Numerical Methods
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Lecture 24
Interpolation Part IX (Hermite's Interpolation with Examples)

Welcome to the lecture series on numerical methods and we are just discussing here this interpolation. In the interpolation we have already discussed this finite difference operators like Newton's forward difference operator, backward difference operator, central difference operator and after that we have covered up this unequal spaced interpolation like Lagrange interpolation and Newton's divided difference interpolation.

So today we will just discuss about this interpellation based on Hermite's hypothesis that is Hermite's interpolation. And then we will just go for some of the examples that we will solve using Hermite's interpolation formula.

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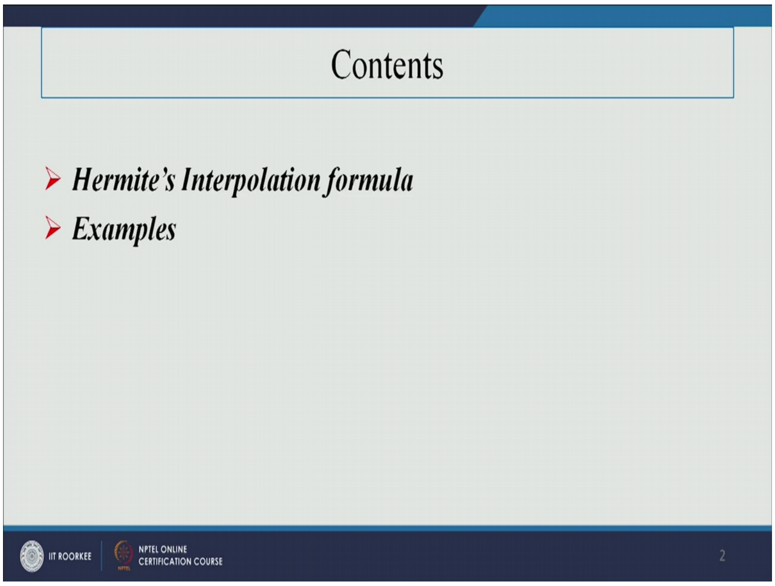

So whenever we are just going for this interpolation, basically we are just dealing here like if a function f of x is defined at the set of data points like x 0 x 1 to x n and this function is approximated by a polynomial of degree n suppose. Since we have defined like points here x 0, x 1 to x n are the tabular points or the nodal points and corresponding to each of these (nabula) tabular points we have associated functional values like f of x 0 x 1, f of x 1 up to x n f of x n.

Sometimes also we are just expressing this tabular points expression as in the form of x 0 y 0, x 1 y 1 up to x n y n. Basically the idea is that we want to approximate this function f of x with a polynomial p of x. Since this function is satisfied at n plus 1 points if you will just see then we can just approximate this function with a polynomial of degree n here. Then if we will just use like Lagrangian interpolating polynomial then we are just using both this equally or unequally spaced interval points.

Basically if we are just considering here x 0, x 1, x 2 up to x n. Sometimes may be it is equally spaced. Sometimes maybe it is unequally spaced. But for both these cases we can just apply Lagrange's interpolation formula. But here in Hermite's interpolation we can just extend these polynomials degree but at the same number of points. This means in a Hermite's interpolation the degree of the polynomial is increased without increasing the number of tabular point here.

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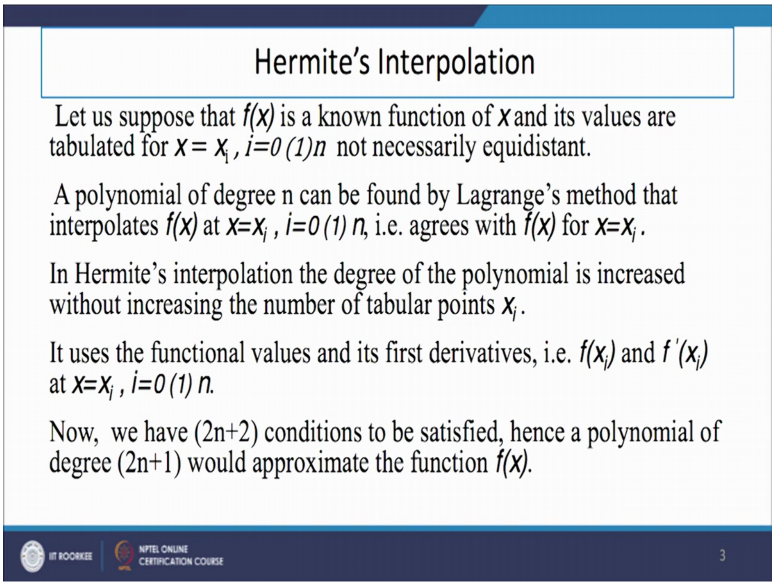

This means it uses this functional value that is f of x 0, f of x 1, f of x 2 and these first order derivatives at that point also. This means that it uses these functional values f of x i and f dash x i. So i is varying from 0 to n here.

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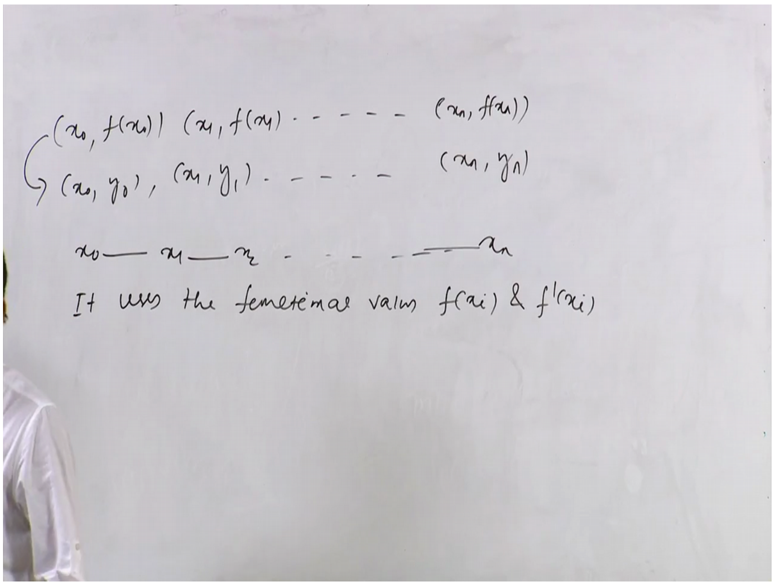

So definitely if we have here all the points 0, 1, 2 up to n and f of x i is satisfied at n plus 1 points and f dash x i is satisfying at n plus 1 points then we will have 2 n plus 2 points here. And if we just formulate a polynomial based on this 2 n plus 2 points then it can just generate a polynomial of a degree 2 n plus 1 here. So let us suppose so y of x be a polynomial of degree suppose 2 n plus 1. Suppose we are just considering here y of x is a polynomial of degree 2 n plus 1.

Since the polynomial involved here consist of 2 n plus 2 points here. So if you will just write this is y of x is a polynomial of degree 2 n plus 1 which approximates this function f of x here. Y of x is approximating this function f of x at the nodal points like x 0, x 1 to x n here.

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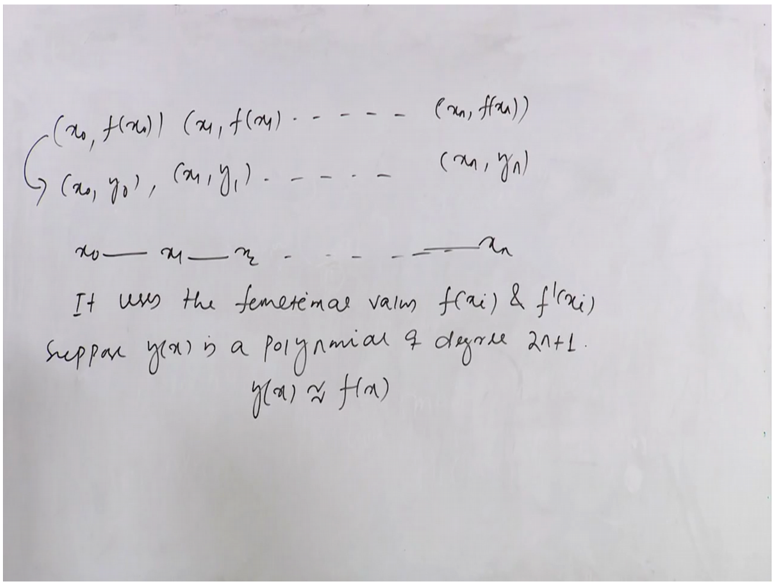

Then definitely we can just write this one as y of x i equals to f of x i and y dash of x i this equals to f dash of x i for i equals to 0, 1, 2 up to n here. And if we want to express it in polynomial form we can just write this polynomial as y equals to summation i equals to 0 to n, u i x y i plus summation i equals to 0 to n, v i x y i dash where u i x and v i x are polynomials of degree 2 n plus 1 here. Definitely we can just say that where u i x and v i x are polynomial of degree 2 n plus 1.

