Numerical Methods

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Lecture No 32

Numerical Integration Part II

Welcome to the lecture series on numerical methods and in the last lecture we have discussed numerical integration. In the numerical integration we have started about this rectangular rule that how we can just implement to find this integration for a particular function. Even if this function is not known to us but the tabular values like functional values at different points if it is known to us then we can evaluate this integration by using this nodal or tabular values.

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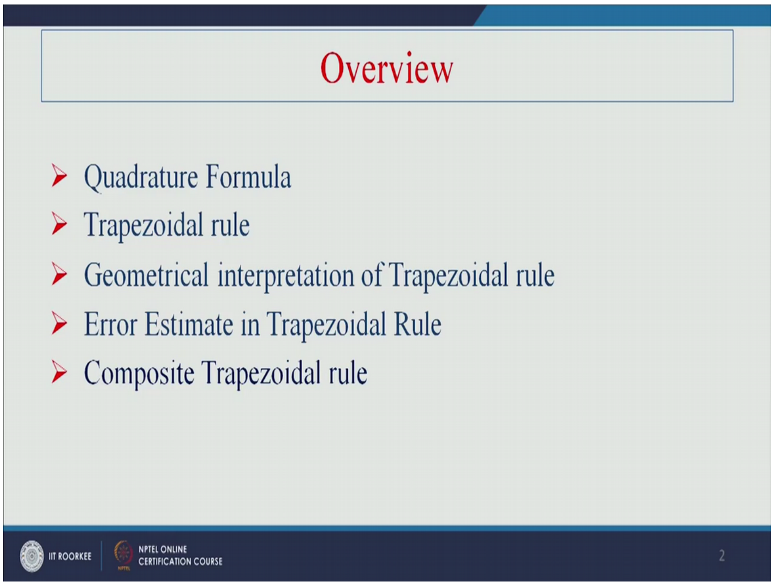

So this class we will just go for this some of this higher-order integration methods, so first will discuss about this Quadrature formula here, then we will just go for 2 point formula that is trapezoidal rule, then we will just go for this geometrical interpretation of this trapezoidal rule here and we will just go for error estimation in a trapezoidal rule, then for composite formula of trapezoidal rule. If the point is like in each of the intervals if you will just use 2 points then the total interval if you will just use this then basically that is called composite trapezoidal rule and how we can just obtain the formula that I will just discuss in this lecture.

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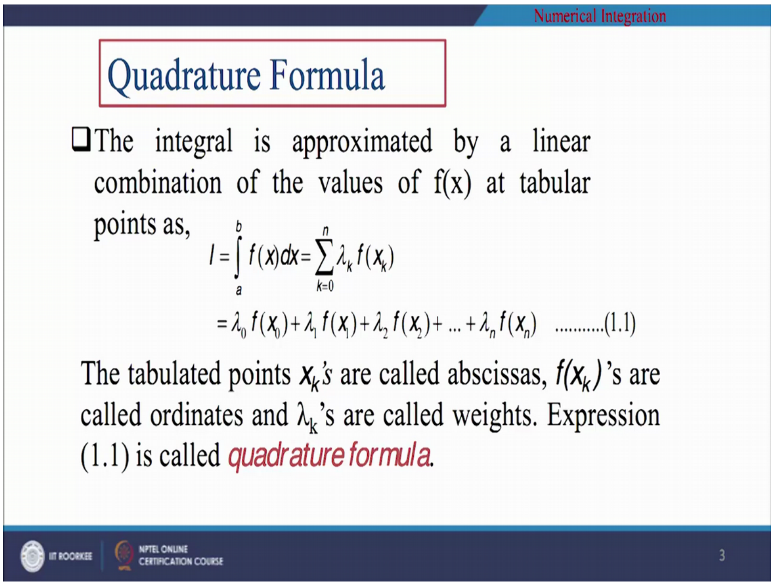

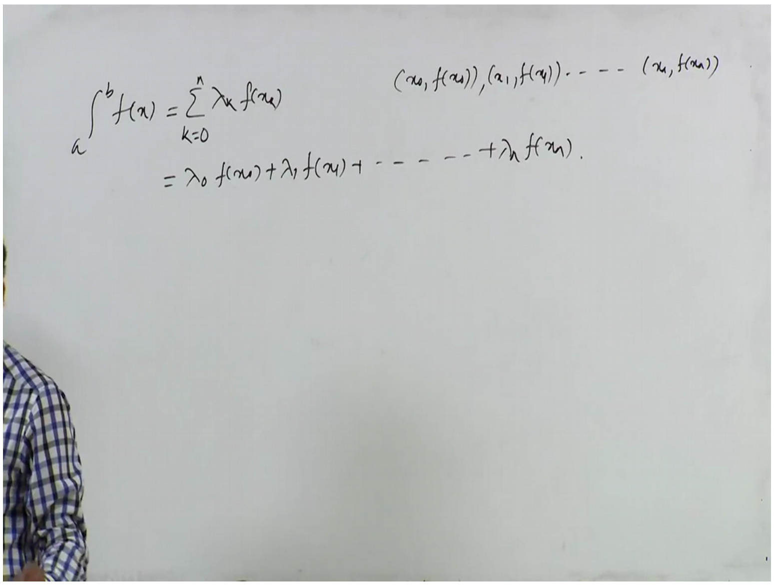

So first if you just go for this Quadrature form here usually this integral is approximated by linear combination of this functional values of f of x at tabular points. This means that integration a to b, f of x into dx it can be expressed in the form of submission k equals to 0 to n, lambda k, f of x k here. Since already it is known that unless always we will have this set of tabular values that is in the form of like x 0, f of x 0, x 1, f of x 1 upto x n, f of x n here. So if this set of tabular values is known to us then we can just implement this Quadrature formula basically evaluate this integration by using this tabular values here.

So especially if you will just expand this formula here that is the linear combination of this functional values are different tabular points then we can just write this one as lambda 0, f of x 0 plus lambda 1, f of x 1 plus upto lambda n, f of x n here. This tabulated point x k’s are called abscissas and f of x k's are called coordinates here or ordinate we can just say and lambda k's are called weights for this function here or this integration here and this complete expression it is called Quadrature formula.

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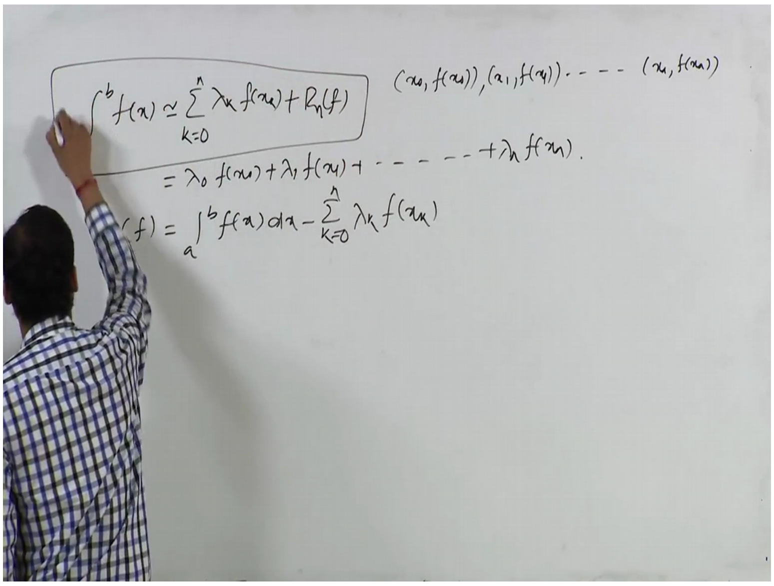

And if we will just go for this competition of this Quadrature formula here usually this error formula R n of f can be written as integration a to b, f of x dx minus summation k equals to 0 to n lambda k, f of x k here. So if we want to evaluate this error always we have to subtract both this terms since especially directly we can just write this is not exactly equals to this is approximation of this term here and we can just write this plus R n of f as the total expression for this complete integration formula here. Then we can just write R n of x or R n of f this equals to integration of a to b, f of x dx minus summation k equals to 0 to n, lambda k, f of x k.

