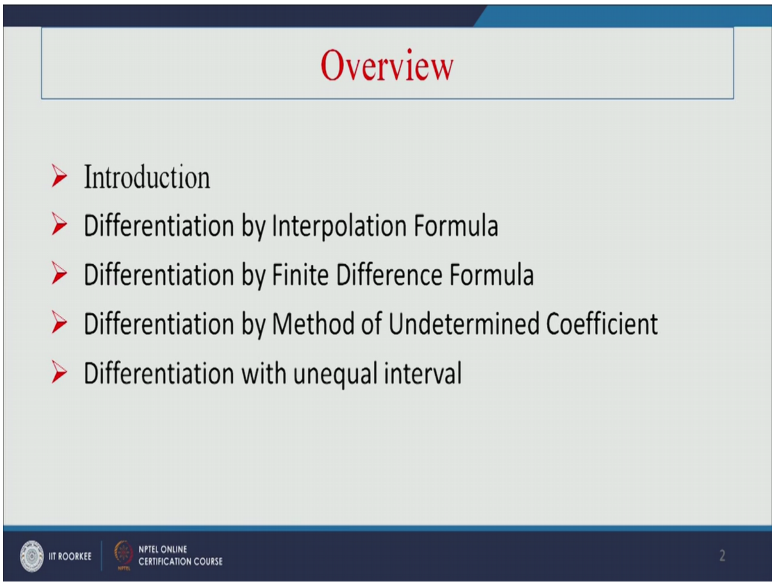
Numerical Methods
Doctor Ameeya Kumar Nayak
Department of Mathematics
Indian Institute of Technology Roorkee
Lecture 25
Numerical Differentiation Part I (Introduction to Numerical Differentiation)

Welcome to the lecture series on numerical methods. Today's lecture we will go for like numerical differentiation. In the numerical differentiation we will discuss about various interpolating polynomials and based on this like polynomial differentiation we can just evaluate these functional derivatives. This means that if we are just approximating a function with a polynomial that the derivative of this function can be written as a derivative of the polynomial also.

So that is why here first we will just go for the introduction section that how we can just approximate a function with a polynomial? Then we will just start this differentiation by interpolation formula, first for finite difference formulas, then we will just go for undetermined coefficient and in the last phase we will just discuss about this unequal interval.

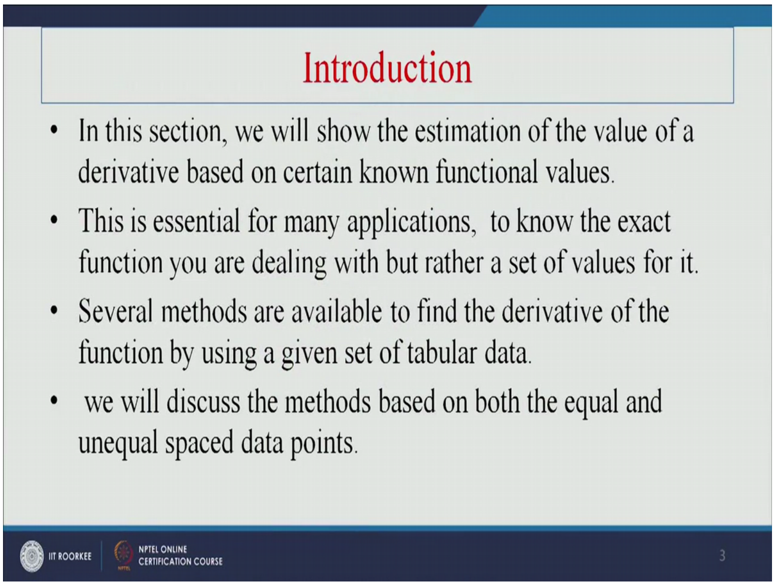
(Refer Slide Time: 01:14)



So in this section if you will just see that the estimation of the value of a derivative based on certain known functional values always certain value has been given and at that point usually we are just evaluating the derivatives. Sometimes if the exact function is not known to us then we can just approximate this function by a polynomial and we can just evaluate the derivatives for the polynomial and then we can just say that this derivative of this function can be represented in this form there.

So several methods are available to find the derivatives of the function by using given set of tabular data. Like if you will have this set of tabular data like x 0 y 0, x 1, y 1 to x n y n, even if the function is not known to us we can just evaluate this derivative of that function at certain points. So here we will discuss both these methods based on both equal and unequal spaced data points.

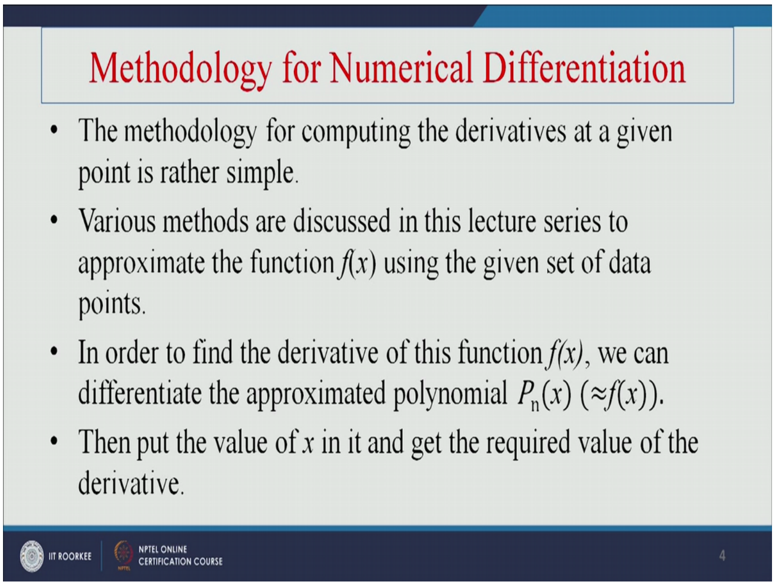
(Refer Slide Time: 02:15)



Since already in the last lectures we have discussed that some of these interpolation polynomials that deals with these equal space points, some interpolation polynomials (bad) are based on both these equal spaced points and unequal spaced points. And we have also discussed, what are the drawbacks? Or what are disadvantages of different interpolation (for) formulas?

So the methodology for computing the derivatives at a given point is rather simple. Since if you will just find these derivatives like d y by d x equals to f of x y, so various methods are available to evaluate these first order derivatives. But if the function is exactly known to you it is easy to evaluate.

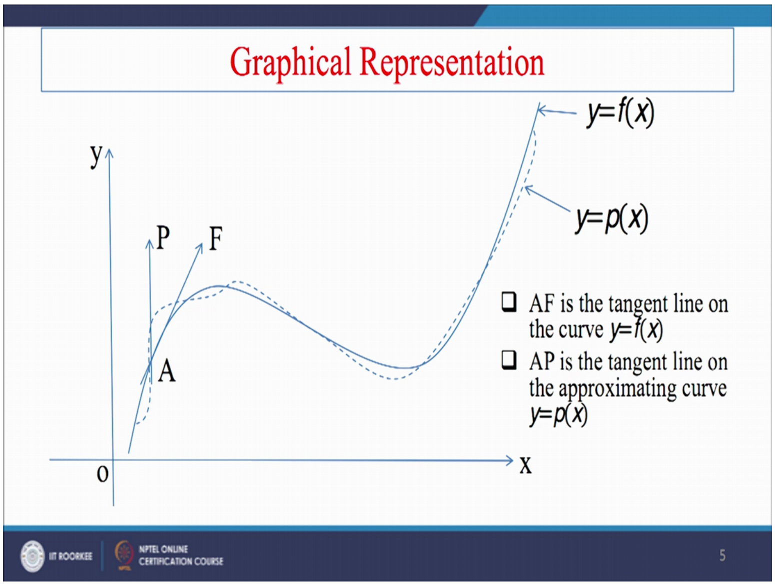
(Refer Slide Time: 02:54)



So we have already discussed like how we can just approximate the function with a polynomial in the last classes. In order to find this derivative of this function here we can just differentiate this approximated polynomial p n x with this function f of x here. This means that if we are just the tabulator points like x 0 y 0, x 1 y 1 up to x n y n here where only these tabular values are known to us but the function is not known to us.

