what we are doing? For calculating the value of Y at X equals to X1 we are using only the value of X at X0. So what we are doing? We are taking a single step for finding the value at next point we are using the value of current point.

So all these methods are single step in the next lecture we will see another method which is different from these methods because that particular method will use multi steps. So for a finding the value at of Y at X equals to X1 it will take the use the value of X at F0 and at some previous points and based on those points it will fit a polynomial and for the next point it will extrapolate the value of the function for the next point. So thank you very much.