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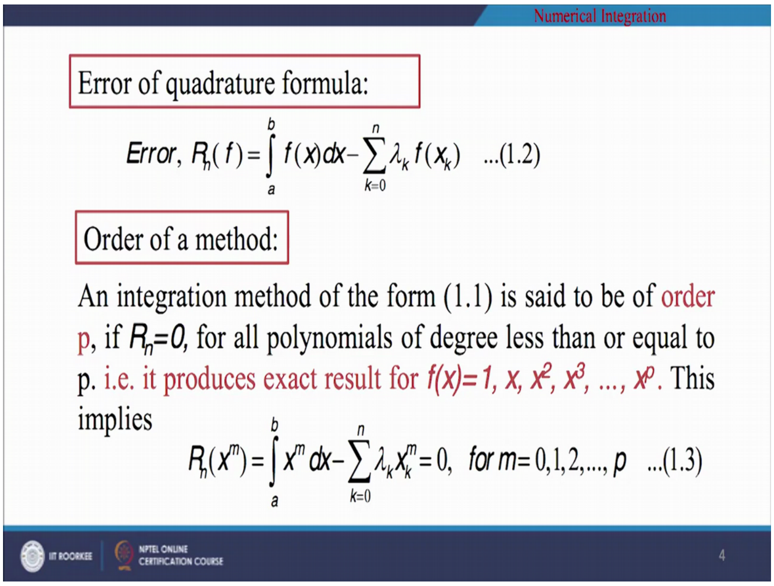

Obviously sometimes we are just approximating this formula like certain order of form here, order of a method we can justify. Suppose this function is exact, suppose some of this order of the polynomial here then we can just say that in integration method of the form like we have just express a to b, f of x this equals to summation of k equals to 0 to n, lambda k, f of x k plus R n f here, this can be of order p i R n equal R n of f equal to 0 for all polynomial's of degree less than or equal to p here.

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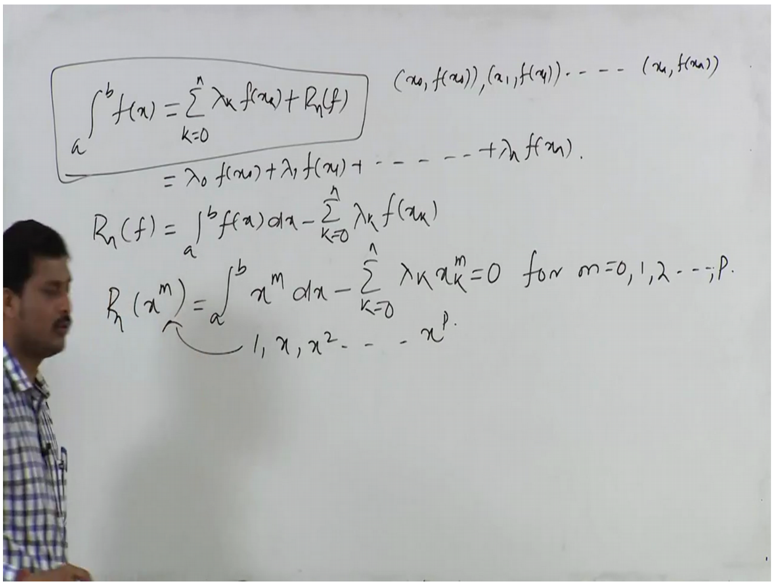

This means that it produces exact results for all polynomials that is in the form of f of x equals to 1, x, x square, upto x to the power p. This means we can just say that R n of x to the power m this equals to integration a to b, x to the power m dx minus summation k equals to 0 to n, lambda k, x k to the power m, this equals to 0 for m equals to 0, 1, 2, upto p here.

So then we can just say that this integration method is said to be of order p here. Exactly if you just put here like 1, x, x square, upto x to the power p, exactly this value will just give you the 0 value there. So that is why it is said to be of order p here. And if you will just try to find out this error term here, if immediate next term if you will just consider then that will just provide the error term here.

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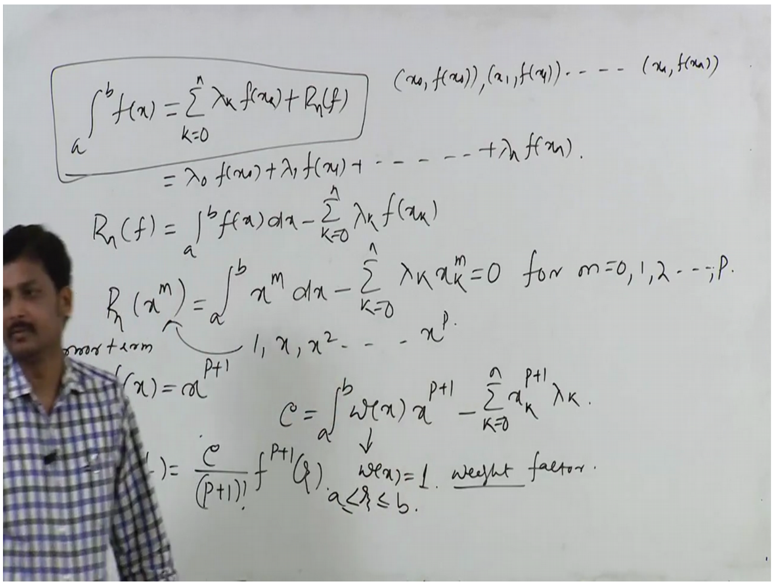

So obviously we can just write the error term for f of x p here, f of x equals to x to the power of p plus 1, this means that the error term for f of x is defined as we can just write C equals to integration a to b we can just write omega x, x to the power p plus 1 minus summation k equals to 0 to n, x k to the power p plus 1, lambda k here, where C is called error constant and omega x especially always it is chosen as 1 this is called weights factor here. So that is why this formula can be written also directly as integration a to b, x to the power p plus 1 minus k summation k equals to 0 to m, x k to the power p plus 1, lambda k here also.

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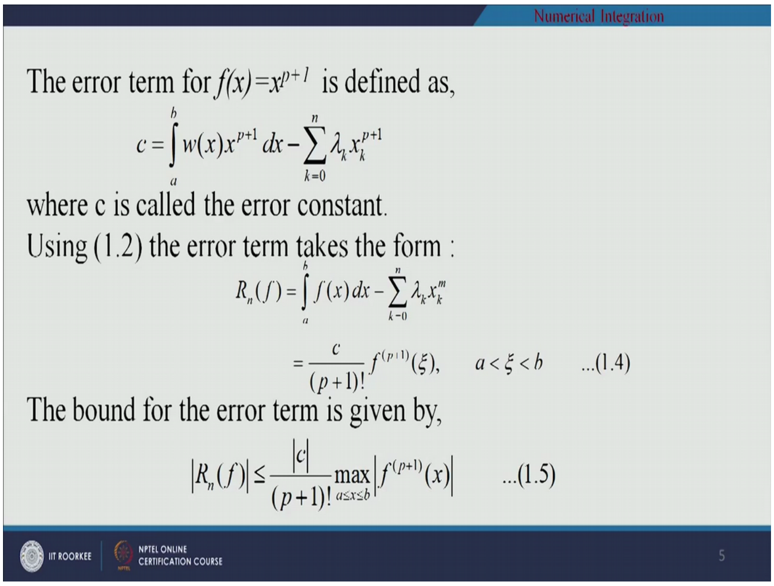

And if we will just calculate this total error term here, so if you will just use this integration formula directly we can just write that one as R n of f this equals to integration or since we have already calculated the difference of this one this can be written as C by p plus 1 factorial, f to the power p plus 1 zeta here, where zeta should be lies between a and b here or you can just write R n of f equal to integration a to b, f of x dx minus submission k equals to 0 to n, lambda k, x to the power x k to the power m this equals to C by p plus 1 factorial, f to the power p plus 1 zeta here, where zeta should be lies between 0 and 1.

