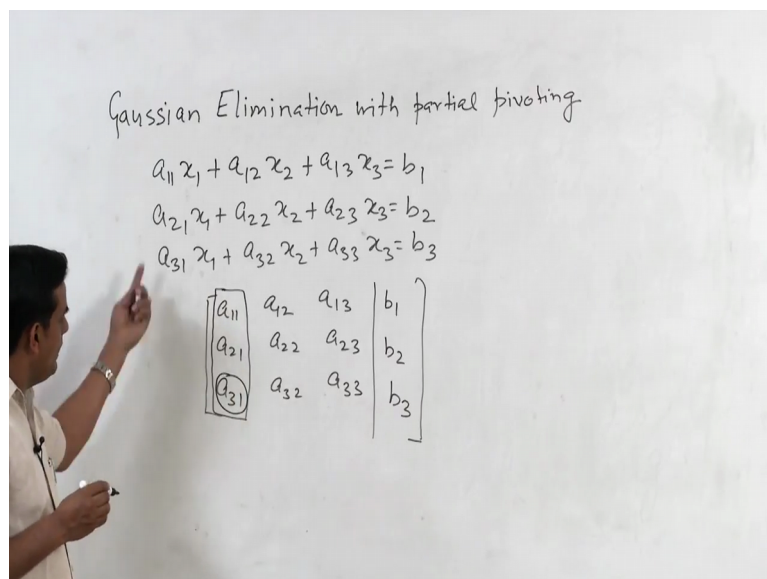


Numerical Methods
Dr. Sanjeev Kumar
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Lecture No 2
Gaussian Elimination with Partial Pivoting

Hello everyone so this is second lecture of this course and in this lecture I will introduce Gaussian elimination method with partial pivoting in the later part of this lecture will discuss about the ill condition systems those are difficult to solve using Gaussian elimination method or Gaussian elimination with pivoting, so in the last lecture we have solved this 3 by 3 system of linear equations using their Gaussian elimination method and after applying the Gaussian elimination and using the back substitution, we got the solution like x_1 equals to minus 0.950, x_2 equals to minus 2.82 and x_3 equals to minus 2.00 wherever the exact solution this x_1 equals to 1, x_2 equals to 2 and x_3 equals to minus 3.

So basically there is a big difference in these 2 solutions one which we obtain using the Gaussian elimination method and the other one which is the true solution. So there were some mistakes when we apply the Gaussian elimination method or something went wrong, so procedure were correct however the reason for this errors is rounding off error propagated to such an extent that the final solution becomes hopelessly inaccurate, so what is the solution to this problem?

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The solution to this problem is the Gaussian elimination method with partial pivoting, so this is a modified version of Gaussian elimination procedure and here we will search for pivot

elements and based on that we will perform the elementary row operations. So again consider a 3 by 3 system having 3 equations with 3 unknowns that is a 13 x 3 equals to b 1 so this is the 2nd equation and the 3rd equation is so in matrix form I can write the coefficient matrix as a 11, a 12, a 13, a 21, a 22, a 23 and coefficient of the 3rd equation, so these are the 9 coefficients of x 1, x 2, x 3 in all 3 equations and then if I had the right-hand side vector here then this becomes the augmented matrix.

In Gaussian elimination we use to perform elementary row operations on this matrix in such a way this matrix reduce into row equivalent form. In Gaussian elimination with partial pivoting what we will do, first of all we will search for the element having maximum absolute value from the 1st column, so I will look for this 3 elements from the 1st column and I will see which element is having the maximum absolute value and if this is the element a 11 I will not do any operation at this particular moment if it is a 21 I will interchange 1st and 2nd equation, if it is a 31 I will interchange 3rd equation with 1st equation. After doing this what I will do let us assume that you maximum element is a 31, so what I will do? I will interchange my 3rd equation with 1st equation.

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Gaussian Elimination with partial pivoting

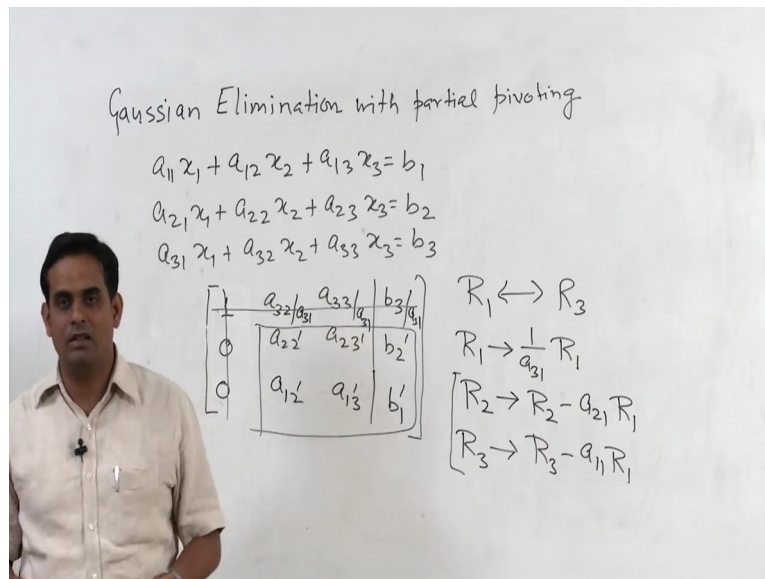
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} a_{31} & a_{32} & a_{33} & b_3 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{11} & a_{12} & a_{13} & b_1 \end{array} \right] \quad R_1 \leftrightarrow R_3$$

So my augmented matrix will become a 11, a 12, a 13 and here it will come b 1 while in this a 31, a 32 and then a 33 equals to b 3. So I have done a operation R 1 interchange with R 3, so now this a 31 is the biggest element in terms of absolute value in the 1st column now what I will do, I will make this element as 1 so what I need to do, I need to divide the 1st row by a scalar that is 1 upon a 31, so I need to replace R 1 with 1 upon a 31 R 1.