Then we can just formulate a polynomial by considering all these tabular points here and from that polynomial we can just find the derivative for the function. And at certain points if it is required to evaluate these derivatives then at that point exactly we can just put in the polynomial derivatives there and we can just evaluate these derivatives for that function at that point. If you will just see this graph here that is y equals to f of x is the curve that we have dotted and this curve is approximated by a polynomial y equals to p x here.

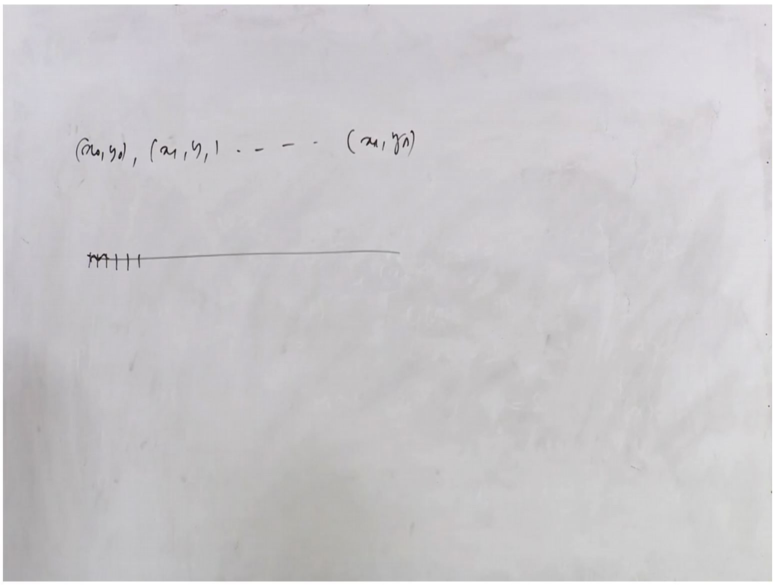
(Refer Slide Time: 04:07)



And if this is approximated if you will just visualise a point A in this graph here and if you will just plot the tangents at that points like AF and AP, we can find that these tangents are completely different. Then we can just say that sometimes these polynomials differentiation is completely differ from these approximated functional derivatives.

So but if you will just use a different technique to find these numerical differentiations by sufficiently closed to these tabular points then we can just obtain this derivative for the function is equals to the derivative for the polynomials. This means that we have to do this domain discretization in a sufficiently closed form that these tangents at a different points for both these functions it should be equal.

(Refer Slide Time: 05:00)

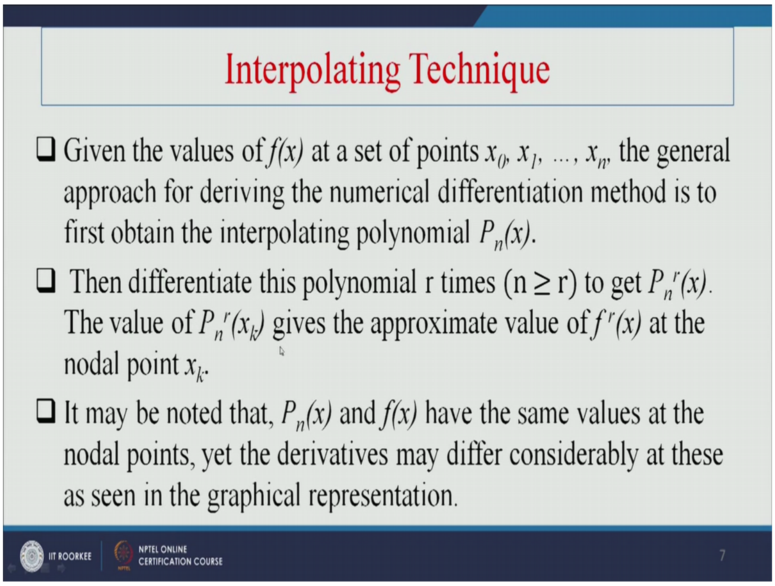


Then we can just say that this derivative for the function and derivative for these polynomials are equal. So first in this discussion we will just consider these equi-spaced points. Whenever the points are or the tabular points are equally spaced how we can use this differentiation there? Then we will just go for finite difference operators. Then we will just go for undetermined coefficients.

Suppose we will have this given set of data values of f of x like x 0, x 1 up to x n. In general approach we can just derive this numerical differentiation method first to obtain interpolating polynomial there. Then difference at this polynomial suppose r times if you will just differentiate this polynomial we can just write this polynomial as p n r x suppose. This means that first we are just approximating. If you will just see f of x is approximating with a polynomial of order p n x here.

This means f of x is approximated with a polynomial p of x of degree n. Then we can just write this nth order difference or this differentiation with respect to p n x with respect to r in the order of r can be written as p n r x here.

(Refer Slide Time: 06:25)

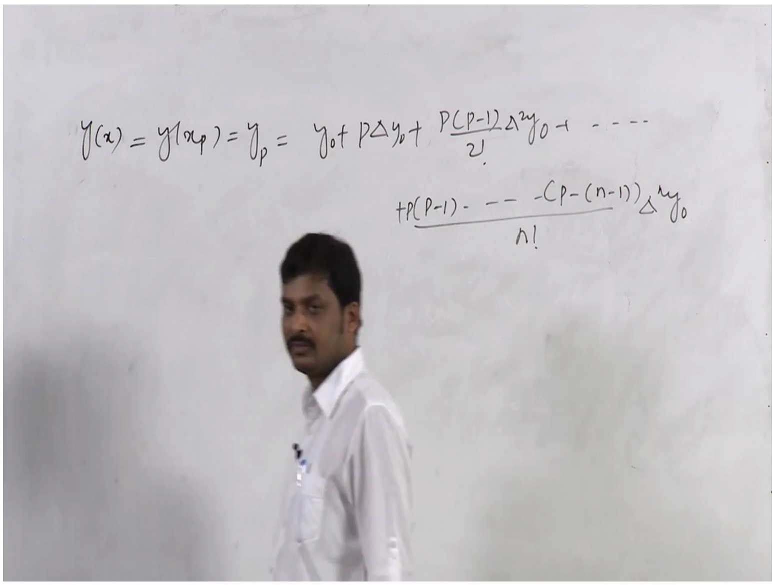


So then at a particular point we can just write this polynomial differentiation of order r as p n r of x k where this differentiation is evaluated at the point x equals to x k of order r there. It may be noted that p n x and f of x maybe sometimes they have the same values at the nodal points but the derivatives are different. That I have already shown in the graph.

First we will just go for this equi-spaced points and the first equi-spaced differentiation interpolation we will just do, sorry this differentiation with the interpolation we will just carry out here is Newton's forward interpolating polynomial. So basically this interpolating polynomial for Newton's forward difference formula is expressed in the form of like y for any point x p or usually you are just writing y of x or y at x p or we are just writing y p.