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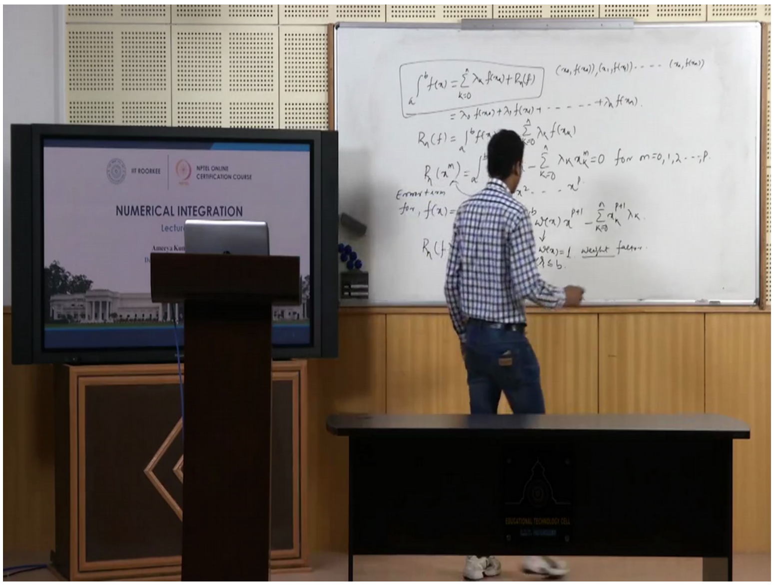

And if you want to find this bound for this error term here we can just find this absolute error term here this should be less or equal to absolute values of C by p plus 1 factorial and then immediate next term we can just find this into maximum of f to the power p plus 1, x for x lies between a to b here, this is completely defines the bound for the error term we can just define that one.

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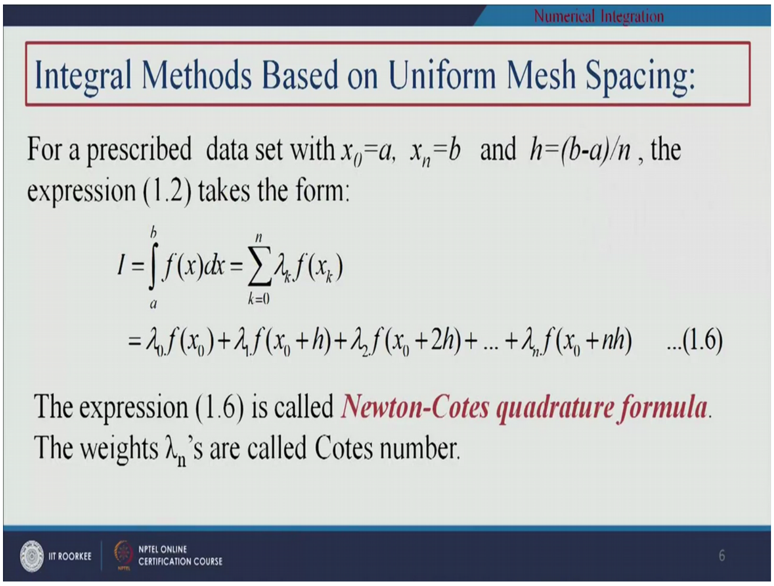

So then we will just go for this integral methods based on uniform mesh spacing, suppose you will have this tabular points like x 0, x 1 to x n here and all points are equi-spaced then how we can just evaluate this error using Quadrature formula that we will just discuss.

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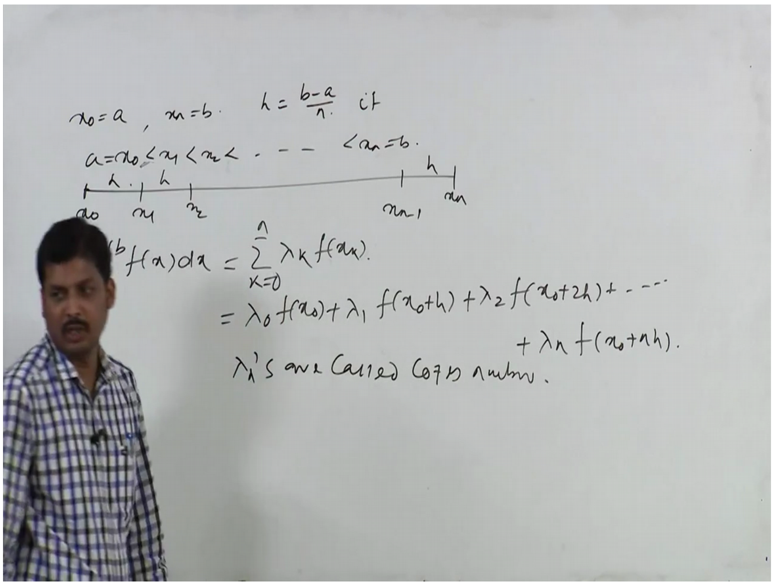

So if you will just go for this uniform mesh spacing here for a prescribe data set points like x 0 equals to a and x n equals to b supposed then we can just define your space size that is h equals to b minus a by n, if we can just define this tabular values that is in the form of like x 0 this equals to a is less than x 1, less than x 2 upto less than x n equals to b and all these points are equi-spaced suppose x 0, then x 1, then x 2 upto x n here, so all are equi-spaced here.

Then we can just write this integration i equals to integration a to b, f of x dx, this equals to summation k equals to 0 to n, lambda k, f of x k here. Usually we can just write this one as lambda 0, f of x 0, lambda 1, f of x 0 plus h, lambda 2, f of x 0 plus 2h, upto lambda n, f of x 0 plus nh here And this expression specially are called Newton-Cotes Quadrature formula and lambda n's are called Cotes number here, Cotes number and this formula is called Newton's cotes Quadrature formula.

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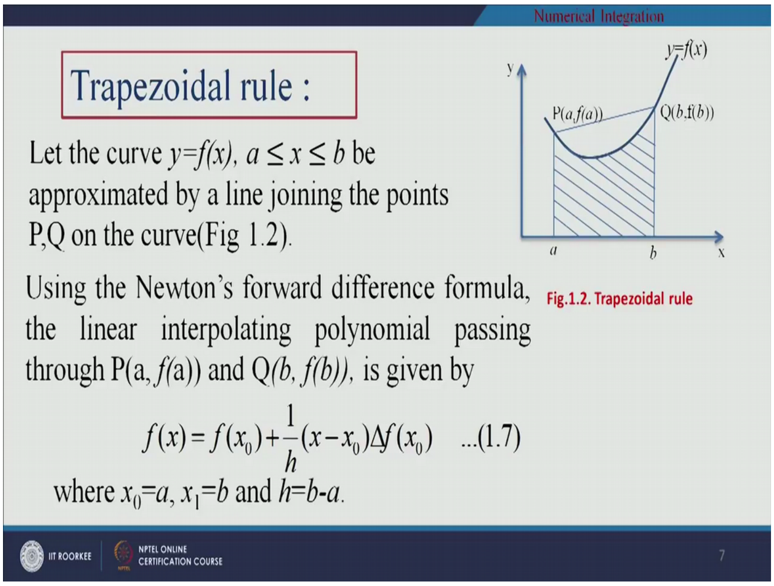