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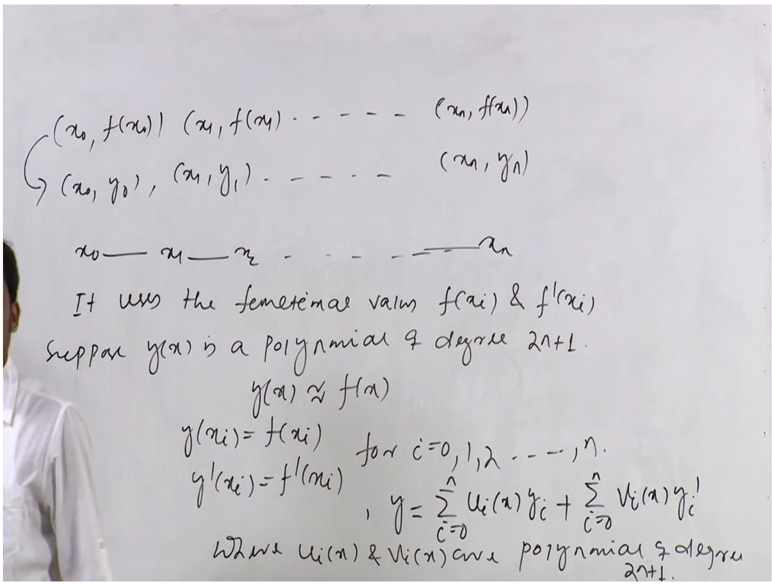

So if we want to express this as in a complete polynomial sense this means that we are just using these n plus 1 nodal points to determine the polynomial of order 2 n plus 1 here. Since we have here n plus 1 (po) points only so then we have to justify that whatever this polynomial we are just generating it can take the n plus 1 points here but it can produce a polynomial of a degree 2 n plus 1 here. That is why we have to consider u i x of polynomial degree 2 n plus 1 and v i x of polynomial degree 2 n plus 1 also.

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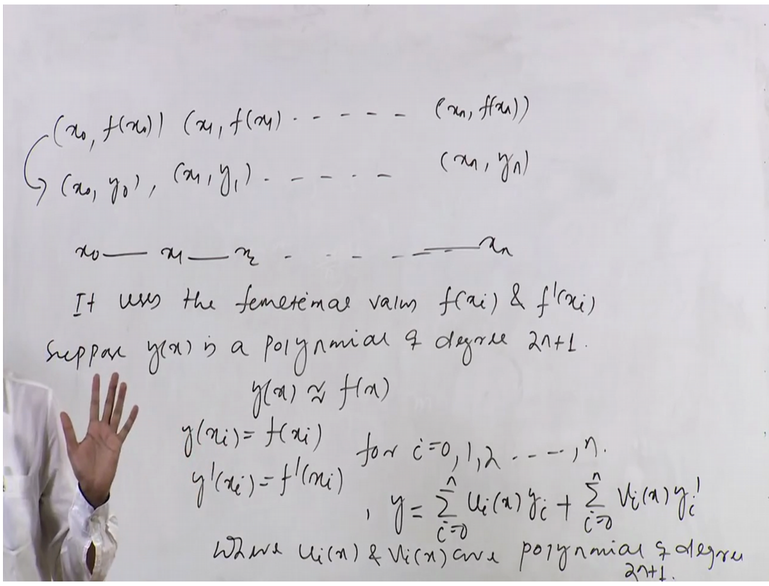

So if we are just writing here y of x i equals to f of x i and y dash of x i equals to f dash of x i and the polynomial is expressed in the form of y of x equals to summation i equals to 0 to n, u i x y i plus summation i equals to 0 to n, v i x y i dash which represents a polynomial of a degree 2 n plus 1.

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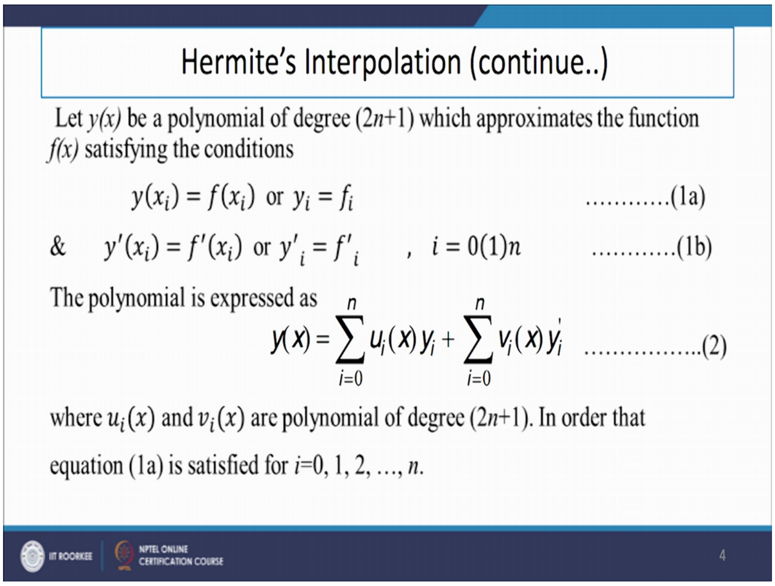

Then since we are just associating this polynomial in a Lagrangian sense so that is why we can just consider that since usually we are just writing this polynomial y this equals to summation of i equals to 0 to n, u i y i plus summation i equals to 0 to n, v i x. Everywhere x is associated here. So u i x y of x, v i x and y i dash x here.

So we can just define u i of x this equals to 1 when x equals to x i or we can just represent this one as u i of x j, this equals to 1 for i equals to j and this equals to 0 for i is not equals to j here. And similarly we can just define v i of x j this equals to 0 for all i and j here.

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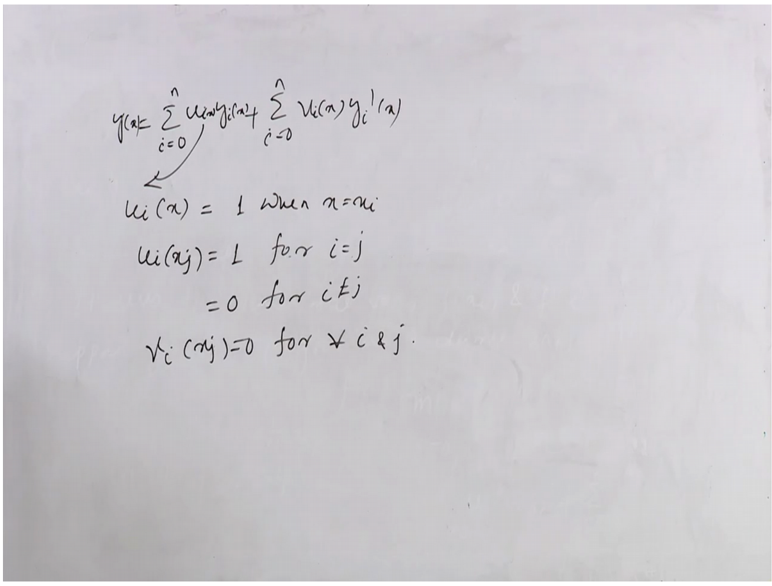

So similarly if we want to express also since it is in a Lagrangian mode here and two functions we are just defining here similarly we can just define that u i dash of x i or we can just write this one as u i dash of x j, this equals to 0 for all i and j. But v i dash of x j this can give you like 1 for i equals to j and this equals to 0 for i is not equals to j here. Then we can just combine this Hermite interpolating polynomial in a 2 n plus 1 degree sense that can generate a polynomial which can be represented as (hermote) Hermite interpolating polynomial here.