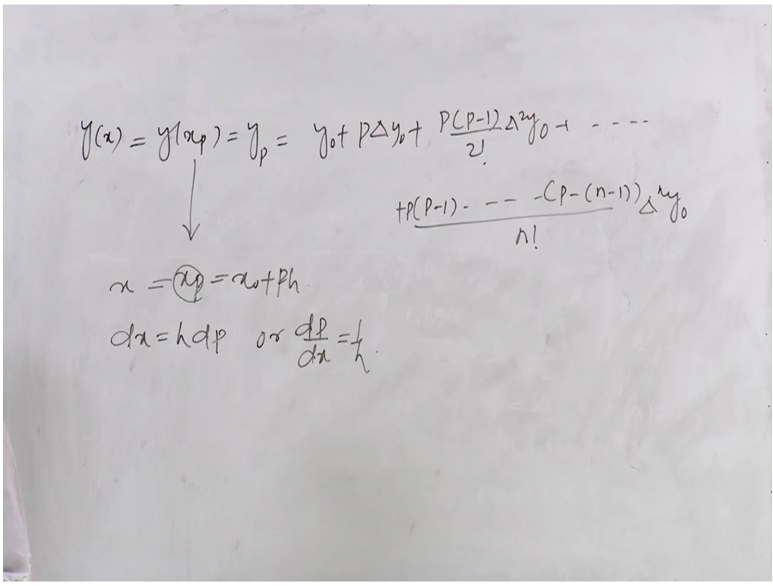
This can be written as in the form of like y 0 plus p delta of y 0, p into p minus 1 by factorial 2 del square of y 0 plus up to like p into p minus 1 up to p minus of n minus 1 divided by n factorial del to the power y 0 here.

(Refer Slide Time: 08:00)



So if we want to differentiate this one first we have to consider this x p point as x 0 plus p h here. And especially this x p is nothing but the undetermined point or the point where we want to find this interpolation (fo) polynomial or the formula. So at that point especially if we want to differentiate we can just write d x equals to h d p here. Or we can just write d p by d x that as 1 by h here.

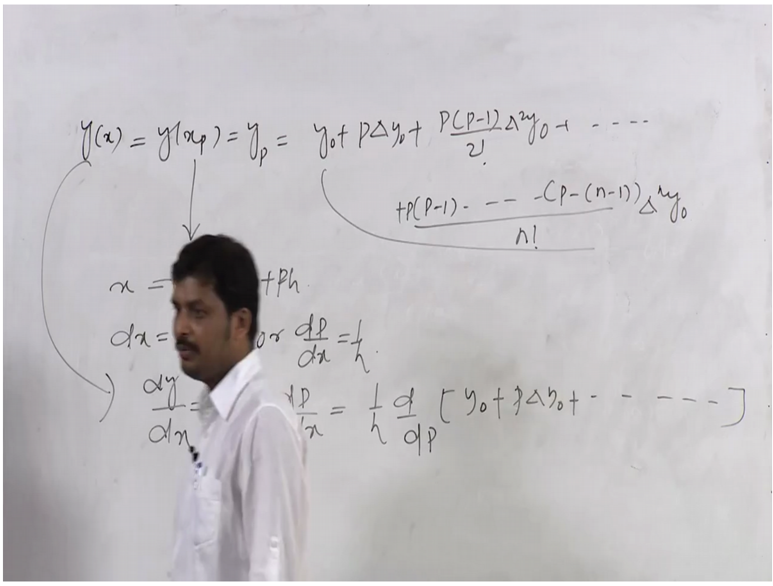
(Refer Slide Time: 08:46)



Either ways you can just define this first order differentiation here. So if you just write this differentiation for this polynomial here we can just write d y by d x, that is first differentiation of y with respect to x here. So we can just write d y by d p into d p by d x here.

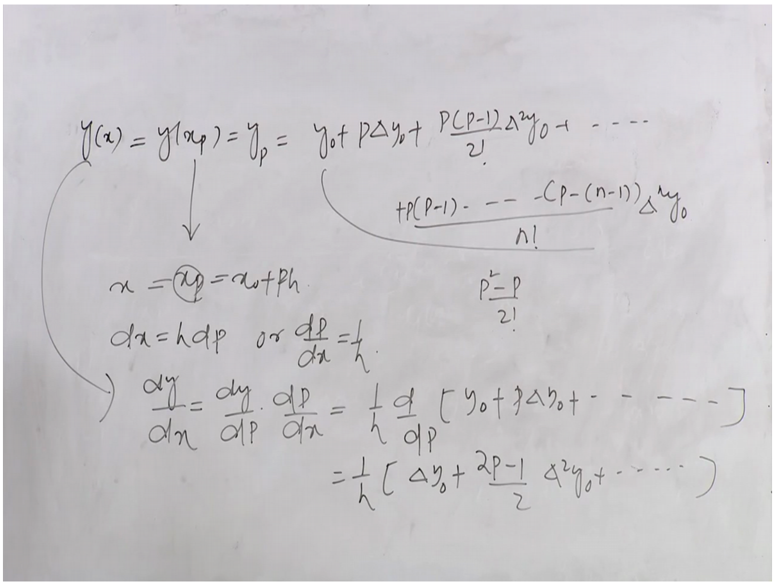
And especially it can be written as 1 by h since all of these points or this function whatever it is expressed here that is variable of p, so that is why it is easy to differentiate this function with respect to p here. So that is why we can just write this one as d by d p of y 0 plus p delta of y 0 plus all these points here.

(Refer Slide Time: 09:38)



And if you just differentiate that one so directly we can just write at this one as 1 by h. First point if you will just differentiate that will just give you 0 here. Second point we will just get as del y 0 here. Third point if you will just differentiate here that is in the form of like p square minus p by 2 factorial here. So it can be expressed as 2 p minus 1 by 2 del square of y 0 here. So likewise all of these points other points you can just differentiate there.

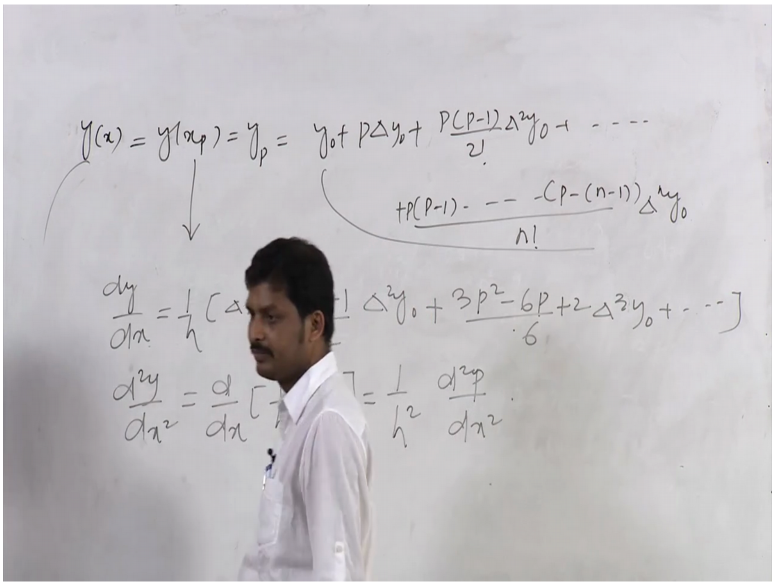
(Refer Slide Time: 10:13)



So in a complete form if you will just write this one then d y by d x can be written as 1 by h. So first point we are just writing here del y 0 plus 2 p minus 1 by 2 del square of y 0 plus 3 p square minus 6 p plus 2 by 6 del cube of y 0 plus likewise. Similarly if you will just go for this differentiation of second order we can just write the second order derivative as d square y by d x square.

This is nothing but we can just write d by d x of 1 by h, d p by d x here. So once more if you will just differentiate this one we can just write this one as 1 by h square, d square p by d x square here.