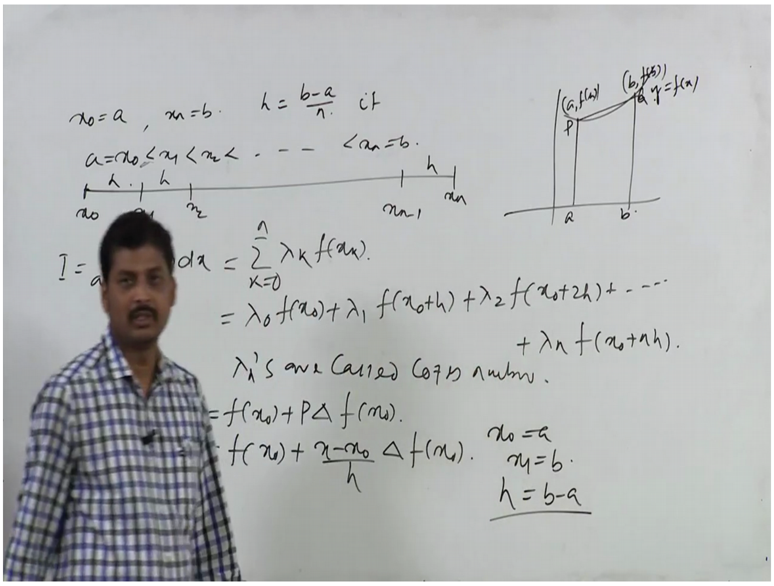

So first will just go for trapezoidal rule here, suppose we will have a curve here and it has this ordinates like a, f of a here and b, f of b here. So suppose this points are like p, q here, so if we want to find this area or this integration from a to b range with this function bounded by this curve y equals to f of x here then we can just estimate this area by a trapezium or trapezoid then we can just evaluate this integration in a complete form here.

Suppose this curve y equals to f of x is given to us and it is asked to find this integration within this range a to b suppose and this can be approximated by joining this line like p to q here in the curve and if you will just use this Newton's forward difference formula for a linear interpolating polynomial passing through this points like a, f of a and b, f of b here, then since we are just approximating this curved region by straight line that, so that is why you can just approximate by a linear interpolating polynomial there.

So if you will just interpolate this linear interpolating polynomial for this curve here then we can just write this formula as f of x equals to f of x 0 plus p delta of f of x 0 here. Since it is a linear interpolating polynomial after that we can just get the 0 values, so that is why we can just write this terms upto this series here. And here in terms of x if you just express this expansion here, so this can be written as f of x 0 plus x minus x 0 by h, delta of f of x 0, where we can just write x 0 equals to a here and x 1 equals to b here and your space size that is h can be defined as h equals to b minus a here.

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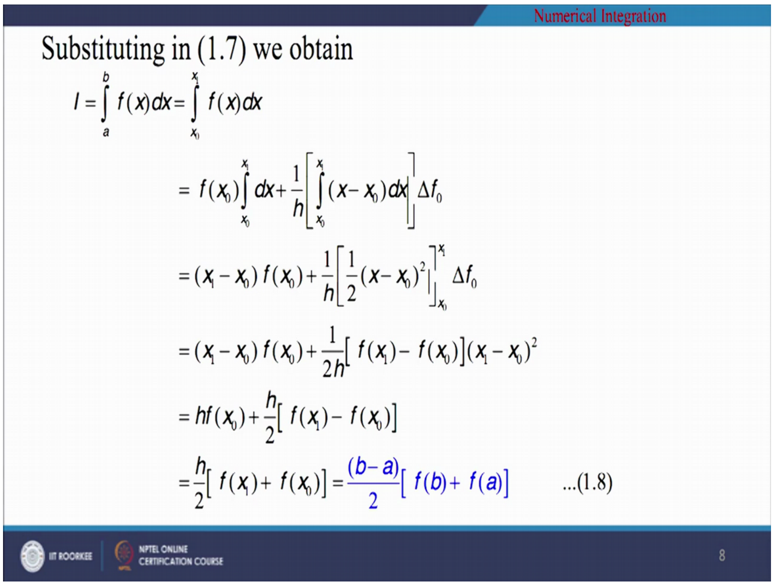

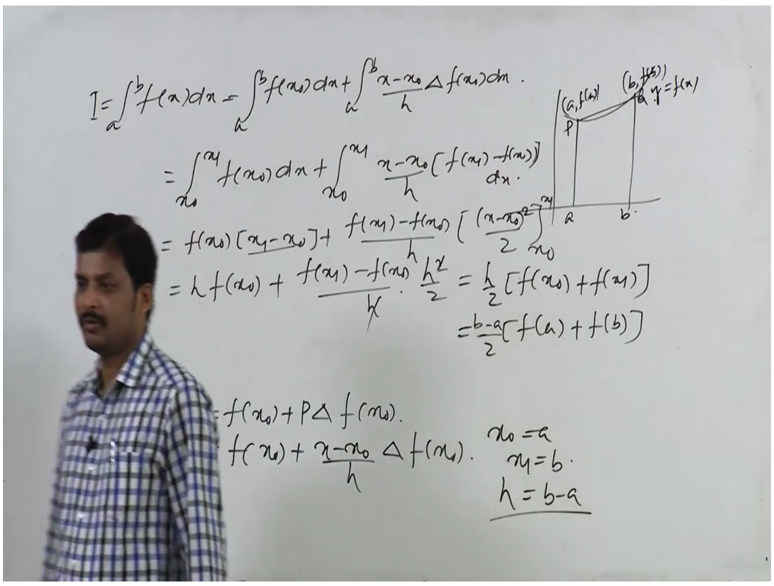

And then if you will just go for this integration of this function here over this range a to b, the integration from a to b, i can be written as integration a to b, f of x dx here and this can be written as integration a to b, f of x 0 dx plus integration a to b, p in terms of x if you just write this one so it can be written as x minus x 0 by h, delta f of x 0 dx here. And if you just replace this one in terms of x 0 and x 1 here, so this can be written as x 0 to x 1, f of x 0 dx plus x 0 to x 1, x minus x 0 by h, f of x 1 minus f of x 0, d of x here.

And if you just integrate this one this can be written as like f of x 0 and x 1 minus x 0 plus this can be written as f of x 1 minus f of x 0 by h and next one it can be written as x minus x 0 whole square by 2, x 0 to x 1. And if you will just put here x 1 minus x 0 equals to h, this can be written as h, f of x 0 plus f of x 1 minus f of x 0 by h into h square by 2. And finally if you will just write this can be written in the form of like h by 2, f of x 0 plus f of x 1 here.

And that it can be represented also in the form of like h by 2, f of a plus f of b if you will just replace here also h in terms of b and a, this can be written as, b minus a by 2 into f of a plus f of b here. So this is basically called 2 point Quadrature formula if you will just consider this linear interpolating polynomial which is interpolating this curve y equals to f of x in a particular region then we can just evaluate this integration here.

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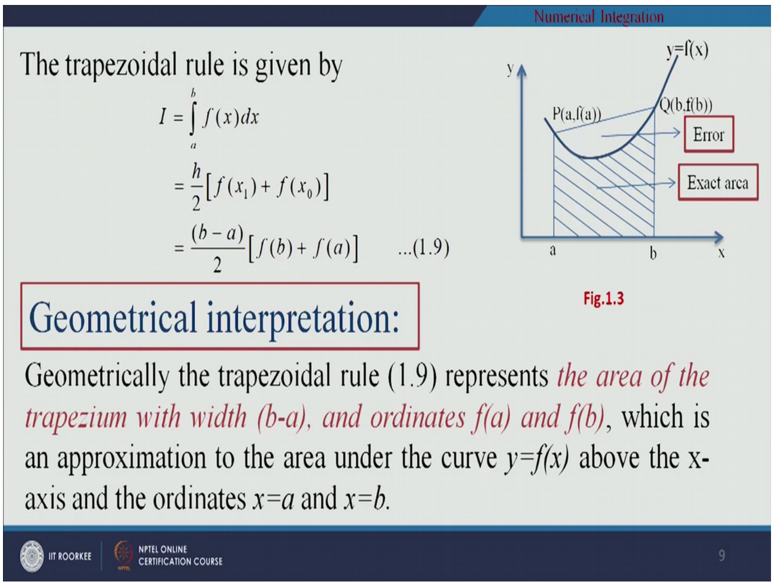

So next if you will just go for this geometrical interpretation that is we can just express this one that the area of trapezium if you will just see here with width b minus a and ordinates f of b and f of a, which is an approximate or which is an approximation to the area under the curve y equals to f of x above the x-axis and the ordinates x equals to a and b, this is nothing but representing the area that is b minus a by 2, f of a plus f of b.