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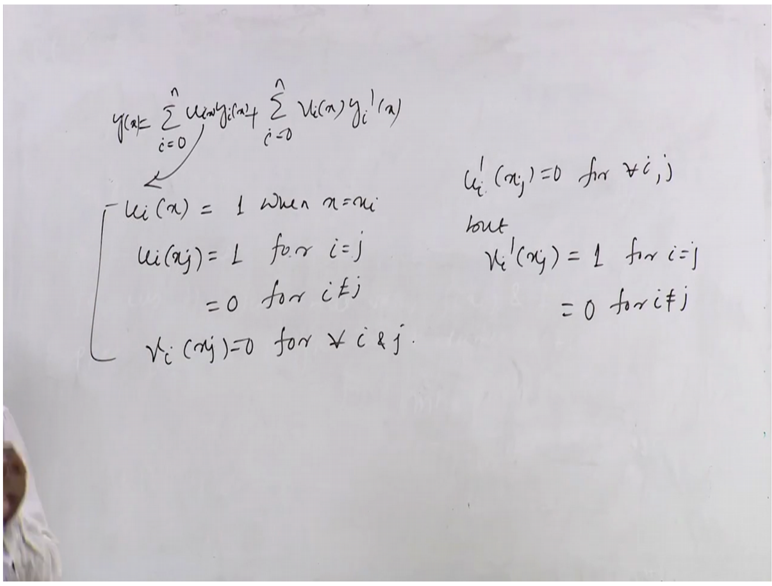

So definitely if it represent the polynomial of a degree n plus 1 then if we want to associate this one with the polynomial of degree n of Lagrangian interpolating polynomial then we can just represent the polynomials of u i x and v i x as, u i x can be written as a i x plus b i into L i x square. Similarly v i x can be written as c i x plus d i L i square of x here.

This means that if we want to associate this Lagrangian polynomial with this variable here or this polynomial u i x and if we want to associate this Lagrangian polynomial L i x with this polynomial v i x then we can just say that if it is multiplied here like L i x into L i x then this can generate a polynomial of a degree 2 n there. And if it is multiplied with this function here a i x plus b i so it can generate a polynomial of a degree 2 n plus 1 there. And similarly since we are just considering L i x is the Lagrangian coefficient of Lagrangian interpolating polynomial.

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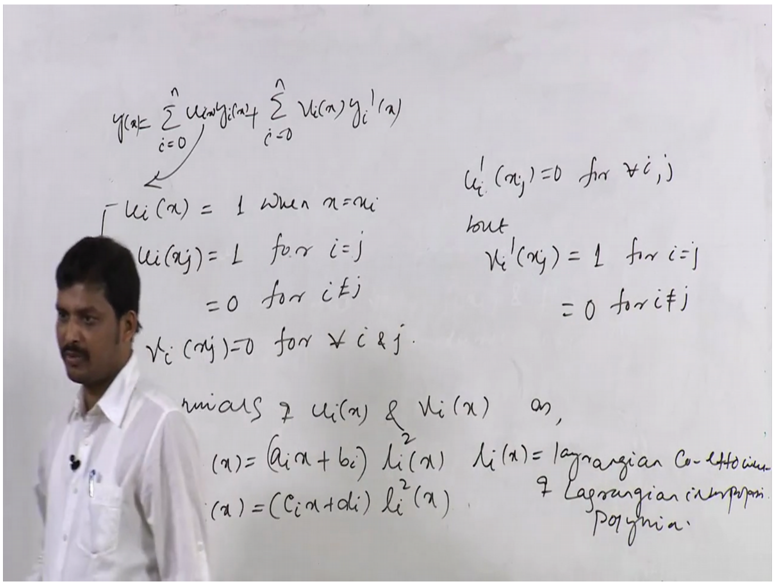

This L i x can be expressed in the form of like x minus x 0, x minus x 1 to x minus x, x i minus x i minus 1 into x minus x i plus 1 up to x minus x n divided by x i minus x 0, x i minus x 1 up to x (mi) i minus x n. So if we will just define this Lagrangian interpolating polynomial where this L i x represents the coefficients of Lagrangian interpolating polynomial here. Using this Lagrangian interpolating polynomial we can just determinant these coefficients a i, b i, c i, d i from these polynomials u i x and v i x here.

So if we will just write this Lagrangian interpolating polynomial here the Lagrangian interpolating polynomial coefficient L i x can be written in the form of like x minus x 0, x minus x 1 up to x minus x i minus 1, x minus x i plus 1 up to x minus x n divided by x i minus x 0, x i minus x 1 up to x i minus x i minus 1, x i minus x i plus 1 up to x i minus x n here.

Obviously sometimes we are just expressing as I have discussed in the earlier classes that we can just express this one in a product form that can be represented as i equals to 0 to n, phi of x divided by x minus x i into phi dash of x i here where phi of x can be represented in the form of x minus x 0, x minus x 1 to x minus x n here. And phi dash of x i especially this is called prime of phi of x. We can just write this one as d phi by d x at x equals to x i here.

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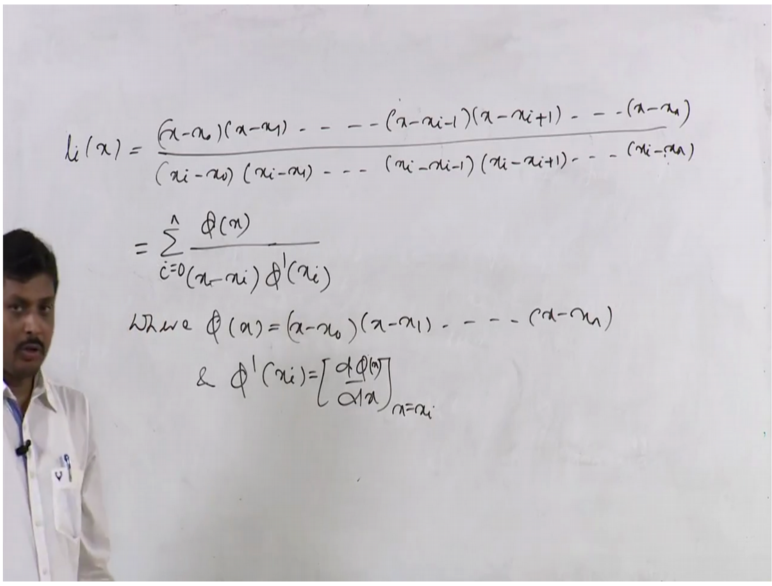

So in both these forms it can be expressed since phi of x, it can also represented in the form of product of i equals to 0 to n, x minus x i also. And some people are also using this one as prime of like product of i equals to 0 to n, x minus x i also.

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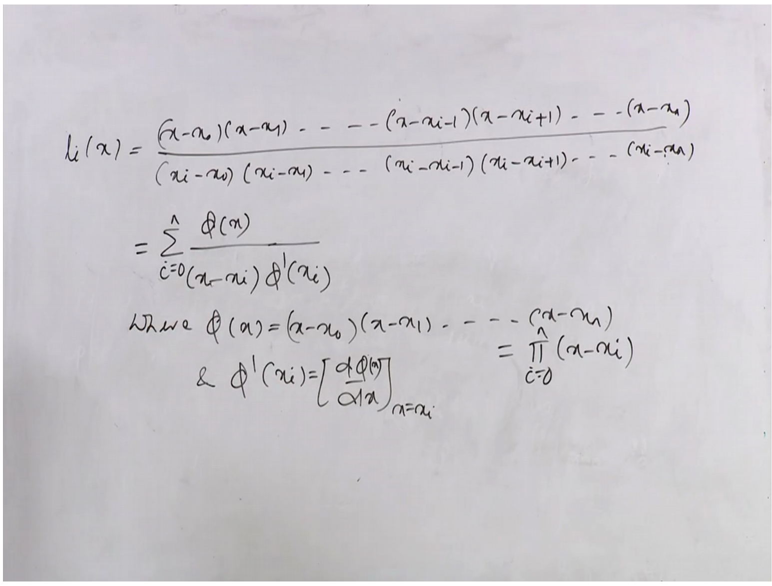

So if you will just express this L i x in a different forms here by considering like n plus 1 points that represents the polynomial of a degree n here then if we will just take this differentiation of this u i dash, whatever we have just expressed for this Hermite's interpolating polynomial we can just write u i x dash as, since it can be expressed in the form of like u i of x is expressed in the form like a i x plus b i L i x square here.