So what will happen this element will become 1, this will become a 32 up on a 31 this will become a 31 and it will become upon a 31. Now what I will do with the help of this 1st equation I will make these 2 elements 0 that is the element in the 1st column of the 2nd row of an element in the 1st column of the 3rd row. For doing this I need to perform to more row operations that is R 2 will be replaced by R 2 minus A 21 times R 1 and R 3 I need to replace by R 3 minus a 11 into R 1. Please note that this element is now a 11 because we have interchanged R 1 and R 3 in our 1st operation.

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Gaussian Elimination with partial pivoting

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\left[\begin{array}{ccc|c} 1 & a_{32}/a_{31} & a_{33}/a_{31} & b_3/a_{31} \\ 0 & a_{22}' & a_{23}' & b_2' \\ 0 & a_{12}' & a_{13}' & b_1' \end{array} \right]$$

$R_1 \leftrightarrow R_3$
 $R_1 \rightarrow \frac{1}{a_{31}} R_1$
 $R_2 \rightarrow R_2 - a_{21} R_1$
 $R_3 \rightarrow R_3 - a_{11} R_1$

So if I will perform these 2 row operations then what I will get these 2 elements will become 0 and I will get according to some other values here. So this is the 1st pass of Gaussian elimination method with partial pivoting. Now in the 2nd pass what I will do I will left out 1st row and 1st column and I will perform same operations and then sub matrix in such a way that this entry they will become 1 and this entry will become 0. So here we are changing finally the matrix in row equivalent form however we are performing some extra operations by searching the pivot elements and we are making the entry 0 with the help of the pivot elements and the diagonal position. So this method is called Gaussian elimination with partial pivoting.

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Partial pivoting

Example-2

Consider a 3×3 system having the following augmented matrix

$$\left(\begin{array}{ccc|c} 0.143 & 0.357 & 2.01 & -5.17 \\ -1.31 & 0.911 & 1.99 & -5.46 \\ 11.2 & -4.30 & -0.605 & 4.42 \end{array} \right)$$

In the left column, the entry 11.2 is pivot because it has largest absolute value.

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So let us take an example of this method so I will consider the same example which I have taken in my previous lecture for that I got a wrong answer using Gaussian elimination, so here the same augmented matrix 3 equations with 3 unknown, so now in the 1st operations of Gaussian elimination with partial pivoting what I will do? I will search the element having the maximum absolute value in the 1st column, here you can see element 11.2 is having the highest value, so what I will do it will interchange my 1st and 3rd equation. This is the same exactly which we have done while I was explaining the procedure.

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Partial pivoting

Example

Interchanging the first and third rows so that pivot is in first row

$$\left(\begin{array}{ccc|c} 11.2 & -4.30 & -0.605 & 4.42 \\ -1.31 & 0.911 & 1.99 & -5.46 \\ 0.143 & 0.357 & 2.01 & -5.17 \end{array} \right)$$

Dividing the first row by 11.2 to make pivot entry 1

$$\left(\begin{array}{ccc|c} 1.00 & -0.384 & -0.0540 & 0.395 \\ -1.31 & 0.911 & 1.99 & -5.46 \\ 0.143 & 0.357 & 2.01 & -5.17 \end{array} \right)$$

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So here I have interchanged my 1st and 3rd row so this has become my 1st equation now and 3rd equation will become 1st equation, 1st equation becomes 3rd equation. Now what I will do? I

will divide the 1st row by 11.2 to make the pivot entry as 1, so what will happen R 1 is replaced by 1 upon 11.2 times R 1, so this will become 1 and subsequently other entries change, this is minus 4.30 up on 11.2 so it is coming out like minus 0.384 and so on. Now what I will do, I will make these elements and this element 0 using the pivot element that is 1 now.

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Partial pivoting

Example

After eliminating x_1 from second and third equations $R_2 \leftarrow R_2 + 1.31R_1$ and $R_3 \leftarrow R_3 - 0.143R_1$, we get

$$\left(\begin{array}{ccc|c} 1.00 & -0.384 & -0.0540 & 0.395 \\ 0.00 & 0.408 & 1.92 & -4.94 \\ 0.00 & 0.412 & 2.02 & -5.23 \end{array} \right)$$

This completes the first pass.