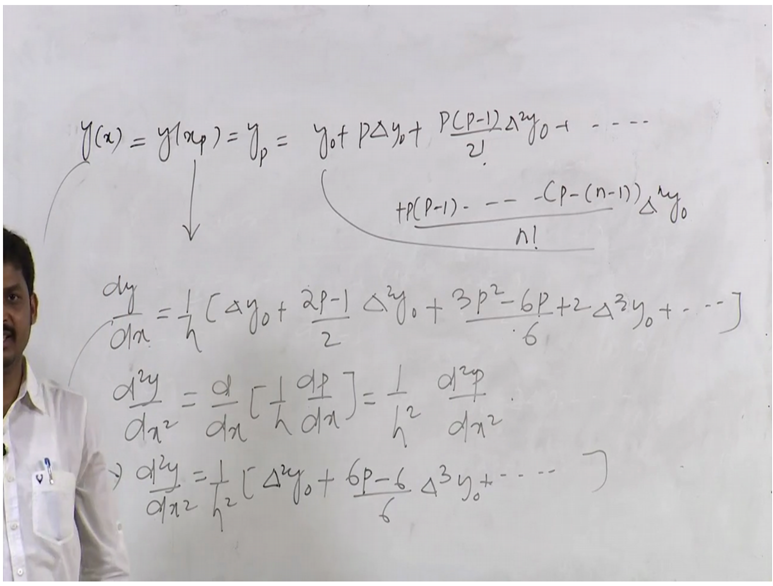
(Refer Slide Time: 11:17)



So if you will just go for this second order differentiation of this formula here, once more if you will just differentiate this one, we can just write this one as d square y by d x square. This equals to 1 by h square. So since once more we are just differentiating this one with respect to p here so this term will just give you 0 here, second term if you will just differentiate this one with respect to p here so 2 by 2 this is one here.

So del square of y 0 is the first term here. Then second one if you will just see here so 6 p minus 6 divided by 6 del cube of y 0 plus all other terms it would be carried out in the same fashion here.

(Refer Slide Time: 12:03)

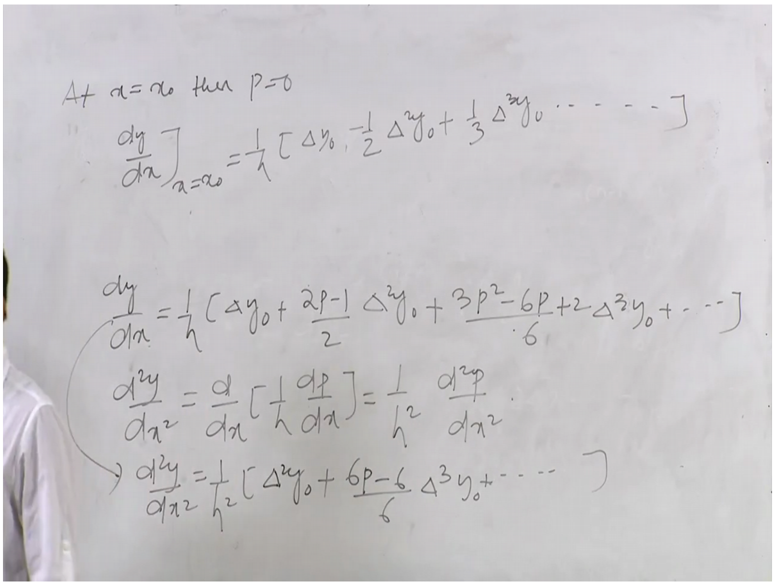


So this formula especially can be used to compute the first and second derivatives respectively near the upper end of the table. This means that at the beginning of the table if the data is supposed asked to find then we can just use this formula at that point. Since already in the previous classes we have discussed already that Newton's forward difference formula especially it is used if the tabular values are asked at the upper end of the table or the beginning of the table there.

So if suppose this tabular point is asked to compute this derivative exactly at suppose the nodal points. This means that if it is asked to compute at x equals to x 0 or x equals to x 1 especially we can just find that at that point exactly p equals to 0. So then we can just reduce these formulas as suppose it is asked to obtain the derivative at x equals to x 0 suppose. Then we can just say that p equals to 0 at that point.

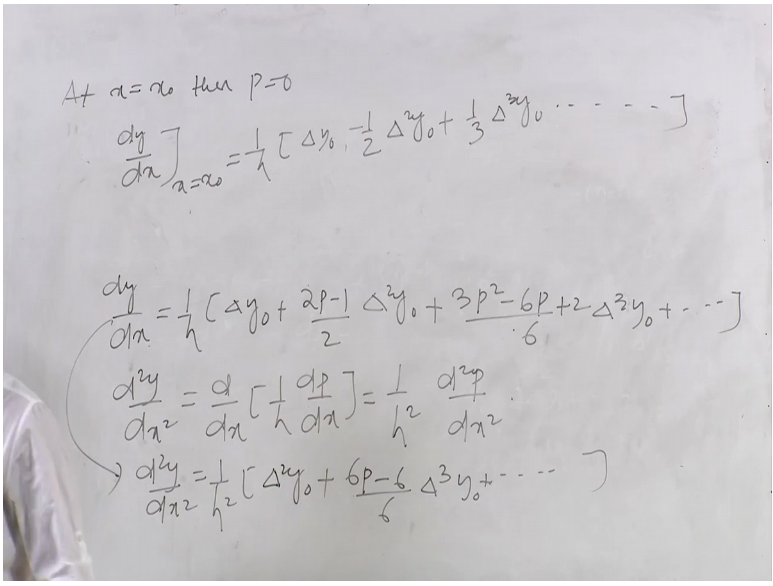
And we can just write d y by d x at x equals to x 0 as 1 by h delta of y 0. Since p is equal to 0 here we can just write this one as minus half delta square of y 0 plus, so 2 by 6 here. So that is why you can just write 1 by 3 delta cube of y 0. So all other points it will be considered in that form only.

(Refer Slide Time: 13:43)



Similarly if it asked to compute this d square y by d x square at x equals to x 0 then this formula can be written in the form like 1 by h square. If you will just see here this means that we are just obtaining o equals to 0 here. So that is why this can be written as del square of y 0 minus del cube of y 0. So all other points can be considered in the same fashion there.

(Refer Slide Time: 14:08)

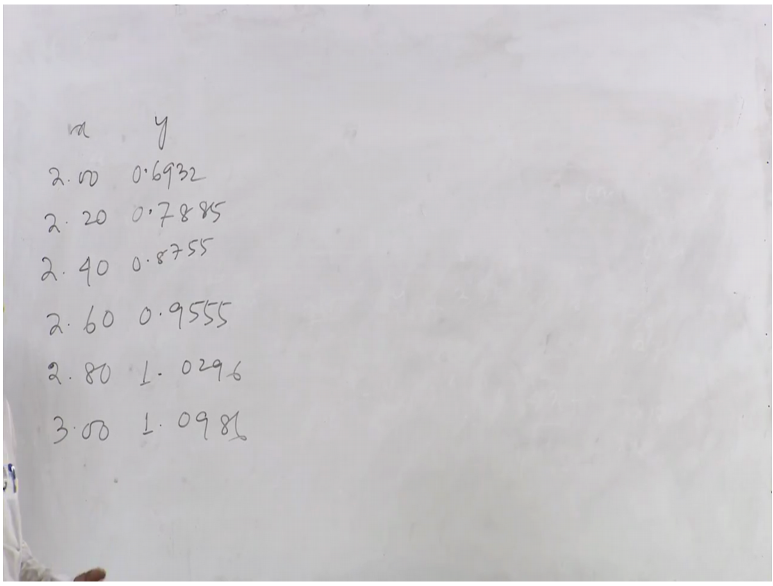


So it can be written in the form of like 1 by h square del square of y 0 minus del cube of y 0 plus 11 by 12 del fourth of y 0 minus all other terms. So first for this type of differentiation we can just consider one example here.

So if you will just consider this example that all the points are equi-spaced here, so if you will just take this tabular points like x y as 2 point 00, 2 point 20, 2 point 40, 2 point 60, 2 point 80 and 3 point 00 suppose and its functional values are expressed as like 0 point 6932 and 0 point 7885, 0 point 9555, then 1 point 0296, sorry one more point I have missed here. Just if you will write this point as 0 point 7885, at 2 point 40 it is 0 point 8755.