So if you will just take the derivative with respect to x here in the first coefficient we can just write this one as a i into L i square x plus a i x plus b i 2 L i x into L i dash of x here. So if you will just take common of this a i from both these terms here we can just write this one as a i L i square of x plus a i x into 2 L i x L i dash of x plus b i 2 L i of x L i dash of x here.

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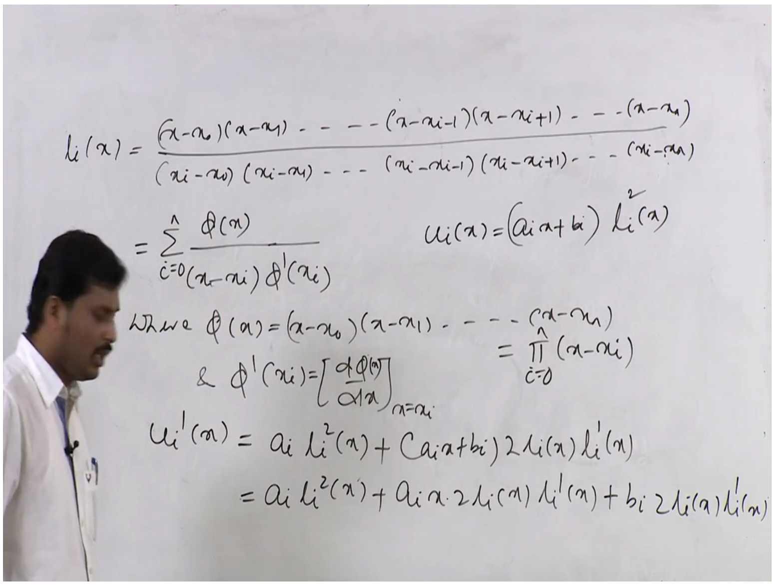

And if I will just take common here a i since we want to separate or eliminate a i from these equations here. So I will just write this one that can be written in the form of like L i square of x plus 2 x L i of x, L i dash of x into a i plus b i into 2 L i x, L i dash of x here.

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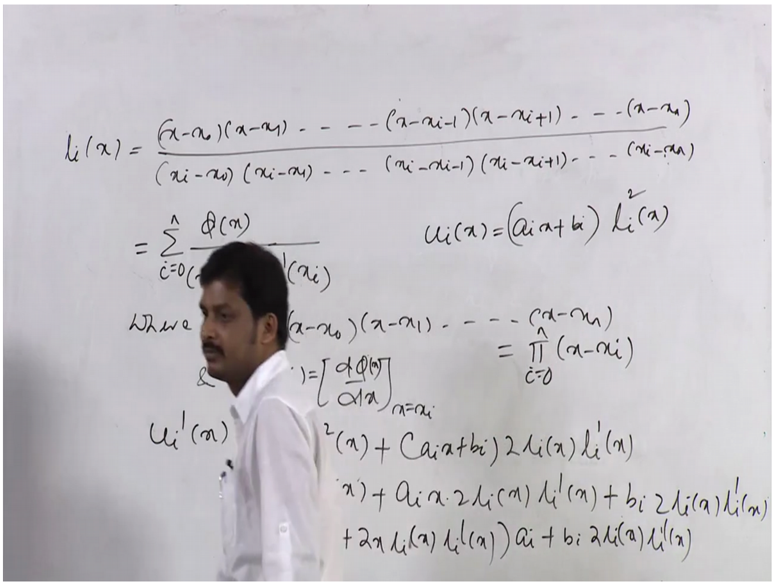

So likewise we can differentiate also v of x and we can just obtain this v i dash of x as, since v i x is usually written in the form of like v i x equals to c i x plus d i into L i x square. And if you will just take the derivative with respect to x here this can be written in the form of c i into L i square of x plus c i x plus d i into 2 L x into L i dash of x here. And if we will just take common of c i from both these terms here, so c i into L i square x plus x 2 L i of x L i dash of x plus 2 L i of x L i dash of x into d i.

So if we will just put here x as x i then we can just eliminate this c i, d i from both these equations and we can get these coefficients that is either in the form of L i x or in the form of x i there.

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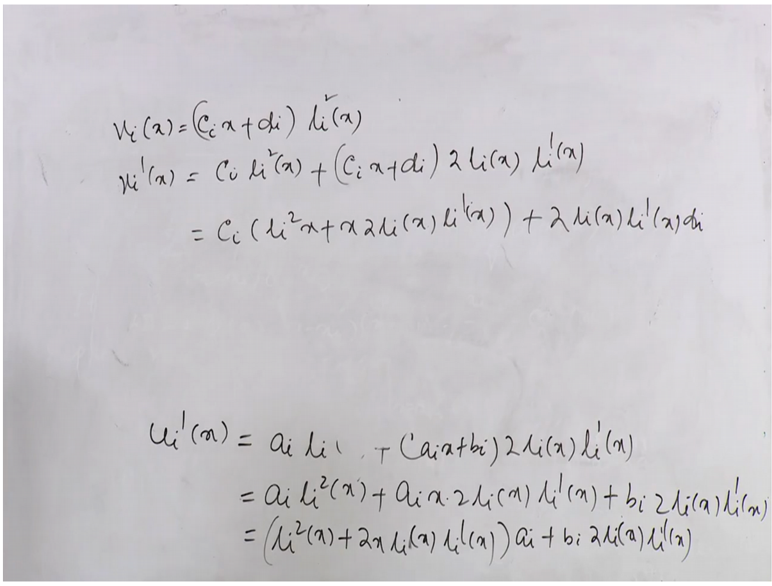

So if we will just put this like the values here, this means that using like equations 3a and 3b here that is as the conditions I have already written. U i of x j this equals to 1 for i equals to j and this equals to 0 for i is not equals to j where for all i and j we can just say that v i of x j this equals to 0 for all i j here. But we are just saying that u i dash of x j this equals to 0 for all i and j but we are just saying v i dash of x j this equals to 1 for i equals to j and this equals to 0 for i is not equals to j.

If you will just use both these conditions in these two equations here that is u i dash and v i dash then easily we can just eliminate a i x and v i x here.

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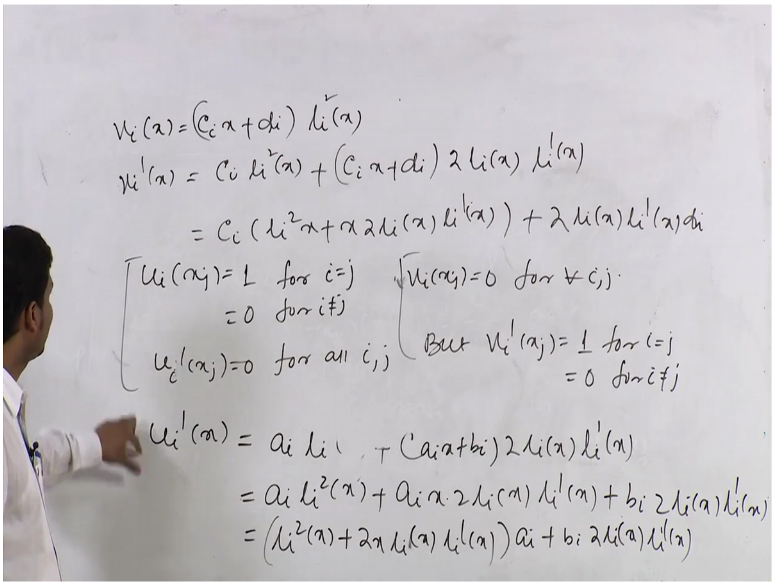

So now if you will just apply these conditions then we can just find here. So first condition we can just get as a i of x i plus b i, this equals to 1. Second condition we can just get as c i of x i plus d i this equals to 0. Since especially if we are just seeing here in the first term here that is u i x equals to a i x plus b i into L i square here, so then if we will just put x equals to x i then we will just get L i square of x i this equals to 1. But if you will just put in a like v i of x i here or x j both the terms we are just getting 0 there.