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So for this I will replace the 2nd row that is R 2 by R 2 plus 1.31 times R 1 and R 3 I will replace by R 3 minus 0.143 times R 1, so after doing these 2 operations I will get this particular matrix, so here you can see in the 1st column in the 1st row am having the pivot entry as 1 and rest of the entry below the pivot elements are 0, so this completes the 1st pass of Gaussian elimination method with partial pivoting. Now what I will do I will do the same on the 2nd and 3rd equation but leaving out the 1st caller because these are 00 already, so for this I will go to the 2nd column, in 2nd column will choose the element having the maximum absolute value or of these 2 that is 0.408 and 0.412, so element 0.412 is bigger than 0.408, so I will replace 3rd equation I will interchange 3rd equation with 2nd equation.

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Partial pivoting

Example

So, interchanging the second and third rows, we get

$$\left(\begin{array}{ccc|c} 1.00 & -0.384 & -0.0540 & 0.395 \\ 0.00 & 0.412 & 2.02 & -5.23 \\ 0.00 & 0.408 & 1.92 & -4.94 \end{array} \right)$$

Dividing the second row by pivot entry 0.412, we get

$$\left(\begin{array}{ccc|c} 1.00 & -0.384 & -0.0540 & 0.395 \\ 0.00 & 1.00 & 4.90 & -12.7 \\ 0.00 & 0.408 & 1.92 & -4.94 \end{array} \right)$$

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Partial pivoting

Example

After eliminating x_2 from the third equation $R_3 \leftarrow R_3 - 0.408R_2$, we get

$$\left(\begin{array}{ccc|c} 1.00 & -0.384 & -0.0540 & 0.395 \\ 0.00 & 1.00 & 4.90 & -12.7 \\ 0.00 & 0.00 & -0.0800 & 0.240 \end{array} \right)$$

This completes the second pass.

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So system will become like this. Now here to make this pivot entry as one I need to divide 2nd row by 0.412, so when I will divide the 2nd equation by 0.412 I get this augmented matrix, so here pivot element is 1 and then again what I will do? I will make this element 0 with the help of this pivot element, so what I need to do? I need to replace 3rd equation that his 3rd row R 3 by R 3 minus 0.408 times R 1, so after performing this particular operation I got the system and this matrix is in row equivalent form however if you look at this element it is not still one. So it is a pivot element and I need to make it 1, so to make it 1 in the 3rd pass I need to divide this equation by this particular number.

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Partial pivoting

Example
To complete the entire procedure, dividing the third row by -0.0800 , we get

$$\approx \left(\begin{array}{ccc|c} 1.00 & -0.384 & -0.0540 & 0.395 \\ 0.00 & 1.00 & 4.90 & -12.7 \\ 0.00 & 0.00 & 1.00 & -3.00 \end{array} \right)$$

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Partial pivoting



Example
Using the back substitution, we get

$$x_3 = -3.00, \quad x_2 = 2.00 \text{ and } x_1 = 1.00$$

Exact solution
The exact solution is

$$x_1 = 1, \quad x_2 = 2 \text{ and } x_3 = -3$$

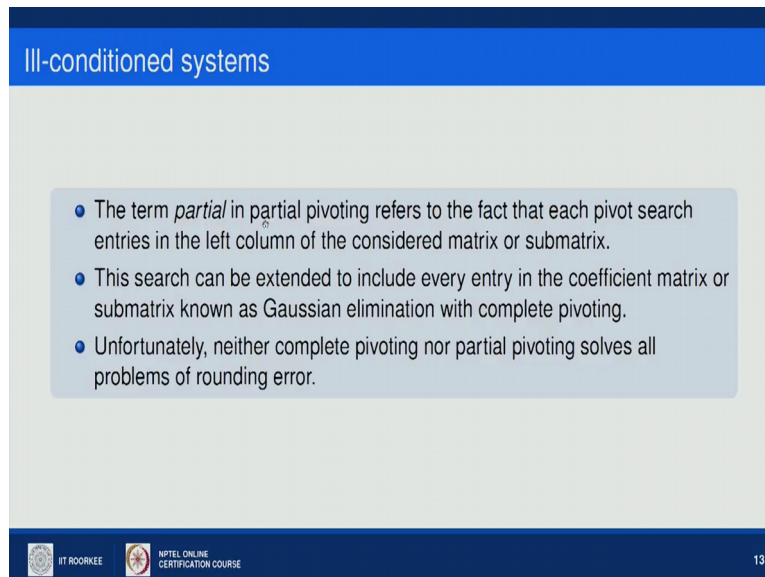
Exact solution agrees with numerical solution when rounded to three significant digits.

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So by doing this dividing the 3rd row by 0.0800 I got this element as 1 now you can see it is a triangular form that is it is an upper triangular matrix or I can say it is in row equivalent form. All the diagonal entries or pivot entries are 1, so from here if I use the back substitution, you can see that x_3 will become minus 2 sorry minus 3 x_2 will become to the 2nd equation after substituting the value of x_3 and x_1 will become 1 after putting the values of x_3 and x_2 . The exact solution is x_1 is 1, x_2 is 2, x_3 is minus 3 which is same which we get both using the Gaussian elimination with partial pivoting hence the exact solution agrees with numerical solution okay when rounded to three significant digits. So this is the modification in Gaussian elimination and this particular method called Gaussian elimination method with partial pivoting.