Then at 2 point 60 if the value is 0 point 9555 and at 2 point 80 suppose value is 1 point 0296 and at 3 point 0 suppose the value is 1 point 0986 suppose. So first differentiation easily we can just obtain since in the last lecture we have already explained that one.

(Refer Slide Time: 16:05)

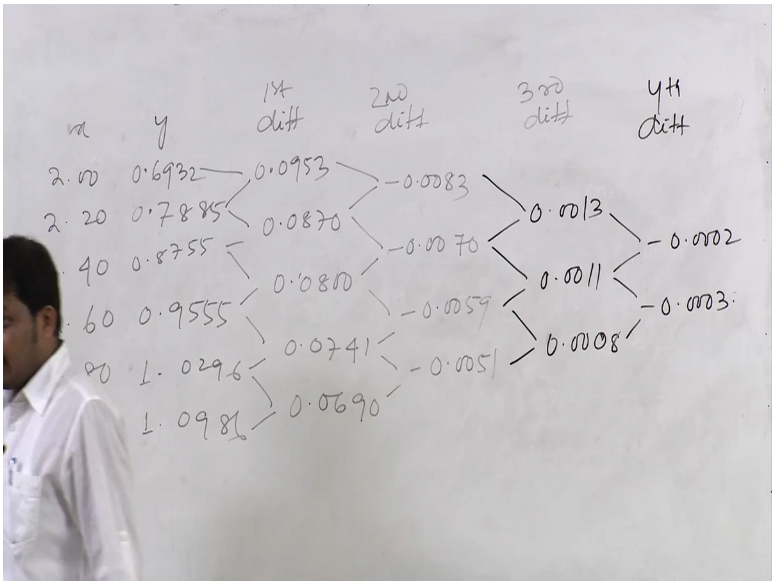


So we will just consider the (diff) difference of these two values here. So first differentiation we can just take the differences like 0 point 7885 minus 0 point 6932. So it can be written in the form of like 0 point 0953. This is the difference of these two values here. Similarly if you will just take the second difference values here like 0 point 0870. If you will just take the difference of these two values here so that can be written in the form of 0 point 0800.

The difference of these two values it can be written in the form of 0 point 0741, the difference of these two values it can be written in the form of like 0 point 0690. Similarly for the second difference we can just consider the difference of these two values here that will come as like minus 0 point 0083. Difference of these two values we can just write this as minus 0 point 0070 here. So third point we can just write minus of 0 point 0059 and last one we can just write minus of 0 point 0051 here.

For third difference if you will just see we can just take the difference of these two here and that can be written in the form of like 0 point 0013 here and if you will just take the difference of these two here, that can be written in the form of like 0 point 0011. If you will just take the difference of these two here that can be written in the form of 0 point 008. And in the fourth difference if you just take the difference of these two here that can be represented as minus 0 point 0002 and last difference this one also minus 0 point 0003 here.

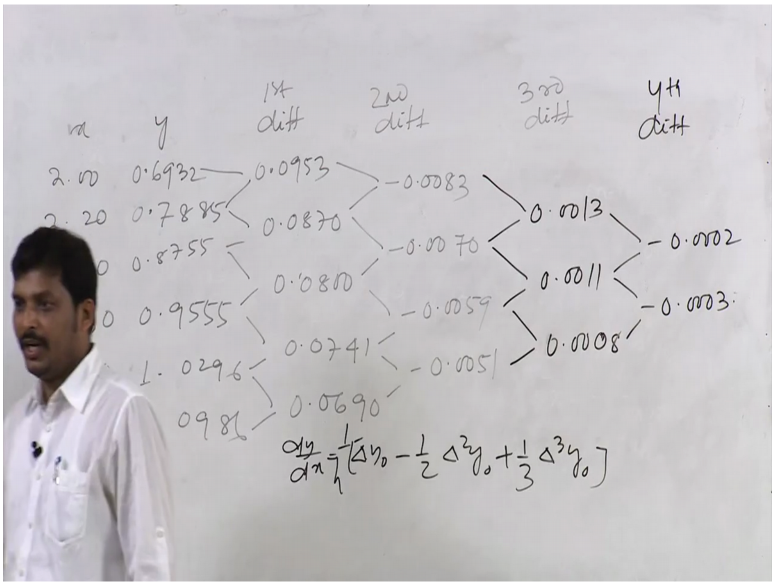
(Refer Slide Time: 18:26)



So if you will just put all these coefficients in the Newton's forward difference formula with differentiation up to third differences here, so fourth difference if you can just see that values are very small here. So that is why you can just consider these differences up to third differences here. Since the question is asked that using this following data find y dash and y double dash at 2 point 00.

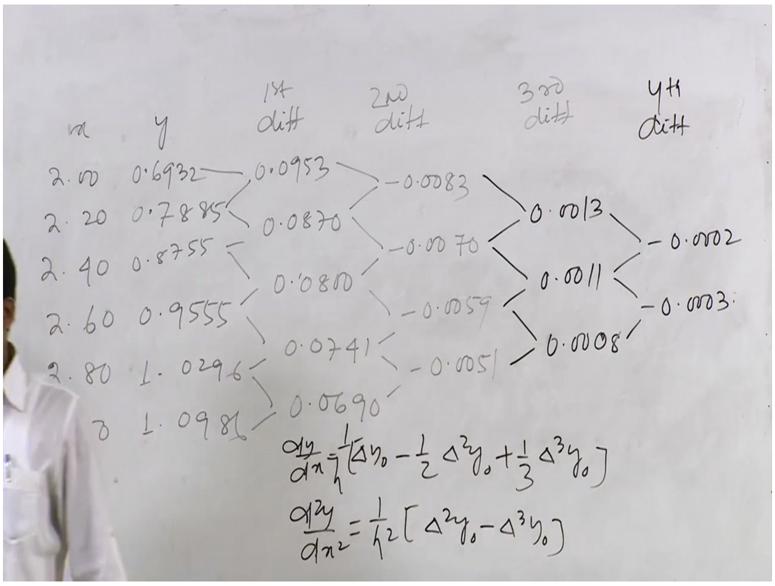
This means that at x equals to x 0 there using up to third difference only. So if you will just use here up to third differences we can just write this formula d y by d x that in the form of like, first term if you will just write here del y 0, first one is 1 by h here minus half del square of y 0 plus 1 by 3 del cube of y 0 here.

(Refer Slide Time: 19:30)



And since it is asked to evaluate up to third differences we can just write this term up to third differences here only. For the second order differentiation we can just write d square y by d x square this equals to 1 by h square. So first differentiation term here we can just get it as del square of y 0 plus del cube of y 0 here. Sorry this one is minus here.

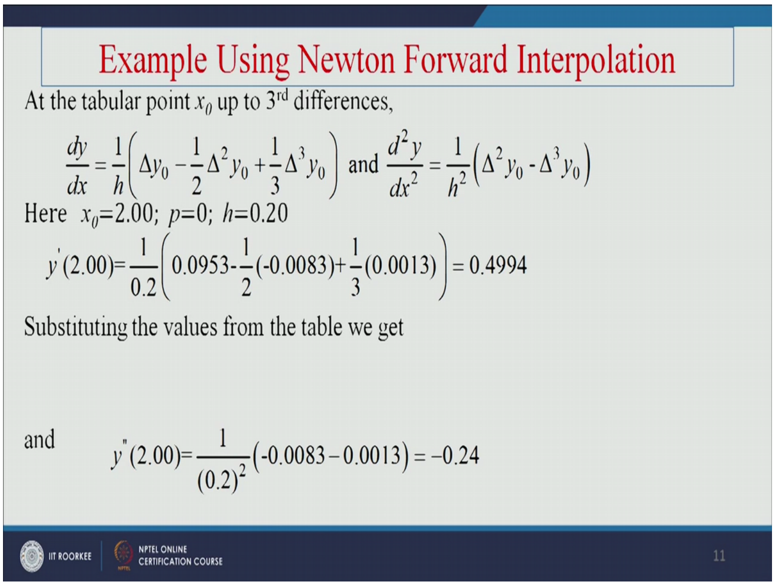
(Refer Slide Time: 20:04)



So if you will just put these tabular values for x 0 equals to like 2 point 0 here then we can just obtain this first difference term as like x 0 equals to 2 point 00. So h equals to 0 point 20 also. So that is why y dash at 2 point 00 it can be written as 1 by 0 point 2 into the first differentiation value that is 0 point 0953 minus half into minus 0 point 0083 plus 1 by 3 into 0 point 0013. This equals to 0 point 4994 here.