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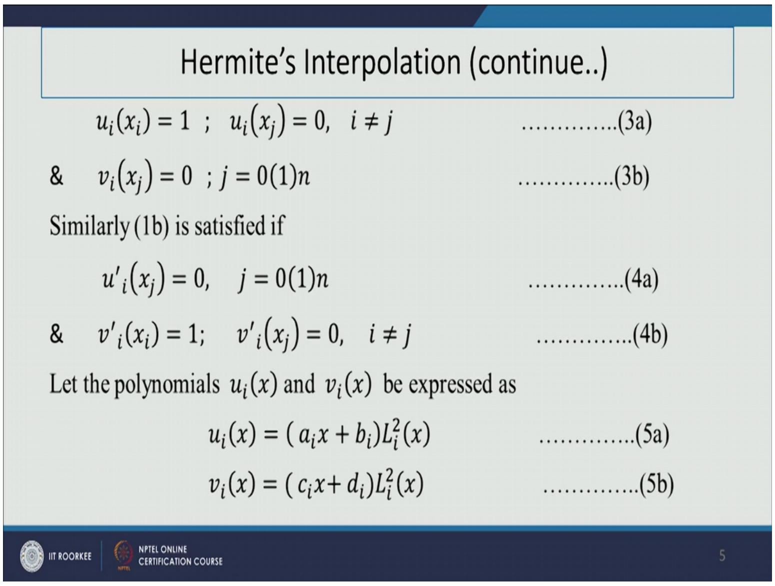

Since we have defined already that u i satisfies this Lagrangian polynomial or i equals to j coefficient only at x equals to x i point. But in the derivative sense if you will just consider then v i will satisfy there. So that is why we can just find this first coefficient as this form here. And if you will just use these same coefficients for the second two equation like 6a and 6b equations here then we can just obtain this one as 2 x i L i dash of x i plus 1 a i plus 2 L i dash of x i b i this equals to 0.

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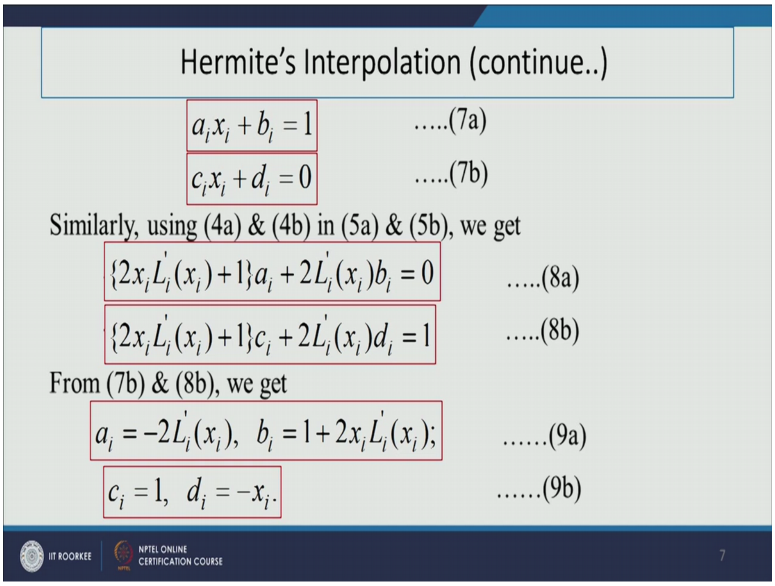

Similarly we can just get 2 x i L i dash of x i plus 1 into c i plus 2 L i dash of x i into d i, this equals to 1 here.

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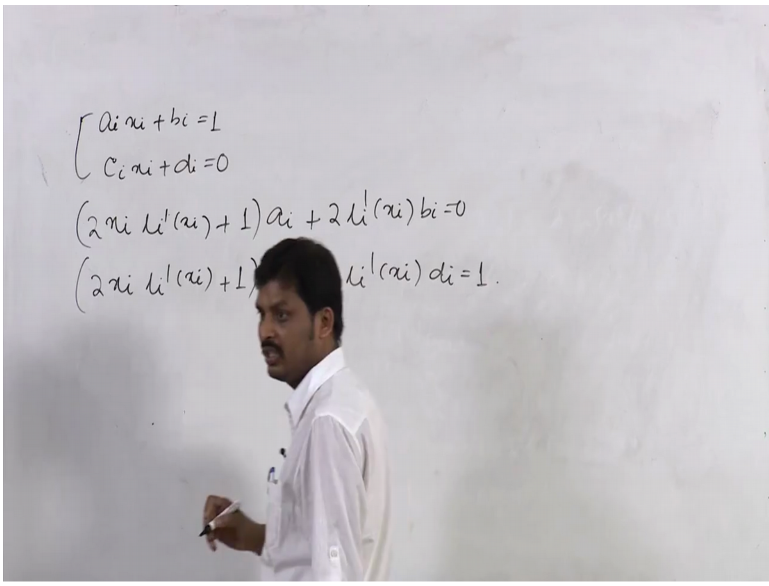

Since now we have like four equations with four unknowns so easily we can obtain these coefficients here. So first if you will just try to eliminate a i and b i from these two equations here then directly if you will just multiply simply since this is 2 L i dash of x i is there, if you will just multiply here 2 L i dash x i in this equation and if you just subtract then we can obtain the coefficient for a i first.

So if you will just multiply like a i x i into 2 L i dash of x i b i sorry this is a i x i into multiplied by 2 L i dash of x i plus we can just write b i into 2 L i dash of x i into b i, this equals to 2 L i dash of x i here. And this equation it can be written in the form of like a i x i L i dash of x i plus a i plus 2 L i dash of x i b i this equals to 0 here. So if you will just subtract then this term will cancel it out and this term also here cancel it out. So that is why we can just obtain a i as, subtract means we can just say that minus 2 L i dash of x i here.

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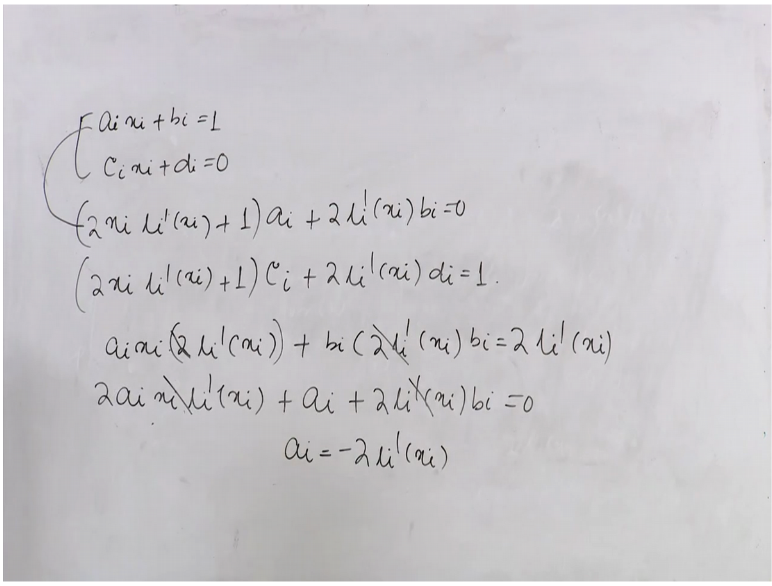

And if you will just put this a i coefficient in the first equation here then we can just obtain b i and b i can be written in the form of like 1 plus 2 L i dash of x i into x i here. Similarly if you will just eliminate this c i and d i from these two equations we can just obtain as c i as 1 and d i as minus x i here.