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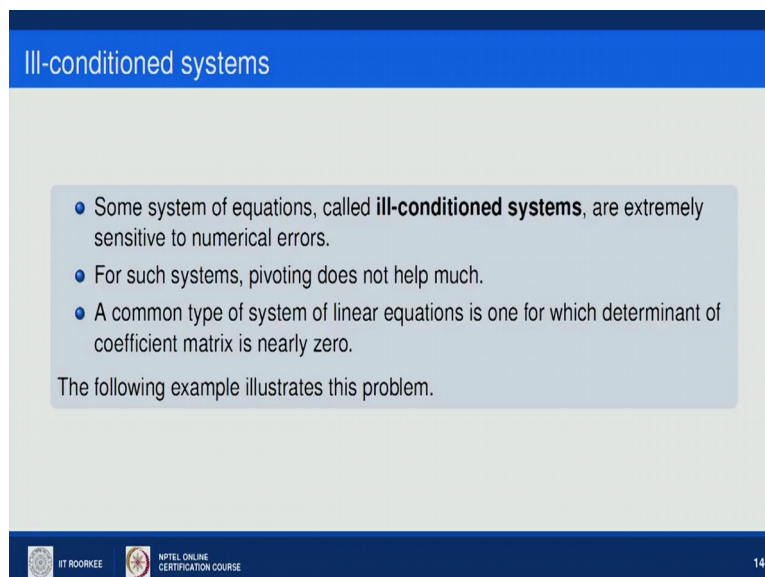
Ill-conditioned systems

- The term *partial* in partial pivoting refers to the fact that each pivot search entries in the left column of the considered matrix or submatrix.
- This search can be extended to include every entry in the coefficient matrix or submatrix known as Gaussian elimination with complete pivoting.
- Unfortunately, neither complete pivoting nor partial pivoting solves all problems of rounding error.

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Here the term partial in partial pivoting refers to fact that we have chosen the biggest absolute entry in 1st column in the 1st pass. So we are looking only at 1st column when we are going through the 1st pass. If we look instead of searching in 1st caller, if we search it in whole matrix or in 2nd pass like whole 2 by 2 sub matrix then the pivoting or pivoting is called complete pivoting or full pivoting. Unfortunately we are having some systems where we cannot solve the equation or system of equations using either partial pivoting or complete pivoting.

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Ill-conditioned systems

- Some system of equations, called **ill-conditioned systems**, are extremely sensitive to numerical errors.
- For such systems, pivoting does not help much.
- A common type of system of linear equations is one for which determinant of coefficient matrix is nearly zero.

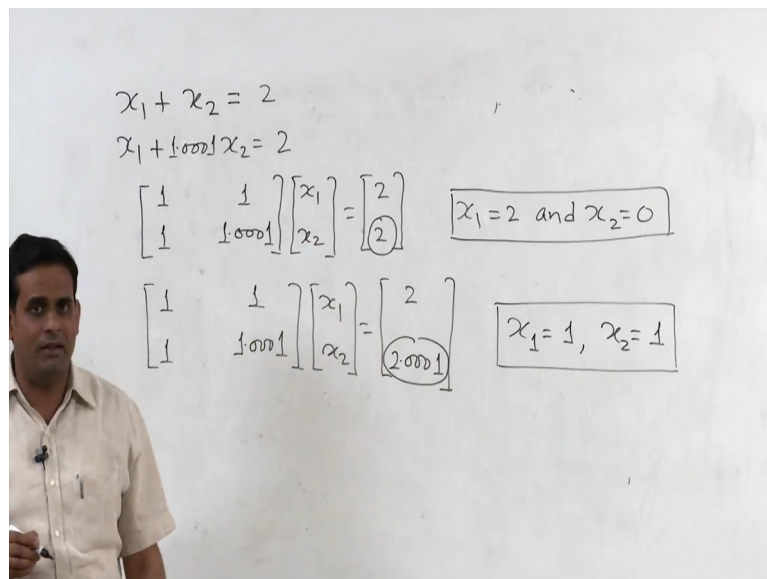
The following example illustrates this problem.

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Now a talk about some of those systems, so 1st of all I will take the system those are called ill conditioned systems, so some linear systems those are ill conditioned system are extremely

sensitive to the rounding off error or any other kind of numerical errors. For such systems pivoting does not help much even though you use partial pivoting, you use full pivoting we can have correct solution using the numerical scheme. So I will illustrate or I will explain such ill conditioned system with the help of an example after that what I will do? I will define matrix norm and I will relate the matrix norm with the conditions when the system becomes ill condition means how to measure whether the system is ill conditioned or it is not.

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Handwritten equations on the whiteboard:

$$x_1 + x_2 = 2$$

$$x_1 + 1.001x_2 = 2$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1.001 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \boxed{x_1 = 2 \text{ and } x_2 = 0}$$