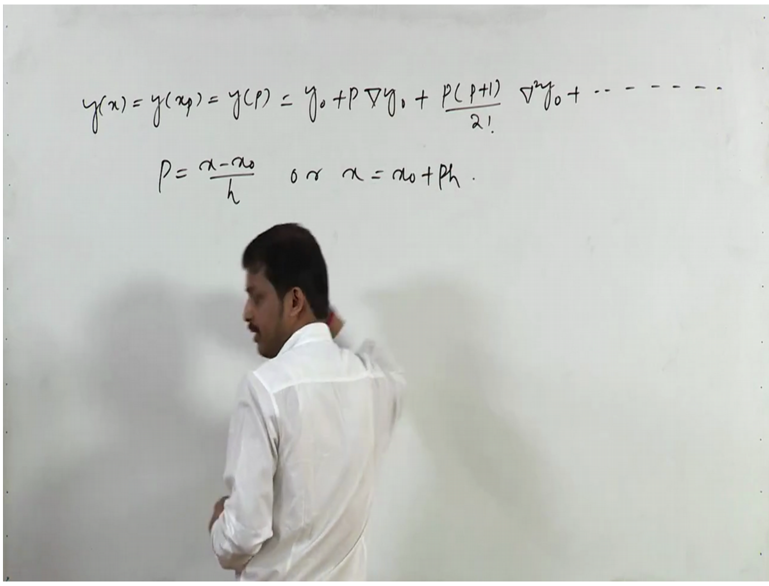
And if you will just use for the second order differentiation here then we can just find this as 1 by h square that is 1 by 0 point 2 whole square into del square of y 0 that is nothing but minus 0 point 0083 minus del cube of y 0, that as minus 0 point 0013 here. So the total value is minus 0 point 24 here.

(Refer Slide Time: 21:05)



So next we will just go for Newton's backward interpolating polynomial. In the Newton's backward difference interpolation formula especially this formula is written in the form like y of x or y of x p or especially y of p. It can be represented in the form of like y 0 plus p nabla of y 0 plus p into p plus 1 by factorial 2 nabla square y 0 plus all other terms are there. So if you will just write here p then p can be represented in the form of p equals to x minus x 0 by h here or x can be written in the form of x equals to x 0 plus p h here.

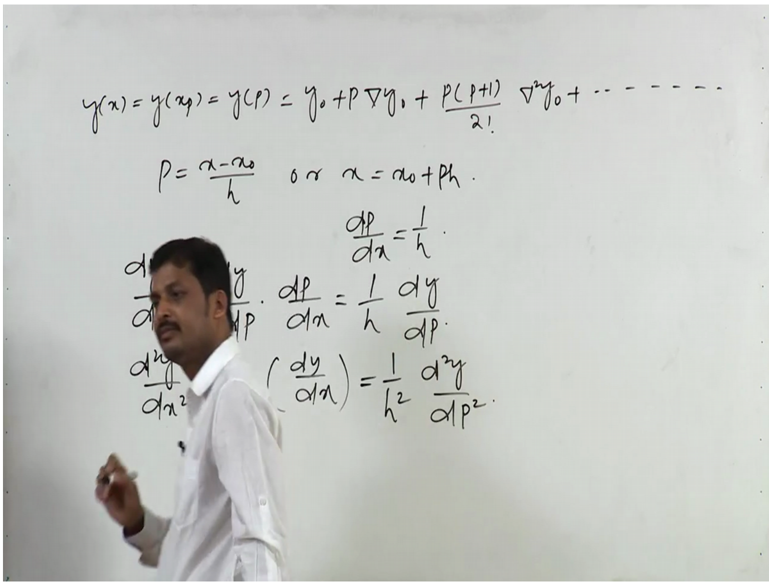
(Refer Slide Time: 22:08)



And if you just define here d p by d x here so d p by d x especially if you will just differentiate both the sides here this can be represented in the form of 1 by h here and if you will just go for this differentiation here for this function y of x here then we can just write d y by d x as d y by d p, since y is a function of p here, this into d p by d x here. And especially it can be written in the form of 1 by h, d y by d p here.

So then if you will just go for like second order differentiation here then this can be written as like d square y by d x square, this equals to d by d x of d y by d x. If you will just see here that is d y by d x can be replaced by 1 by h, d y by d p and then we can just write this one as 1 by h square, d square y by d p square here.

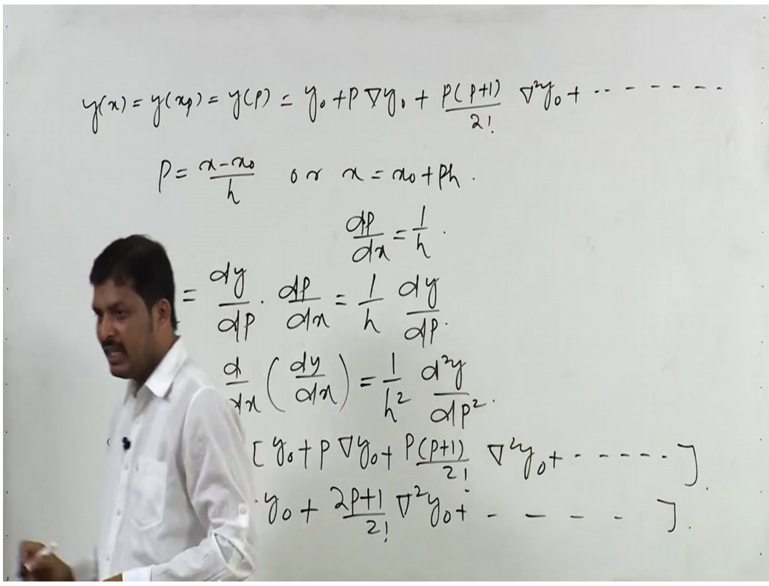
(Refer Slide Time: 23:23)



And if we can just put this formula or if we will just write this one in the form of like y of x there then we can just represent this one as d y by d x as 1 by h, d by d p of the complete formulation that as y 0 plus p nabla of y 0 plus p into p plus 1 by factorial 2 nabla square y 0 plus all other terms are there. And if you will just differentiate this one then we can just get this one as 1 by h.

So first differentiation this will just give you del y 0 since y 0 is a constant. That will just give you 0 value here. So then next one we can just write that as 2 p plus 1 by 2 factorial del square of y 0 plus the third order term we can just differentiate. We can just write there.