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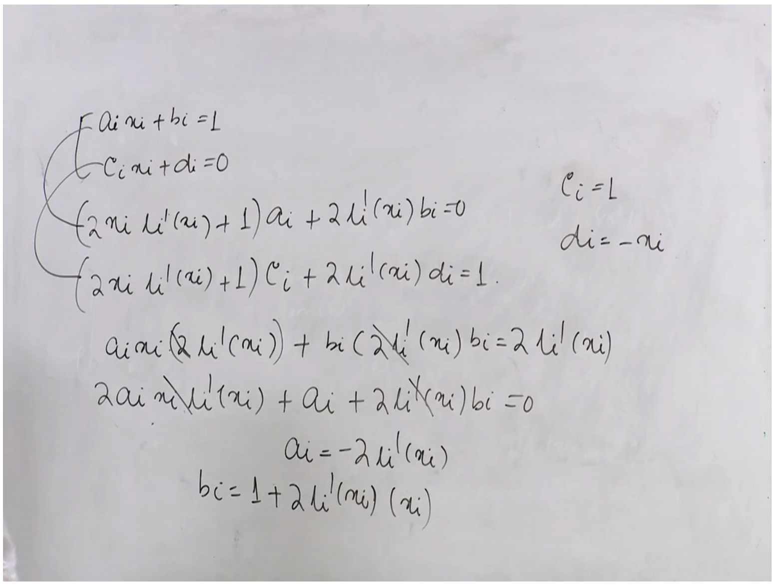

So just simple multiplication if you will just do then we can just find these coefficients a i and b i and c i and d i from these four equations. So in a complete form if you will just write this one that it can be represented in the form of y of x equals to summation of i equals to 0 to n, 1 minus of 2 L i dash of x i, x minus x i into L i square of x into y i plus summation of i equals to 0 to n, x minus x i into L i square of x here into y i dash. This is the complete formula.

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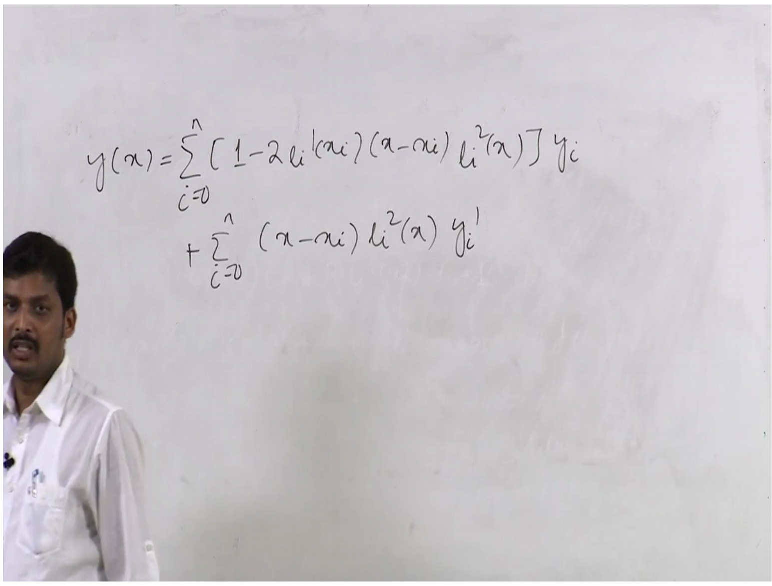

Sometimes also you can just remember these coefficients as h i x here, this one as h i dash of x also here. Since L i square x is easy to remember here and both these coefficients first one is multiplied. Since we want to generate this polynomial of a degree 2 n plus 1 here, so that is why y i can be multiplied there and y i dash can be multiplied here. And the beauty of this method is that only n plus 1 points are used to get a polynomial of order 2 n plus 1 here.

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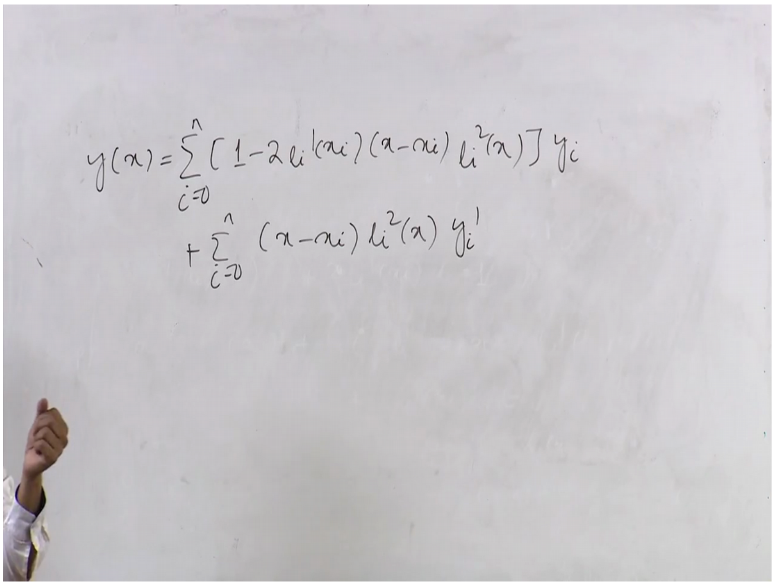

And if we want to find this error terms here, the error terms can be written in the order of 2 n plus 1 here that as, since this polynomial is involved here the degree of 2 n plus 1 here, the error term will be just represented as f to the power 2 n plus 2 zeta by 2 n plus 2 factorial into product of x minus x i square here.

Since if you will just see here so both these polynomials that has been just multiplied here so that is why if you will just consider like the earlier Lagrangian interpolation error term we are just used to write like f to the power n plus 1 zeta by n plus 1 factorial into product of like x minus x i there.

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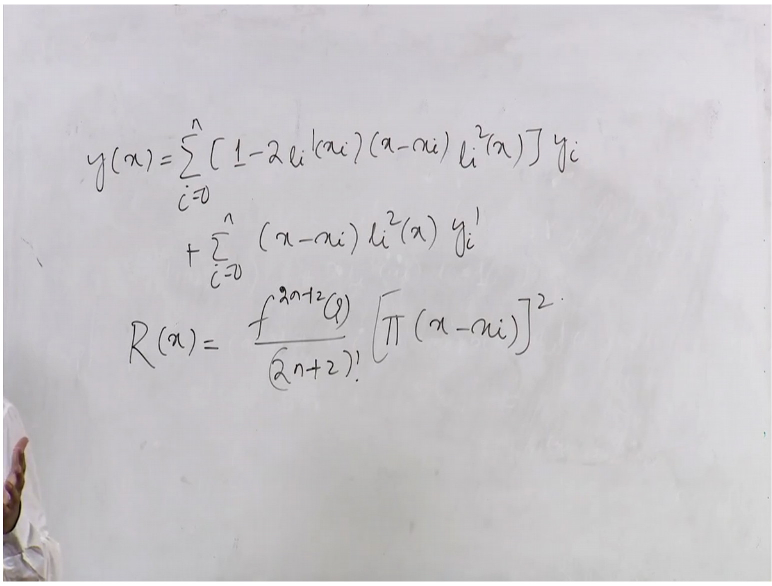

So if L i square of x if we are just considering so it can be taken as a square sense there and the derivative can be taken as the order of 2 n plus 2 here. So based on this if you will just go for a problem here that is suppose if the question is asked like using Hermite's interpolation formula find the value of sin of 0 point 5 from the following data. So we can just write first like L i of x 0 sorry L 0 of x 0, L 1 of x 1, L 2 of x 2. Then we can just find their derivatives and you can just put in the formula.

Then you can just obtain this interpolation in that problem. So if this problem is given like sin x data is just given and it is asked to find sin of 0 point 5 here. So suppose these tabular values are given as x, sin x, cos x here and the values are like minus 1, 0, 1 and like minus point 8415, this is 0 and one value is 0 point 8415 here. Similarly cos values are like 0 point 5403, cos 0 is 1, then this value is also 0 point 5403 here.