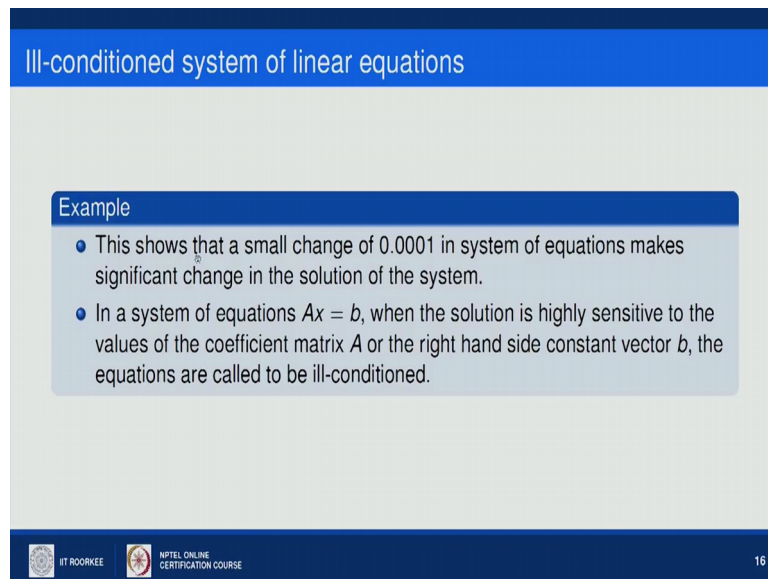
$$\begin{bmatrix} 1 & 1 \\ 1 & 1.001 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.001 \end{bmatrix} \quad \boxed{x_1 = 1, x_2 = 1}$$

So let us take an example which is having 2 equations and 2 unknowns, so a very small example and then we will see that this particular example of an ill conditioned system. So let us take 1st equation as $x_1 + x_2 = 2$ and the 2nd equation is $x_1 + 1.001x_2 = 2$, so here in the matrix form I can write the coefficient matrix as 1, 1, 1 and 1.001 into the unknown vector that is x_1 and x_2 and right hand side vector is 2, 2. If we solve this particular 2 by 2 system just looking at the system we can say the solution of this system is $x_1 = 2$ and $x_2 = 0$, so this particular solution satisfied our stem of equations and hence it is an exact solution of the system.

Now what I will do? I will not change my coefficient matrix but there is a little change in right hand side vector in the entry right-hand side entry of 2nd equation let us say earlier it was 2 but it has become let us say 2.001 due to some error, it may be some numerical error or some error in the measurement let us say some sensor error. So what will happen? I am having a very small change in the input data earlier it was 2 and now just here it is 2.0001 that is a change of the order of 10^{-4} .

Now if you look the solution of this particular system this new system what I will get? x_1 equals to 1 and x_2 equals to 1. Now look at these 2 solutions and the 2 systems, in input data I am having a very little change, a change of the order 10^{-4} , here it was 2 it is 2.0001 however if you see the change in the solution we are having a large deviation, so if we are having large deviation in the solution just due to a small change in the input data such a system is called ill condition system, so it is a perfect example of an ill conditioned.

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The slide has a blue header with the title "Ill-conditioned system of linear equations". Below the header is a light gray area containing a blue box labeled "Example". Inside the "Example" box, there are two bullet points. The first bullet point states: "This shows that a small change of 0.0001 in system of equations makes significant change in the solution of the system." The second bullet point states: "In a system of equations $Ax = b$, when the solution is highly sensitive to the values of the coefficient matrix A or the right hand side constant vector b , the equations are called to be ill-conditioned." At the bottom of the slide, there is a dark blue footer containing logos for "IIT ROORKEE" and "NPTEL ONLINE CERTIFICATION COURSE", and the page number "16" on the right.

Ill-conditioned system of linear equations

Example

- This shows that a small change of 0.0001 in system of equations makes significant change in the solution of the system.
- In a system of equations $Ax = b$, when the solution is highly sensitive to the values of the coefficient matrix A or the right hand side constant vector b , the equations are called to be ill-conditioned.

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Now the same example and this particular example shows that a small change of 0.0001 in system of equations makes significant change in the solution of the system, so in a system of equation $Ax = b$, when the solution is highly sensitive to the value of the coefficient matrix A or the right-hand side factor b , the equations are called to be ill conditioned.

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Matrix Norm

Definition
A matrix norm $\|\cdot\| : A_{n \times n} \rightarrow R$ is a real valued function on the set of all $n \times n$ matrices satisfying the following conditions:

- 1 $\|A\| \geq 0$;
- 2 $\|A\| = 0 \Leftrightarrow A$ is a zero matrix;
- 3 $\|\alpha A\| = |\alpha| \|A\|$ for $\alpha \in R$;
- 4 $\|A + B\| \leq \|A\| + \|B\|$;

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A matrix norm is a real valued function which is defined on a set of let us say n by n matrices and satisfying the following conditions, condition one the norm of a matrix will be always non-negative. The 2nd condition is it will be 0 if and only if A is a null matrix or 0 matrix, if you multiply the matrix A by some scalar α then norm of the α times A will become absolute value of α times norm of A for R α belongs to set of real numbers. Now 4th condition is triangle inequality that is basically if you are having 2 matrices A and B of order n by n then norm of A plus B will be always less than equals to norm of A plus norm of B , so any real valued function satisfying these property on a set of n by n matrices is called a norm.

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Matrix Norm

Definition: Natural/Induced matrix norm
The induced norm of a matrix A is defined as

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

where, $\|x\|$ is the vector norm of x on R^n .

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We are having different types of norm the 1st one is natural or induced matrix norm, it is defined for a matrix A by this equation so norm of A will be maximum of norm of A times x where x is a vector in our end and it is a vector having unit norm, vector of unit length. So here if you having a vector of unit length you multiply this vector with A, so norm of A equals to maximum of norm of x.