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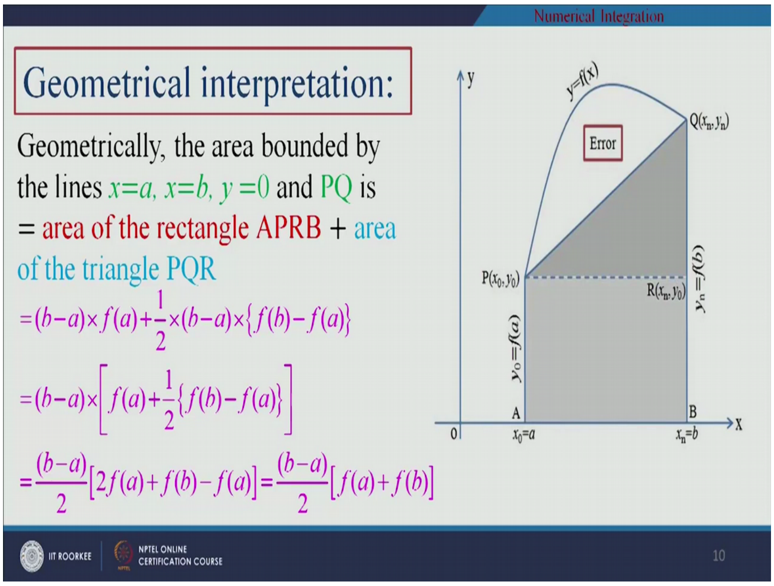

So if we can just go for the geometrical calculation of this area here we can just say that we will have this 2 regions here that is a rectangular region if you will see this one and and this is a triangle region here also. So if you will just calculate the area for this rectangular here and the area of the this triangle in that region then also we can just get this total area under this 2 sections are as, b minus a by 2 into f of a plus f of b here.

So if you will just see here geometrically if you just calculate this area then the area bounded by this lines x equals to a, x equals to b, y equals to 0 and pq line that is area of the rectangular APRB if you will just see plus area of the triangle PQR. So this is nothing but first this is total area of this rectangle we can just write b minus a into this height is f of a here, so that is why b minus a into f of a plus your triangular area that is half h into like b minus a. So usually we are just writing that is half into b minus a into f of b minus f of a here. So finally we can just get it as b minus a by 2 into f of a plus f of b here.

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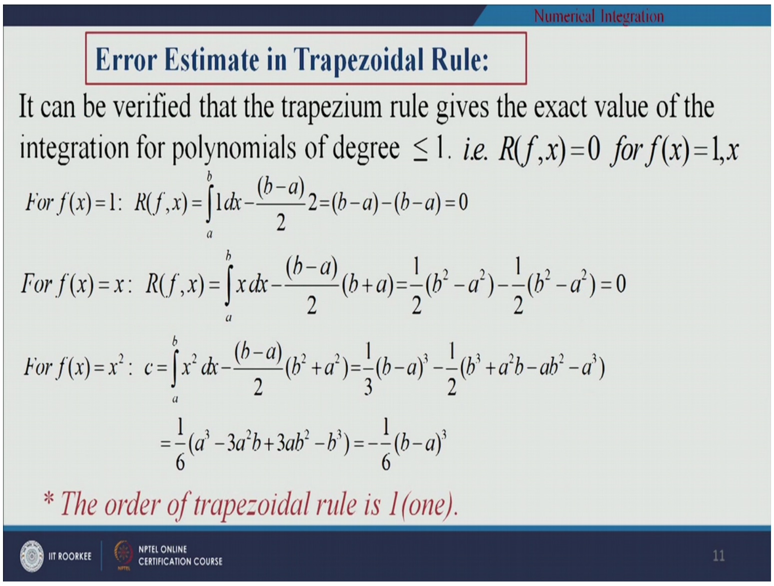

So if you will just go for this error estimation for this trapezoidal rule, so we can just verify that this error as R of f, x equals to 0 for fx equals to 1 and x. If already I have discussed that one, an integration method is said to be order p, if p is the largest positive real number for which the less numbers existing there that means 0, 1, 2 upto p all will give us the zero values there for the error term.

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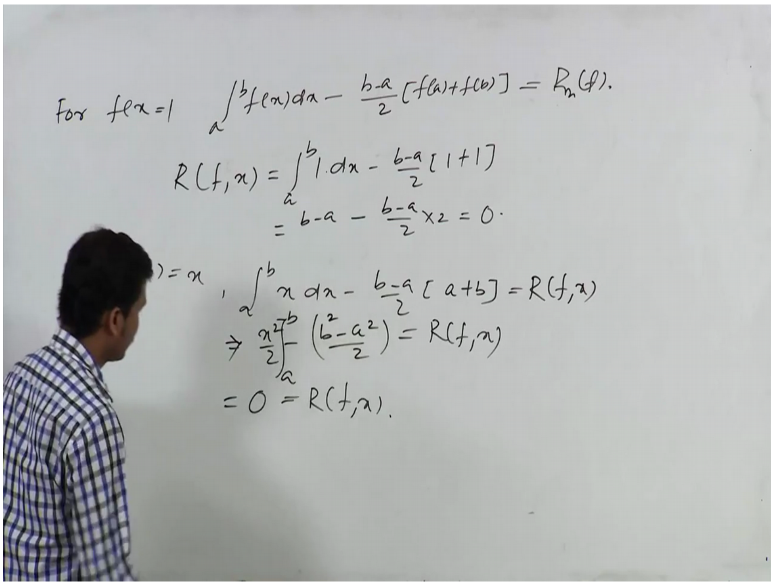

So then we can just express here for suppose f of x equals to 1 if you just consider then we can just express this one a to b, f of x dx minus our formula that is expressed as minus b minus a by 2, f of a plus f of b, or the value it is just providing that is nothing but the error function here. So if you will just see here that is R of f of x here which can be written as like x 0 to x 1 or a to b whatever you want you can just write there. So that is if you just write a to b, 1 into dx minus b minus a by 2, since f of a is 1 here, f of b is 1 here, then we can just express this one as b minus a minus b minus a by 2 into 2 here, so this is just giving you 0 here.

Similarly, if will just go for f of x equals to x here then also we can just express a to b, x dx minus b minus a by 2 and f of x equals to x here, so f of a equals to a here and f of b is equal to be here. So then if you will just evaluate this one for R of f, x here this implies that this can be written in the form of x square by 2 minus this is a plus b into b minus a here, so b square minus a square by 2, this range is a to be here, this is nothing but R of f, x here, which can be written as 0 also for R of f, x here.

So if you will just consider like f of x equals to x square then we can just obtain the value for C here. This means that for f of x equals to x square since we have considered this polynomial of degree one year or we are just approximating this function with polynomial of degree one here, then we can just get this exact 0 values or this polynomial just providing us also a polynomial of degree one which is exact or error term is zero for this polynomial of degree one here.