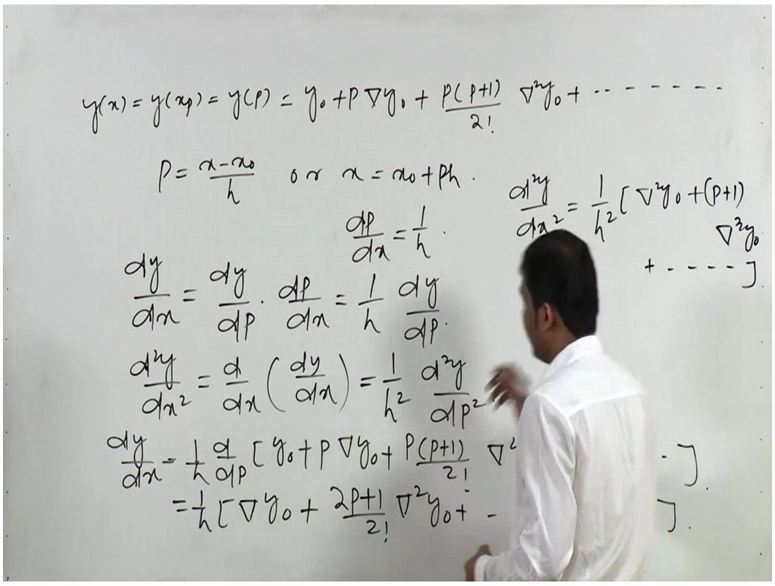
(Refer Slide Time: 24:27)



And similarly if we will just go for the second order differentiation here then we can just write d square y by d x square as 1 by h square. And one more differentiation for these functions if you will just write out here then the first value since del y 0 is a constant here then we can just write the second one as del square of y 0 here, since the differentiation if you will just take here 2 p means this is 2 by 2 it will just cancel it out.

So first term will be like del square of y 0 here plus if you will just see here then immediate next term it can be just represented as p plus 1 into nabla cube of y 0 plus all other terms are there.

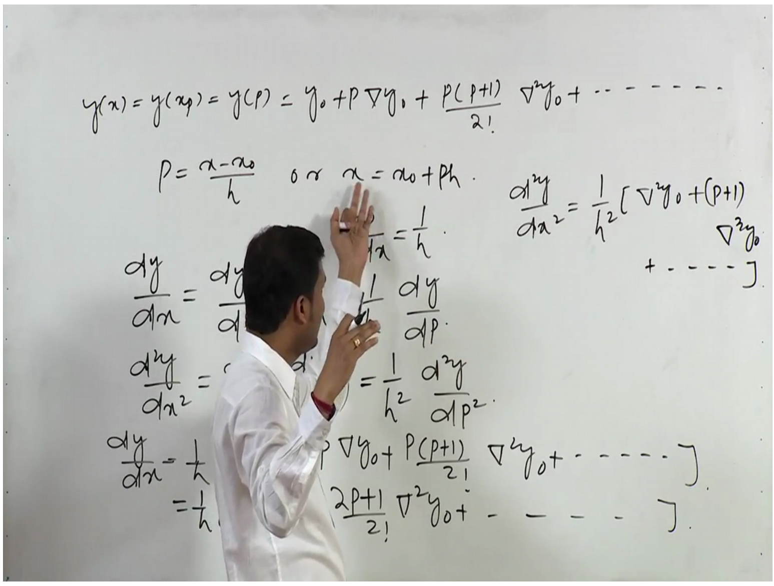
(Refer Slide Time: 25:15)



And especially if we want to find this formula at the upper end of the table since usually I have explained you in the previous lectures that this backward difference formula it is used at the end of the table. So that is why if it is asked to find the value near the upper end of the table or at the lower end of the table then we can just use Newton's forward difference formula and the backward difference formula there. Suppose if you will just use this tabular point at x is equal to x 0 suppose where p equal to 0.

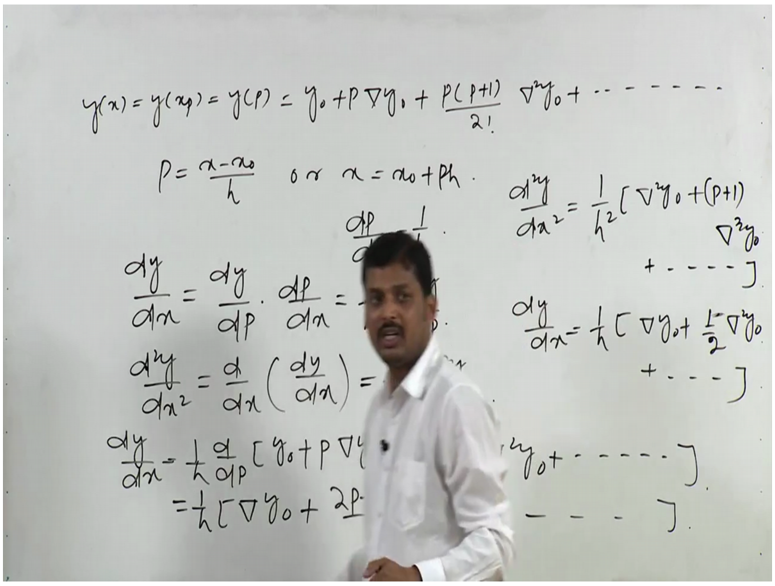
This means that we are just shifting this point to the lower end of the table and at that point only we are just using these p values. So that is why if you will just put at the lower end of the table x 0 then upper points it will be represented in the form of like x of minus 1, x of minus 2 up to x of minus n there. And if we will just put this means that if x equals to x 0 exactly then we can just put here p equals to 0.

(Refer Slide Time: 26:17)



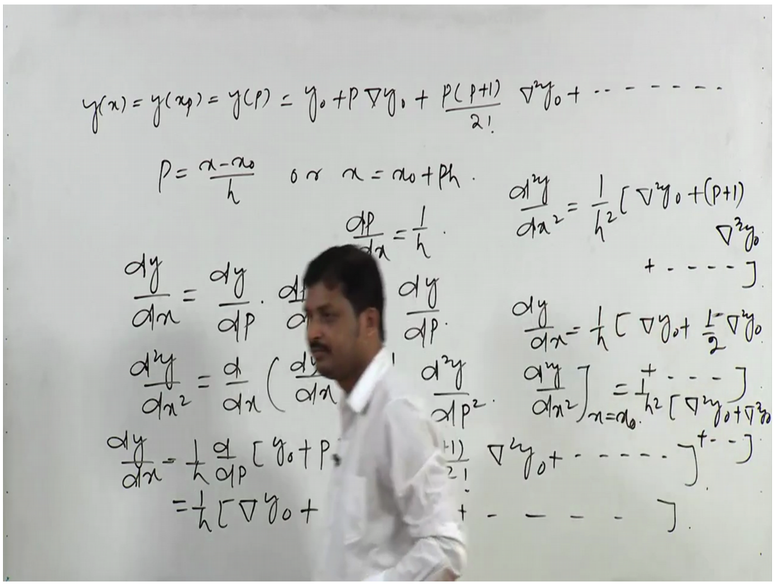
So that is why your formula can be reduced in the form of like d y by d x it can be represented as 1 by h, nabla of y 0 plus since p is equal to 0 here then we can just write this one as 1 by 2, nabla square of y 0 plus rest of the terms it can be represented from the formulation there itself.

(Refer Slide Time: 26:45)



Similarly if you will just put this p equals to 0 in the d square y by d x square here then we can just obtain this value d square y by d x square at exactly x equals to x 0 at the lower end of the table. And at that point we can just say that 1 by h square, so first value it will be nabla square y 0 plus if p equals to 0 here then this will just give you nabla cube of y 0 plus all other terms are there.