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Matrix Norm

Definition: Maximum row norm

For any $n \times n$ matrix A, the maximum row norm is defined as

$$\|A\| = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$$



Definition: Euclidean norm

$$\|A\|_2 = \sqrt{r_\sigma(A^T A)},$$

where,

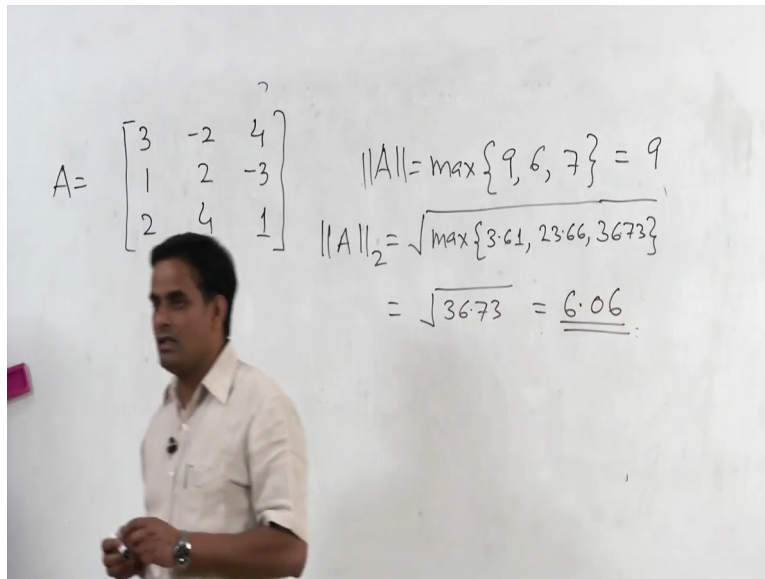
$$r_\sigma(A) = \max_{\lambda \in \sigma(A)} |\lambda|$$

Here, $\sigma(A)$ represents the spectrum of A.



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The 2nd type of norm we are going to define here maximum row norm, for any n cross n matrix A, the maximum row norm is defined as norm of A equals to maximum over the absolute sum of all row means absolute sum means we are having entries of each row, let us take entries in 1st row take the absolute value of each element of the 1st row, take the ((23:50) sum similarly take the absolute sum of 2nd row and so on and then maximum of those sums will give you the maximum row norm. The 3rd norm we are going to define here over the set of matrices is Euclidean norm and that is gained by square root of r of ATA, where r of A is given by maximum absolute value of lambda and lambda is a eigen value of A, it means maximum of the all eigenvalues in terms of absolute value.

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Handwritten on the whiteboard:

$$A = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 2 & -3 \\ 2 & 4 & 1 \end{bmatrix}$$
$$\|A\| = \max\{9, 6, 7\} = 9$$
$$\|A\|_2 = \sqrt{\max\{3.61, 23.66, 36.73\}}$$
$$= \sqrt{36.73} = \underline{\underline{6.06}}$$

Let us take a matrix 3 minus 2, 4 as the 1st row, 1, 2 minus 3 as the 2nd row and 2, 4, 1 as the 3rd row, so here if you look the maximum row norm of this matrix let us say A is given by maximum of so in the 1st row 3 the absolute value of minus 2 will be 2, so 3 plus 2 plus 4 it will become 9, for the 2nd row 1 plus 2 plus 3 so it will become 6 and in the 3rd row it will become 2 plus 4 plus 1 that is 7 so it is 9. On the other hand Euclidean norm for this particular matrix will be square root of maximum of eigenvalues of transposed A, so if this is the A transposed A will be having the eigen value as 3.61, 23.66 and 36.73, so it means square root of 36.73 which is maximum among these 3 eigen value and these are the eigen value of a transposed A and this is coming out as 6.06.

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Matrix Norm

Example

Consider a 3×3 matrix,

$$\begin{pmatrix} 3 & -2 & 4 \\ 1 & 2 & -3 \\ 2 & 4 & 1 \end{pmatrix}$$

Here, maximum row norm $\|A\| = 9$, while the Euclidean norm is

$$\|A\|_2 = \sqrt{\max\{3.61, 23.66, 36.73\}} = 6.06$$

Here, the Euclidean norm induces the matrix norm.

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Ill-conditioned system and matrix norm

Theorem

Let A be nonsingular matrix. Then, the solution x and y of the systems $Ax = b_1$ and $Ay = b_2$, respectively, satisfy

$$\frac{\|x - y\|}{\|x\|} \leq \|A\| \|A^{-1}\| \frac{\|b_1 - b_2\|}{\|b_1\|}$$

The multiplying coefficient $\|A\| \|A^{-1}\|$ is interesting. It depends entirely on the matrix in the problem and not on the right-side vector, yet it shows up as an amplifier to the relative change in the R.H.S. vector.



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So here you can see the Euclidean norm induces the matrix norm. Now to relate this norm with the ill conditioned system we are having an important result in this theorem. In this theorem let A be non-singular matrix, then the solution x and y of the system $Ax = b_1$ and $Ay = b_2$ respectively satisfy this particular inequality.