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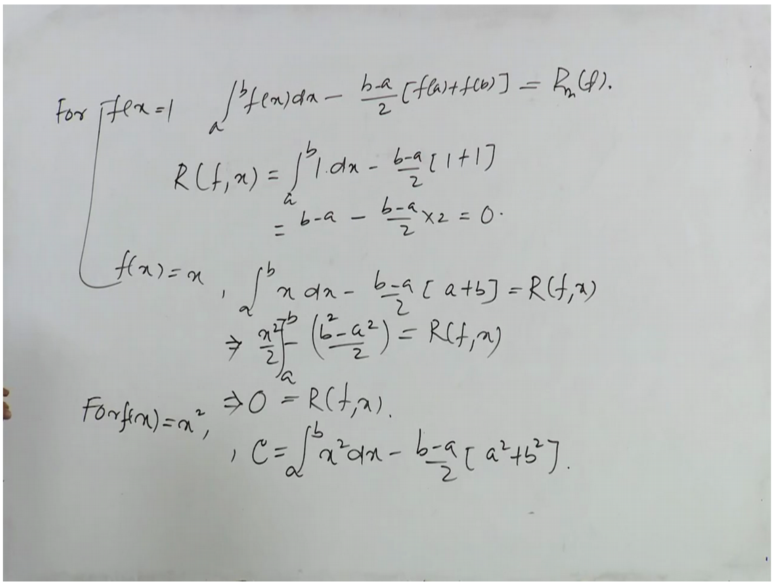

So that is why immediate to the next if you just consider this polynomial that is in the form of x equals to x square here or f of x equals to x square here then we can just get this error term here. If you just put f of x equals to x square here, C can be written as integration a to b, so first function that is x square dx minus summation or you can just use this formula for this trapezoidal rule here that can be written also as, b minus a by 2 and f of x is x square here, so we can just write a square plus b square here.

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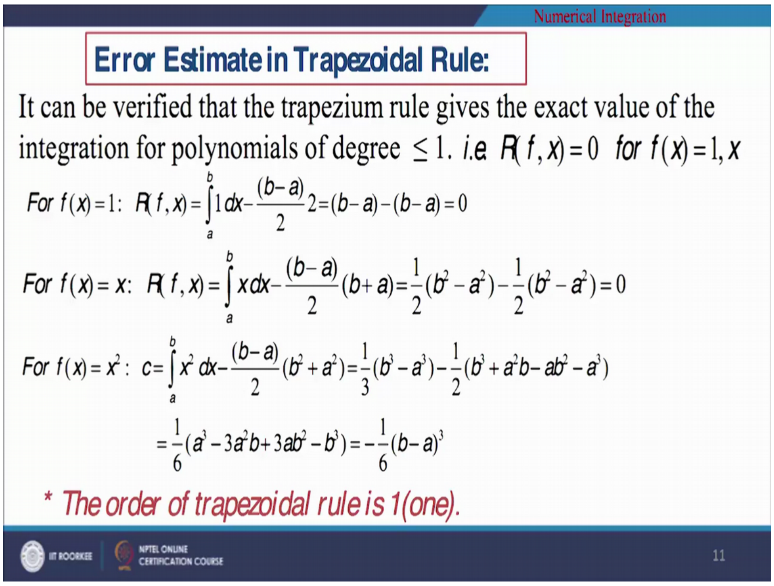

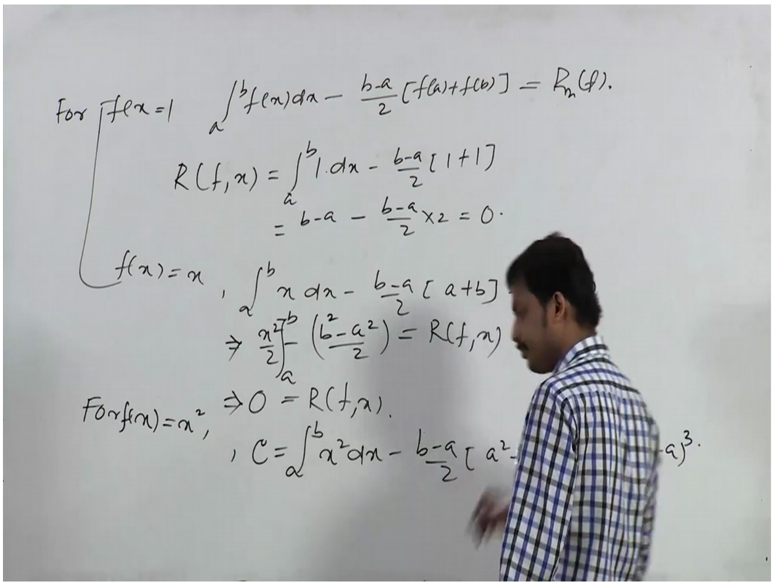

If you will just solve this equations here that is x cube by 3 it will just come, so you can just write b cube by 3 minus a cube by 3, first one or if you will put this values all of this values here that is in the form of like x to the power 3 by 3 here, then put all of this ranges here then you can just obtain the C value as minus of 1 by 6, b minus a whole cube here.

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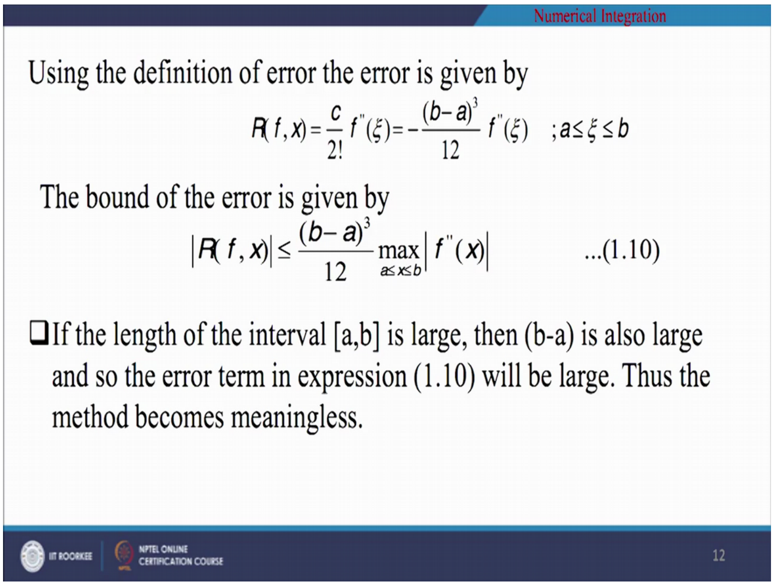

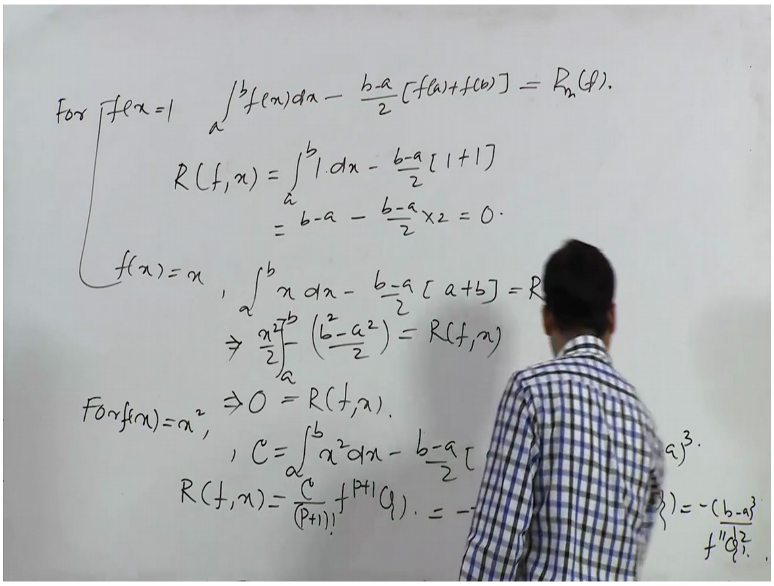

And if you will just put this C value in the error term here, so usually this are R of f of x for this one it can be written as, C by p plus 1 factorial, f to the power p plus 1 zeta. So p is especially 1 here, so that is why we can just write that one as, minus 1 by 6, b minus a whole cube, 1 by 2 here, f to the power double dash of zeta. Which can just give you here minus of if you will just see here minus of b minus a whole cube by 12, f double dash of zeta, where zeta should be lies between a and b.