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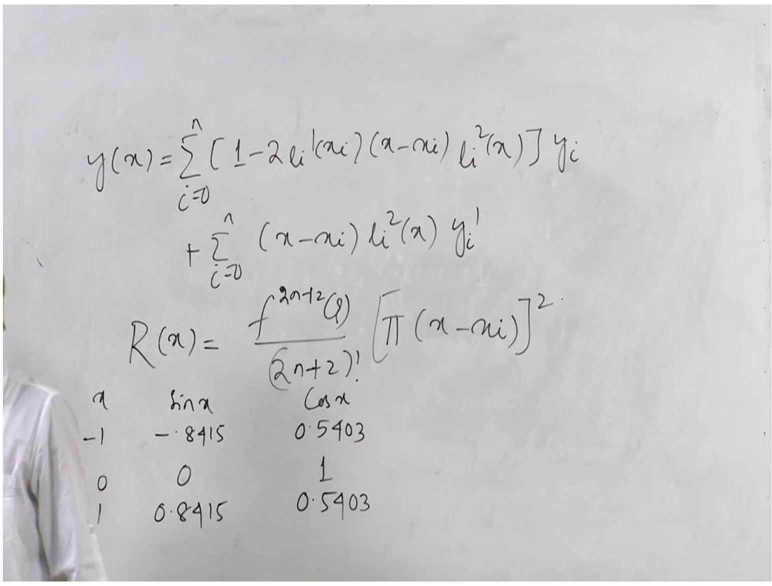

And we have here if you will just see f of x is sin x, f dash x is your cos x here. And if we want to express this one in the formula like here y of x if we want to find so directly we can just put y i of x i or y of x i as f of x i that is sin of x i here. And if we want to put y i dash of x i here so that can be written as cos of x i there.

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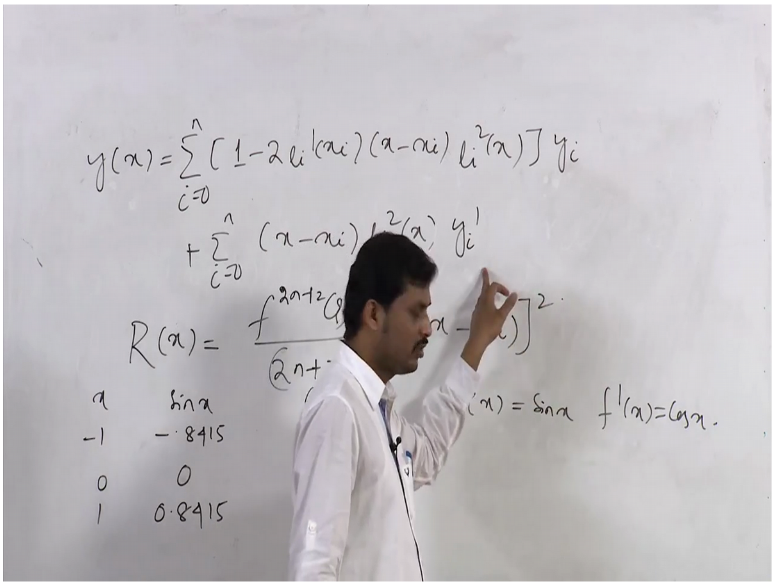

So first we have to find here like L 0 x we will just find. So if we want to find here L 0 x first, L 0 x can be written in the form of like x minus x 1, x minus x 2 by x 0 minus x 1, x 0 minus x 2 here. Similarly L 1 x can be written as x minus x 0, x minus x 2 by x 1 minus x 0, x 1 minus x 2 here. And L 2 x can also be written as x minus x 0, x minus x 1 divided by x 2 minus x 0, x 2 minus x 1 here.

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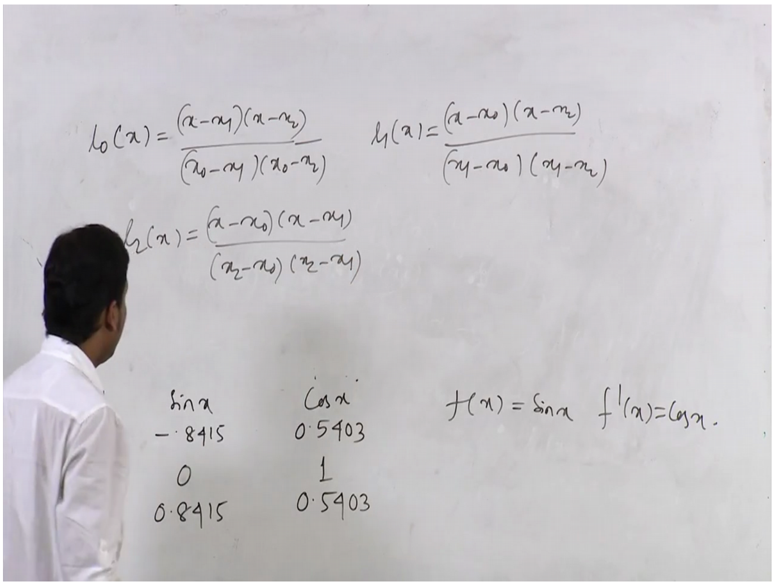

So if you will just put these values here like L 0 of x here it can be written as first value like here 0 it is there, so x into x minus 1 divided by, so x 0 is like minus 1 here so minus 1, minus 0 then like minus 1 minus 1, so minus 2 it is there. So definitely it is minus-minus so that is why it will come as 2 here.

Similarly if you will just determine L 1 of x here, that can be written as like x minus of minus 1. So x plus 1 into x minus 1 here divided by, if you will just see here since we have these coefficients like x 0 equals to this one, x 1 equals to this one, x 2 equals to this one here.

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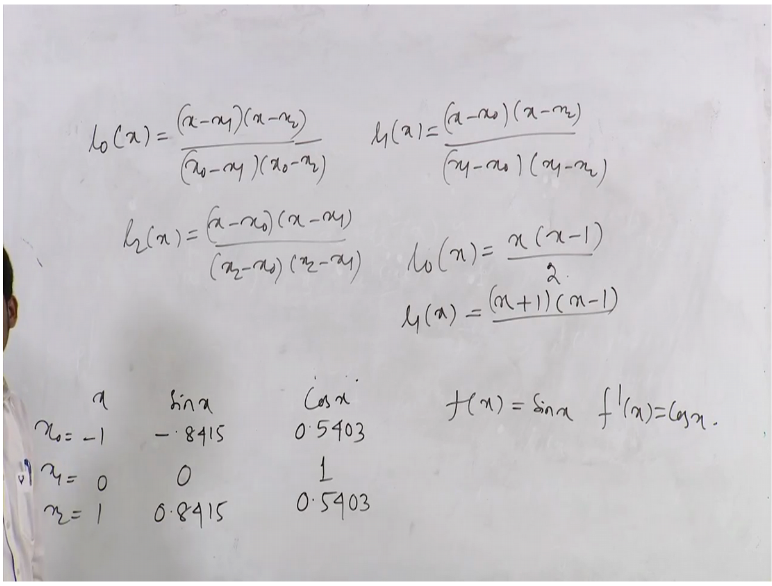

So we can just write this one as like minus of x plus 1 by x minus 1 divided by 1 it will just come. So similarly you can just obtain like L 2 x, that as x minus x 0. So that is why you can just write x plus 1, then x minus 0. This means that x only, so divided by if you will just see it will come as 2 here.