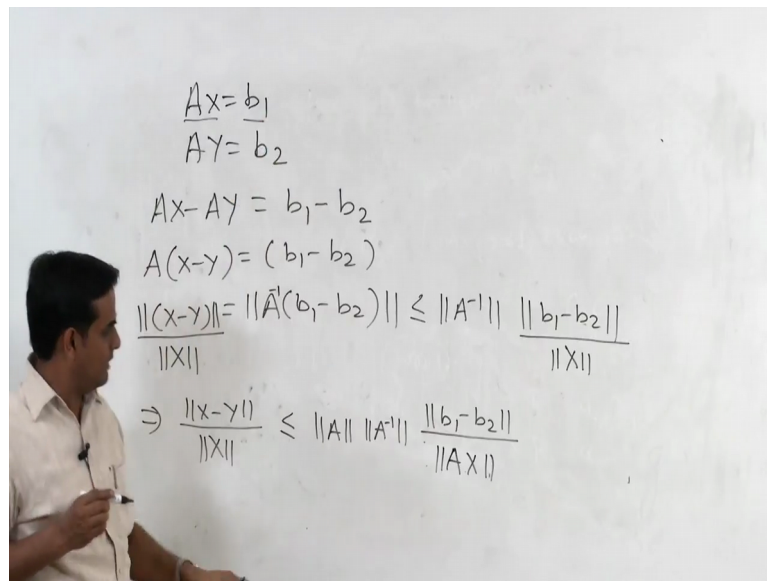
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$$\begin{aligned}
 Ax &= b_1 \\
 Ay &= b_2 \\
 Ax - Ay &= b_1 - b_2 \\
 A(x - y) &= (b_1 - b_2) \\
 \frac{\|A(x - y)\|}{\|A\| \|x - y\|} &= \frac{\|A(b_1 - b_2)\|}{\|A\| \|b_1 - b_2\|} \leq \|A\| \|A^{-1}\| \frac{\|b_1 - b_2\|}{\|b_1\|} \\
 \Rightarrow \frac{\|x - y\|}{\|x\|} &\leq \|A\| \|A^{-1}\| \frac{\|b_1 - b_2\|}{\|b_1\|}
 \end{aligned}$$

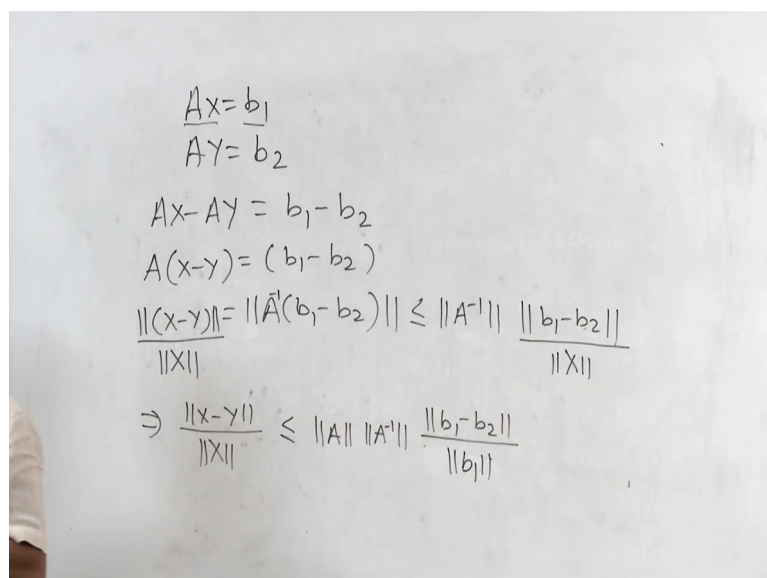
So I am having to system $Ax = b_1$ on other system is $Ay = b_2$, so I can write $Ax - Ay$ if I subtract the 2nd from the 1st equals to $b_1 - b_2$ or it can be written as $Ax - Ay = b_1 - b_2$ of our $x - y$ equals to $A^{-1}(b_1 - b_2)$, it is given that A is non-singular so A^{-1} exist. Now norm of this particular vector equals to norm of this vector and by the inequality I can write norm of $Ax - Ay$ less than equals to norm of A into norm of $x - y$.

So I can write norm of A inverse into norm of $b_1 - b_2$, so norm of $x - y$ less than equals to norm of A inverse $b_1 - b_2$, if I divide it by the norm of x this will also divide divided by the norm of x . Now I can write it the left-hand side as such $x - y$ norm upon norm of x , this is less than equal to if I multiply in the right inside in the numerator as well as in denominator by the norm of A so it will become norm of A, norm of A inverse into norm of $b_1 - b_2$ upon norm of A since I multiplied in the numerator so I have to multiply in denominator also into norm of x .

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$$\begin{aligned} Ax &= b_1 \\ Ay &= b_2 \\ Ax - Ay &= b_1 - b_2 \\ A(x - y) &= (b_1 - b_2) \\ \frac{\|x - y\|}{\|x\|} &= \frac{\|A^{-1}(b_1 - b_2)\|}{\|x\|} \leq \|A^{-1}\| \frac{\|b_1 - b_2\|}{\|x\|} \\ \Rightarrow \frac{\|x - y\|}{\|x\|} &\leq \|A\| \|A^{-1}\| \frac{\|b_1 - b_2\|}{\|Ax\|} \end{aligned}$$



$$\begin{aligned} Ax &= b_1 \\ Ay &= b_2 \\ Ax - Ay &= b_1 - b_2 \\ A(x - y) &= (b_1 - b_2) \\ \frac{\|x - y\|}{\|x\|} &= \frac{\|A^{-1}(b_1 - b_2)\|}{\|x\|} \leq \|A^{-1}\| \frac{\|b_1 - b_2\|}{\|x\|} \\ \Rightarrow \frac{\|x - y\|}{\|x\|} &\leq \|A\| \|A^{-1}\| \frac{\|b_1 - b_2\|}{\|b_1\|} \end{aligned}$$