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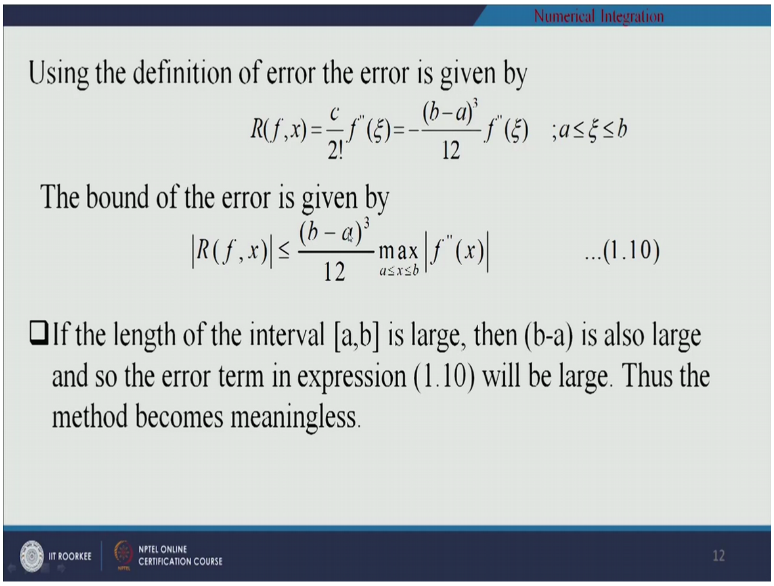

And if you will just go for this maximum bound of this error this can be written as R of f of x in magnitude form this will be less or equal to b minus a whole cube by 12, maximum of f double dash of x. And from this formula we can just visualize that if the length is larger suppose then we can just obtain this b minus a difference is also large and in that case this method becomes meaningless, if the like the section is very small then we will have good result.

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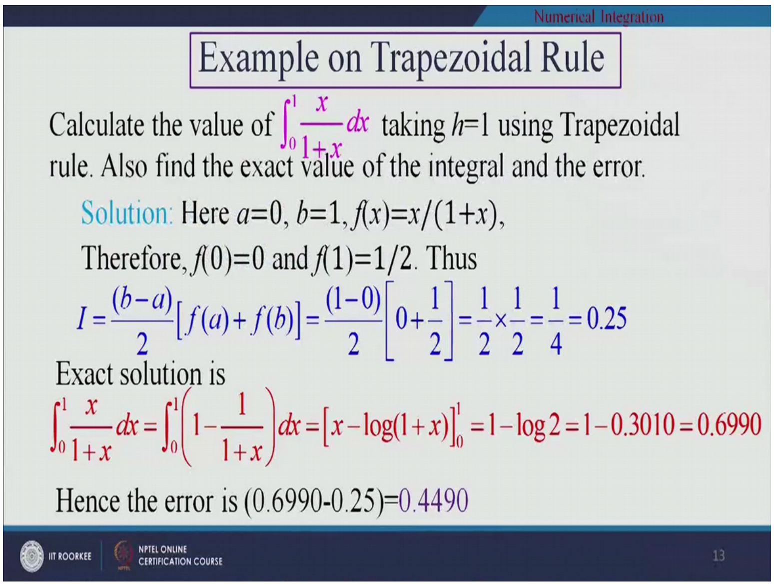

So then if you will go for example of trapezoidal rule here, so if the question is asked suppose find the value of integration 0 to 1, x by 1 plus x dx. We have just consider a simple example here taking h equals to 1 using trapezoidal rule than how we can just implement this formula that I will just discuss, also we can just find this exact value of integrals since it is a simple function I have just consider we can easily get this exact value and then we can just obtain the error by taking this approximated value calculated by using trapezoidal rule and exact value calculating in numerical form or analytical form.

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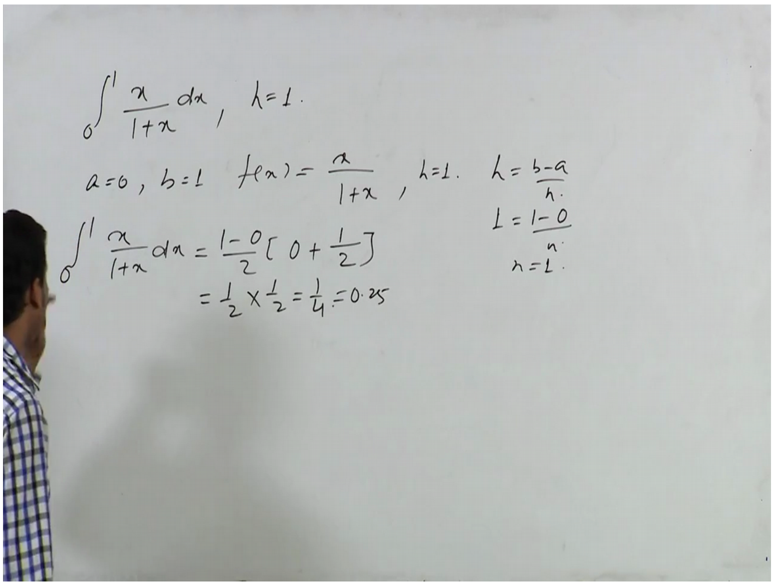

Suppose this integration is asked 0 to 1, x by 1 plus x dx here and this space size that is given as I think one here. So using trapezoidal rule solve this integral equation here if you just see here, so a equals to 0 here, b equals to 1, f of x equals to x by 1 plus x and space size h equals to 1 means n equals to 1 here. Usually we are just defining h equals to b minus a by n here, so that is h is given as 1, so 1 minus 0 by h here, so that is why we can just consider sorry n, so n equals to 1 here.

So we can just use trapezoidal rule by considering all this values whatever it has just given to us. So if you just use them we can just write integration 0 to 1, x by 1 plus x into dx, using trapezoidal rule this can be b minus a by 2, 1 minus 0 by 2, f of a, that is if you if I will just put here 0 that will especially give you 0 value here plus 1 by 2 here. And this can be written as 1 by 2 into 1 by 2 that is 1 by 4 here or especially we can just write this one as 0.25.

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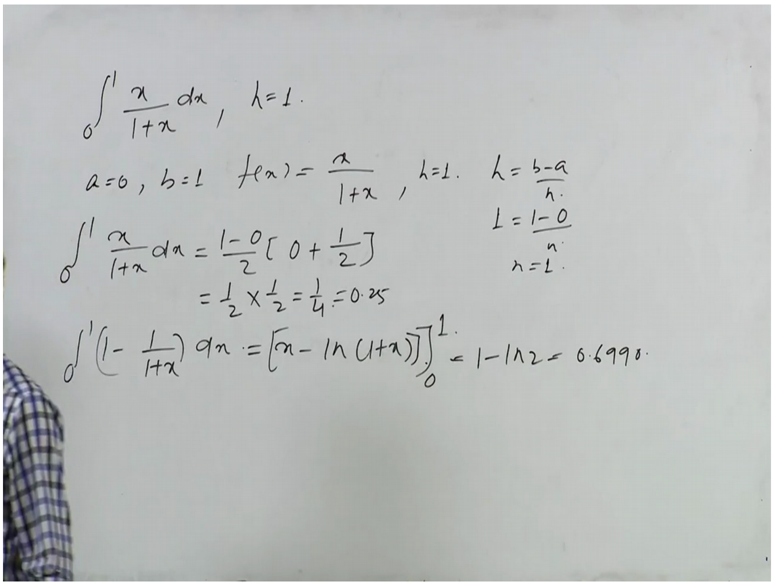