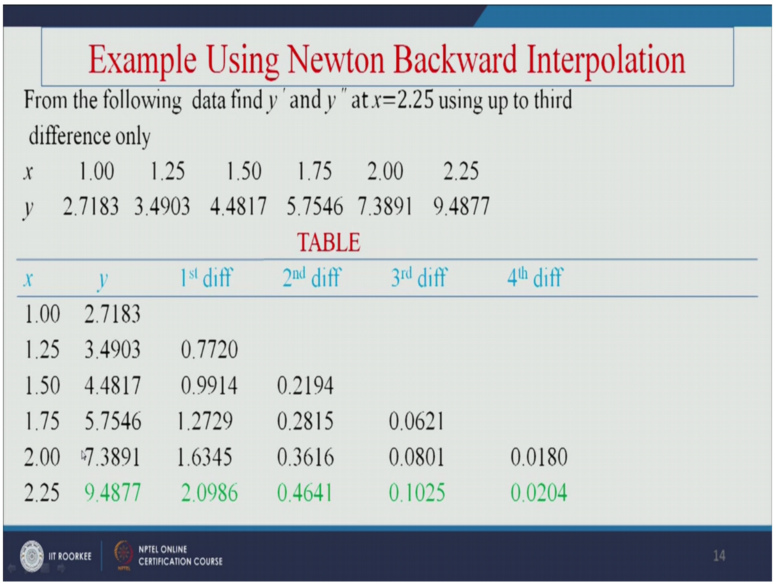
(Refer Slide Time: 27:20)



And immediate next term if you will just write this one then it can just give you 11 by 12 del to the power 4 of y 0 here. And if you will just go for the set of data points like the data points is given like 1 point 00, 1 point 25, 1 point 50, 1 point 75, 2 point 00, 2 point 25 and their corresponding y values are like 2 point 7183, 3 point 4903, 4 point 4817, 5 point 7546, 7 point 3891, 9 point 4877.

And if you will just use this backward difference formula so in the backward difference formula if you will just see first we are just taking this first difference that is nothing but the difference of first two values. But we have to consider these values like if we can just write x 0 at the end of the table here then our tabular values will be followed like from the bottom to the up of the table. So that is why we can just consider these tabular values that in the form of.

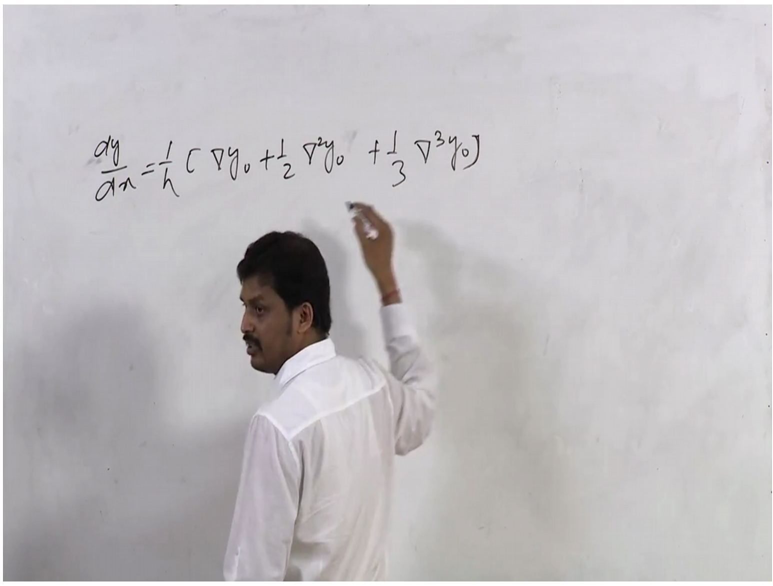
(Refer Slide Time: 28:29)



So if you will just use these tabular values that in the form of like 1 point 00 as 2 point 71 and take the differences like 3 point 4903 minus 2 point 7183 then it can just provide the values as 0 point 7720 here. If you take the difference 4 point 4817 minus 3 point like 4903 then it can just provide us the value 0 point 9914. So likewise the differences we can just calculate.

And if you will just use the differentiation formula here since the question is asked to find y dash and y double dash at x equals to 2 point 25, since it is at the end of the table using up to third differences. So up to third difference formula we will just consider. Up to third differences this formula can be written in the form of like d y by d x that as 1 by h, nabla of y 0 plus half nabla square of y 0 plus 1 by 3 nabla cube of y 0 here.

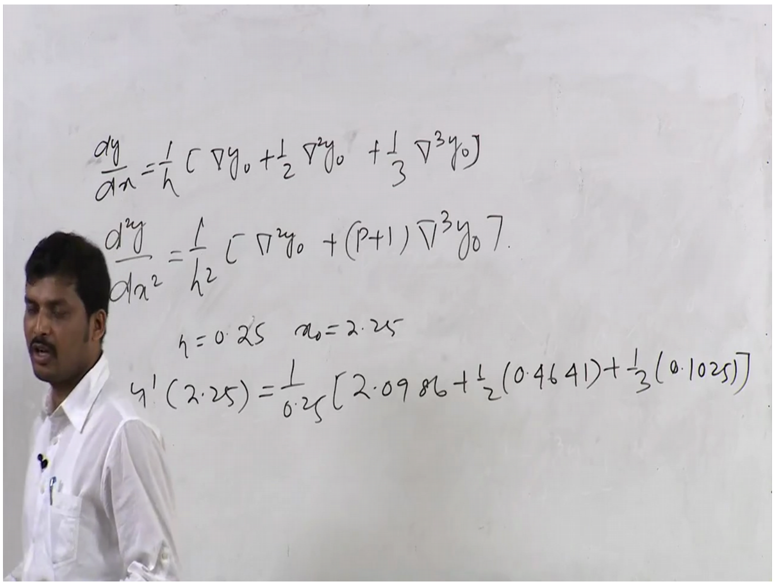
(Refer Slide Time: 29:35)



And similarly for second order differentiation we can just write these forms in the Newton's backward difference formula up to third order terms as 1 by h square nabla square of y 0 plus p plus 1 nabla cube of y 0. And if you will just put here h equals to 0 point 25 and x 0 equals to 2 point 25 here then we can just obtain this derivative as y dash at 2 point 25, that as 1 by 0 point 25.

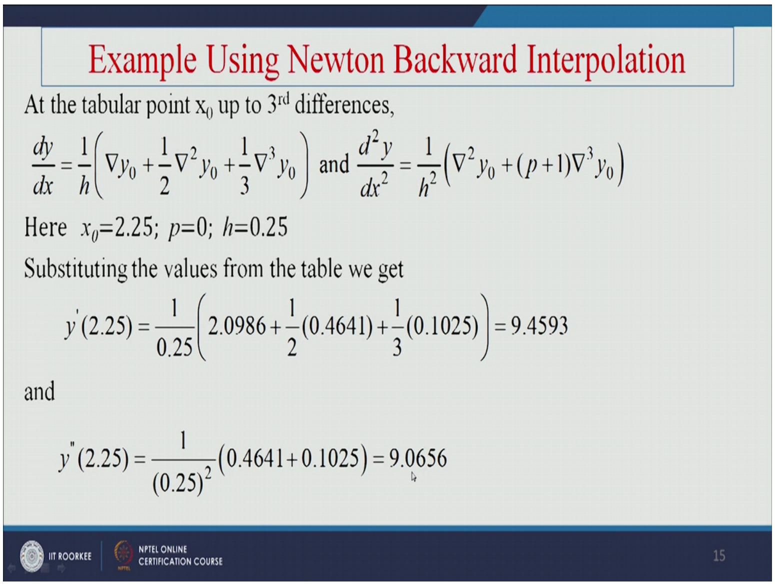
If you will just see these tabular values, del y 0 it is just giving you 2 point 0986 plus half into the second value that is nabla square of y 0 that is coming as 0 point 4641. And last value 1 by 3 that is nabla cube of y 0 that is coming as 0 point 1025.

(Refer Slide Time: 31:01)



So if you just evaluate these values that will just come as 9 point 4593. Similarly we can just obtain this derivative for second order that is at 2 point 25 also and it can be written as like 1 by h square. So that is why we can just write 1 by 0 point 25 whole square into your nabla square of y 0 that as 0 point 4641 plus 0 point 1025 that as 9 point 0656 here.

(Refer Slide Time: 31:32)



So maybe next class we will just continue about this differentiation using Lagrange interpolating polynomial that is both for equi-spaced points and unequi-spaced points. Thank you for listen this lecture.