Ill-conditioned system and matrix norm

Theorem

Let A be nonsingular matrix. Then, the solution x and y of the systems $Ax = b_1$ and $Ay = b_2$, respectively, satisfy

$$\frac{\|x - y\|}{\|x\|} \leq \|A\| \|A^{-1}\| \frac{\|b_1 - b_2\|}{\|b_1\|}$$

The multiplying coefficient $\|A\| \|A^{-1}\|$ is interesting. It depends entirely on the matrix in the problem and not on the right-side vector, yet it shows up as an amplifier to the relative change in the R.H.S. vector.



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We know that norm of A into norm of x will be always greater than equals to norm of Ax , so if I replace it this inequality will not change and you know that norm of Ax equals to here Ax equal to b_1 so norm of Ax will be norm of b_1 , so I can replace here it norm of b_1 and this is the result of the theorem. Here we can see that multiplying coefficient norm of A into norm of A inverse it is interesting, it depends entirely on the matrix in the problem and not on the right-hand side vector, yet it shows up as an amplifier to the relative change in the right-hand side vector. So if it will be a high-value there is norm of A into norm of A inverse for matrix there will be large change in the solution due to a small change in the right-hand side vector and hands what will happen? The system will become ill conditioned.

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Ill-conditioned system and matrix norm

Definition: Condition number

For a given non-singular matrix $A \in \mathbb{R}^{n \times n}$ and a given matrix norm $\|\cdot\|$, the condition number of A with respect to the given norm is defined by

$$\text{cond}(A) = \|A\| \|A^{-1}\|$$

When the condition number of a matrix is very large, even a small variation in the RHS vector can lead to a drastic variation in the solution. Such matrices are called ill-conditioned matrices. The matrices with small condition number are called well-conditioned matrices.



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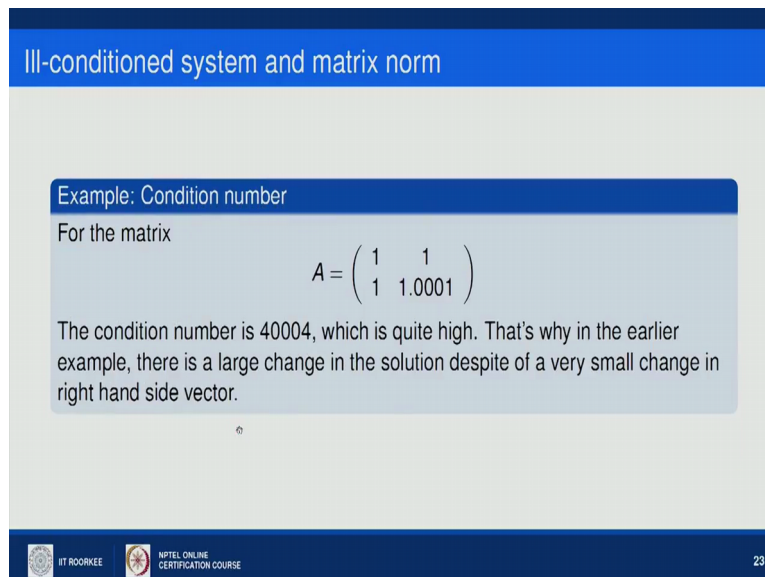


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So based on this idea we can define condition number, so for a given non-singular matrix A which is a real having real entries and of size n by n and a given matrix norm, the condition number of A with respect to the given norm is defined by condition number of A is norm of A into norm of A inverse means this particular term, so if condition number for a matrix is large even a small variation on the right-hand side vector that is in b_1 or b_2 can lead to a drastic variation in the solution and such matrices are called ill conditioned matrices.

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The slide has a blue header with the text "Ill-conditioned system and matrix norm". Below the header is a light gray area containing a blue-bordered box. Inside this box, the text "Example: Condition number" is followed by "For the matrix" and the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1.0001 \end{pmatrix}$. Below the matrix, it states: "The condition number is 40004, which is quite high. That's why in the earlier example, there is a large change in the solution despite of a very small change in right hand side vector." At the bottom of the slide, there is a dark blue footer containing logos for "IIT ROORKEE" and "NPTEL ONLINE CERTIFICATION COURSE", and the page number "23" on the right.

So if you see the earlier example which we have taken 1, 1, 1, 1.0001, so I was so this example was taken in the beginning when I introduced the ill conditioned system, so norm for if this is the matrix M , the condition number of this matrix will be something around which is quite large and hence the system was the ill conditioned system. So I will stop this lecture here, in the summary of this lecture 1st I introduced the Gaussian elimination with partial pivoting and in the 2nd phase of the lecture I have introduced an ill conditioned system and the idea of condition number to measure the whether a system is ill conditioned or not and in the next lecture I will continue with the direct method of solving the linear system and there I will introduce a technique called LU decomposition. Thank you very much.