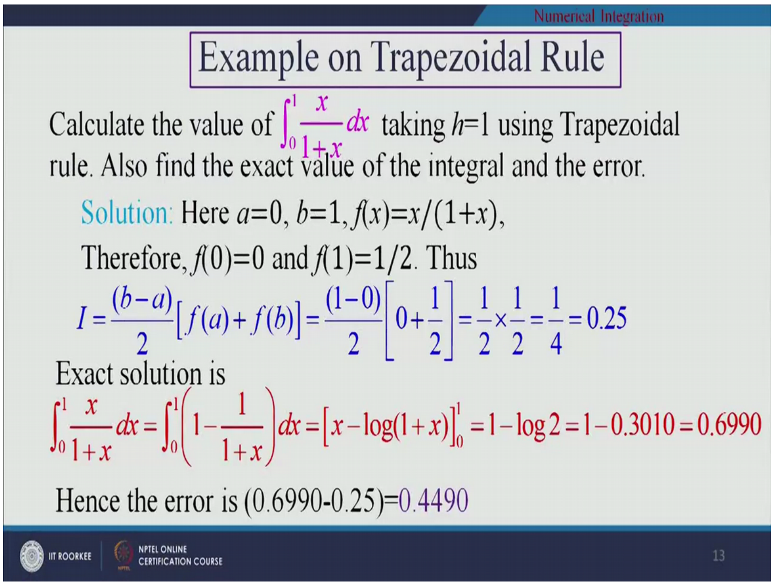

And if you will just go for this analytical solution here I can just write this one as 0 to 1, 1 minus I think 1 by 1 plus x here into dx and this can be written as like x minus l n of 1 plus x, 0 to 1 range here. So it can be written as like 1 minus (())(27:24) or L n 2, I can just say and this value is just giving you like 0.6990 here. And if you will just this difference then we can just find this error that is 0.6990 minus 0.25 the error is coming as 0.4490 here, since we are just taking h sizes is larger here that is why this error is very high.

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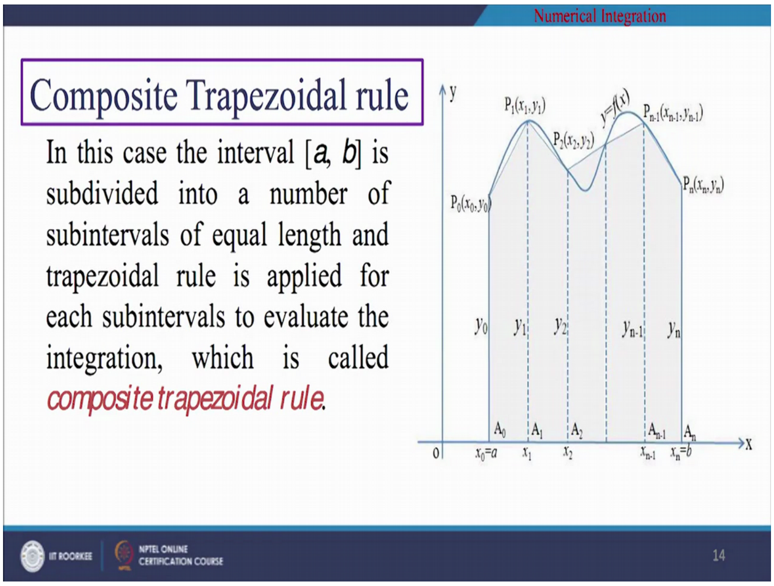

So then we will just go for composite trapezoidal rule, so in each of this interval we can just form a trapezoid it and then we can just approximate this integration in each of this ranges there that is basically called the composite trapezoidal rule here. So graphically you can just say that in each of this section a trapezium is formed there and area this trapezium is approximated in each of this intervals and then we can just combine each of this intervals and then in a composite form we can just obtain a composite formula.

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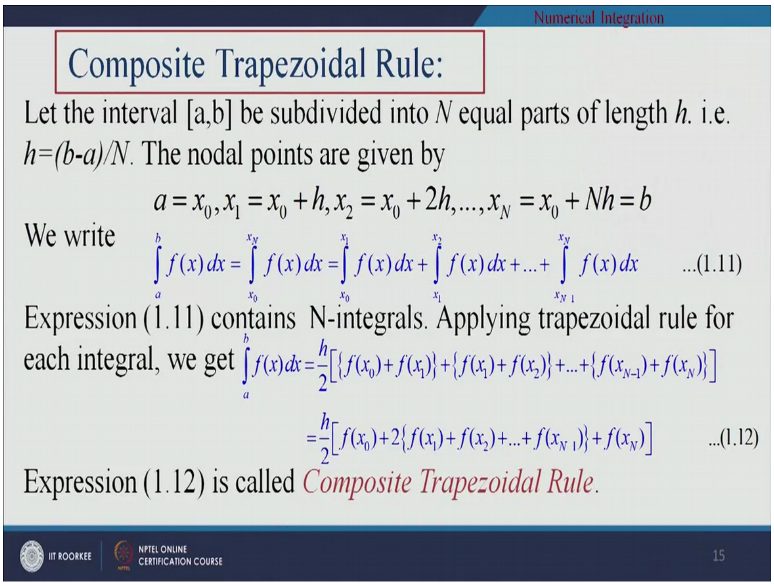

So if you will just consider this space sizes are all are equal here this means that if you will just starts from x 0, x 1, to x n here, so if you will just use this one this formula like a to b, f of x dx here, then we can just write this one as x 0 to x 1, x 1 to x 2 upto x n minus 1 to x n, f of x dx here.

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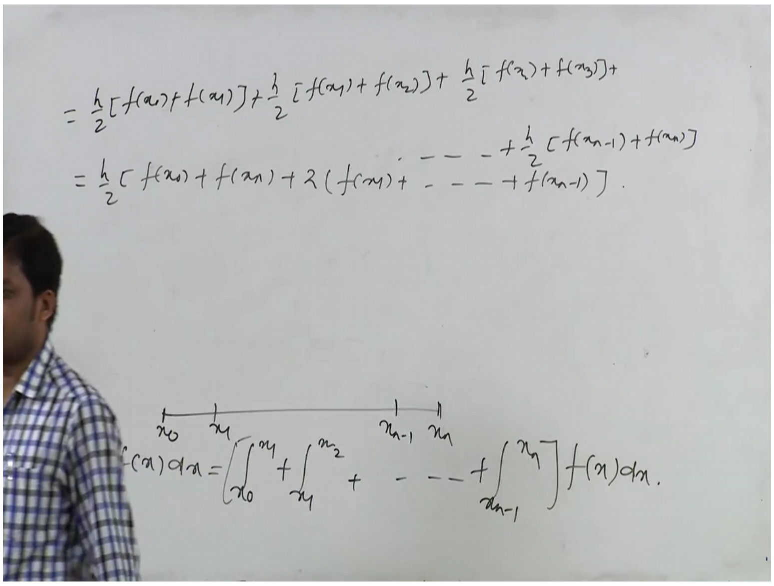

And each of intervals if you just write this formula here then we can just rewrite this formulation as, so 1st interval we can just write h by 2, f of x 0 plus f of x one here, 2nd interval we can just write h by 2, f of x 1 plus f of x 2, 3rd interval we can just write h by 2, f of x 2 plus f of x 3 plus the last interval if you will just write here f of x n minus 1 plus f of x n here.

So if you will just add up all this terms here we can just write this one as h by 2, f of x 0 plus f of x n plus 2 into f of x 1 to f of x n minus 1, this is basically called composite trapezoidal formula here. So with this I am just completing this lecture next lecture I will just go for this error computation of trapezoidal rule, thank you for the listening the lecture.