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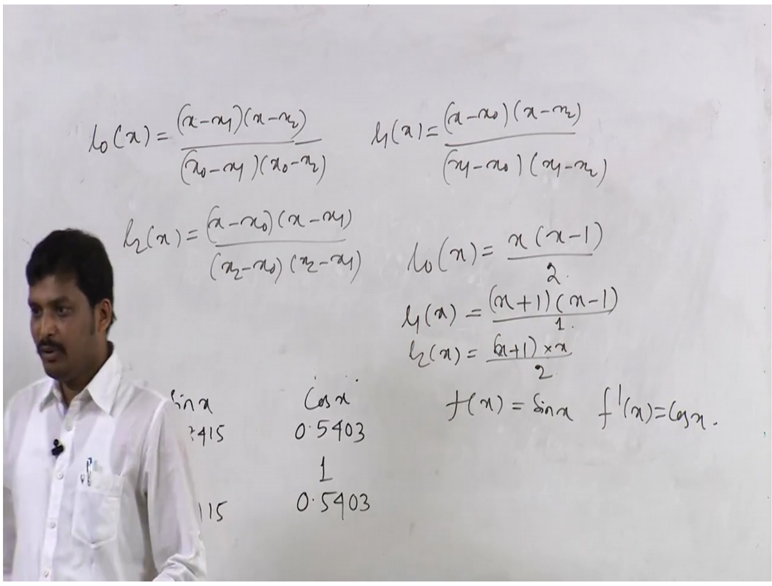

So directly we can just put the values there like L 0 of 0 point 5 if you will just put here then we can just obtain this value as minus 0 point 125 and L 2 of 0 point 5 if we will just put we can just get this one as like 0 point 75. And if you will just put these values of 0 point 5 here then we can just obtain these values as 0 point 375.

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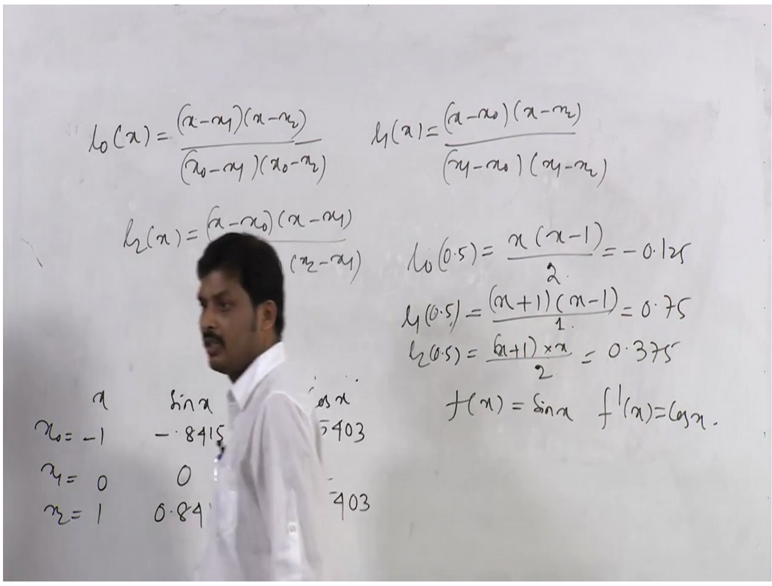

So similarly we can just obtain like L 0 dash of x that is directly if you will just differentiate here, since this is a polynomial we are just getting here so L 0 dash x we can just write as 2 x minus 1 divided by 2 here.

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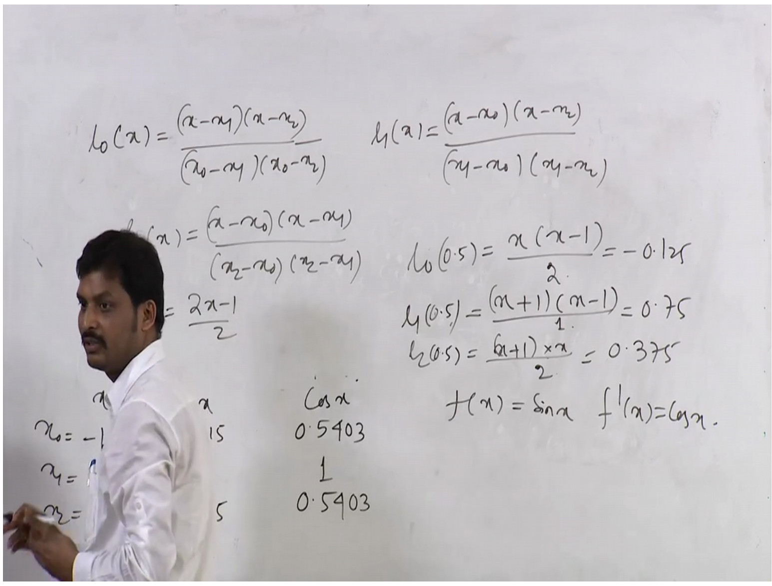

And L 0 dash of 1 if we want to find so sorry minus 1 here since we have the points like minus 1 here, we can just obtain that one as minus 3 by 2 there. So then again if L 1 dash of 0 if we want to find that can be written as 0. Then L 2 dash of 1 if we will just find that will be represented as 3 by 2 here.

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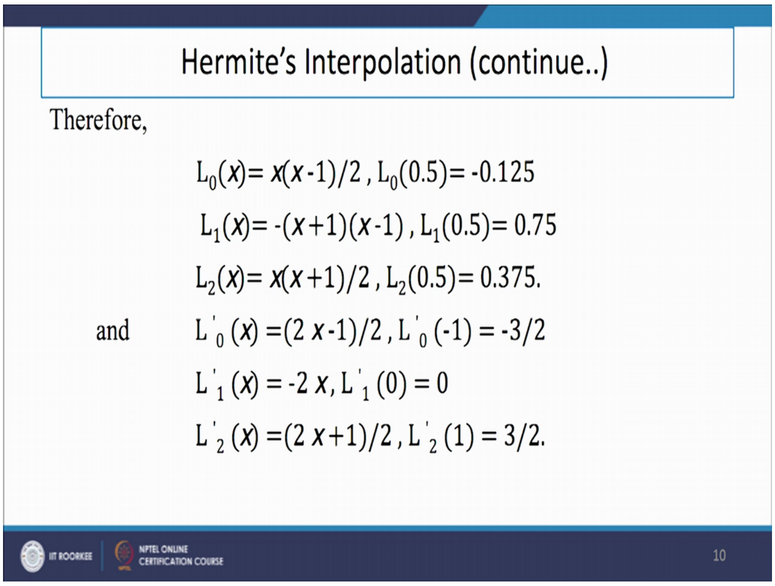

So finally if you will just put these values then we can just get these values as 0 point 3743 here. Therefore y of 0 point 5 we can just write it as 0 point 3743 and the exact value of sin 0 point 5 it can be represented as 0 point 4794. If the Lagrangian formula can be used we get the sin 0 point 5 as 0 point 4207.

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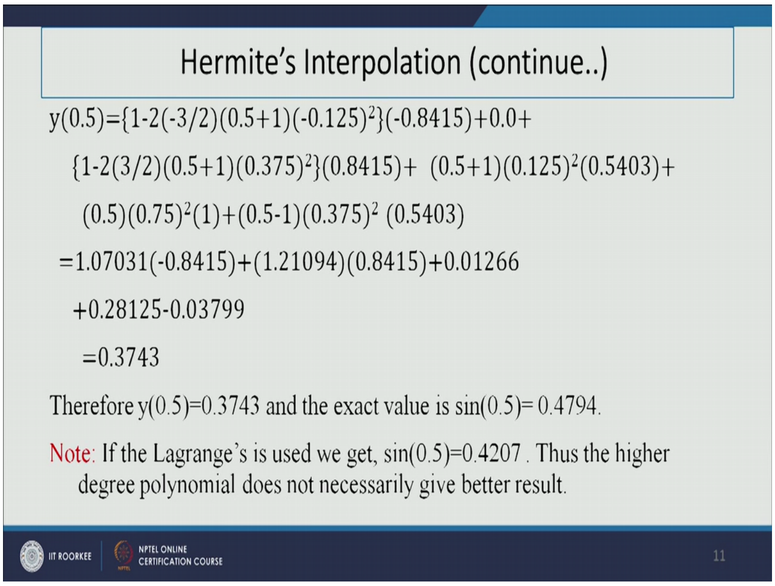

But for the higher degree polynomial it is not necessary to, it can just provide a better result. Thank you for listen this lecture.