Numerical Methods

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Lecture No 17

Interpolation Part II

Welcome to lecture series on numerical methods, last class we have discussed this interpolation on finite difference operators. So in the finite difference operator first we have started about this Newton's forward difference operator like Delta operators, Nabla operators then this Central difference operators and Average operators.

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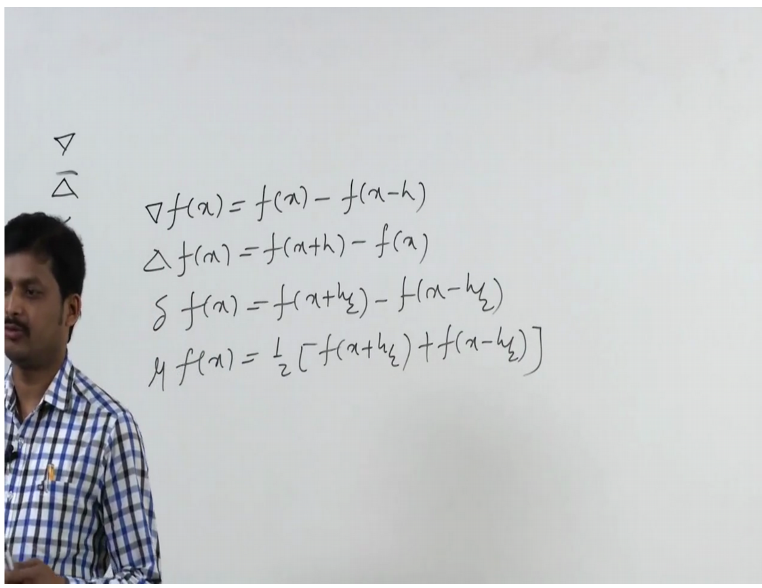

So, this finite difference operators whatever we have discussed that includes like Nabla, Delta and this Central difference operator Delta and the Average operator Mu and in this presentation first I will just give few introduction about this few operators that I have discussed in the last class then we will just go for Newton's forward difference formula and Newton's backward difference formula and then the error approximation.

So this forward difference operator, backward difference operator, whatever it is operated in the last lecture that are basically expressed in the form of Nabla f of x that is represented as f of x minus f of x minus h and delta of f of x it is expressed in the form of f of x plus h minus f of x and this Central difference operator delta of f of x it is expressed in the form of f of x plus h by 2 minus f of x minus h by 2. And this Average operator this is expressed in the form of half of f of x plus h by 2 plus f of x minus h by 2.

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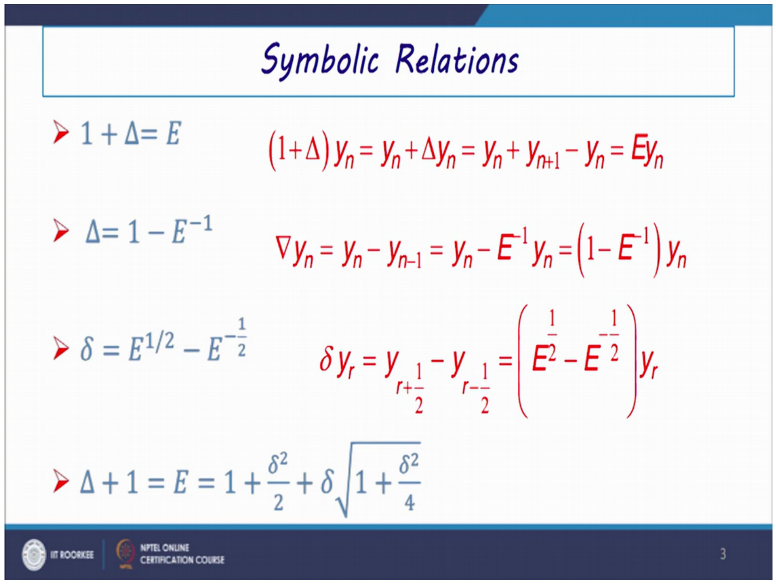

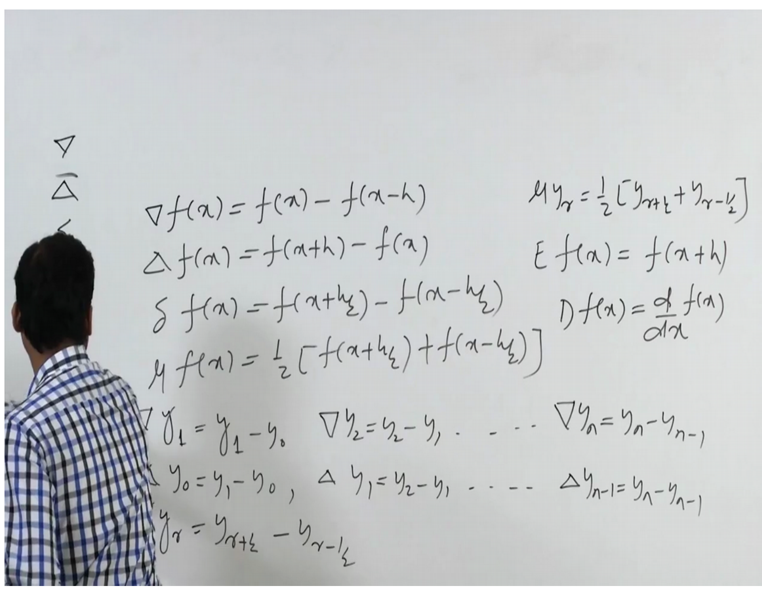

So whenever we will go for this application of this 4 difference operators, so then we have to go for this Shift operators and Differential operator here. Basically if we are just using sequentially these operators, so sequentially this can be applied like if Nabla of y 1 can be applied so it can be written in the form of y 1 minus y 0 here. So then Nabla of y 2 can be written as y 2 minus y 1, so likewise we can just write Nabla of y n can be written as y n minus y n minus 1.

Similarly, if we will just apply this forward difference operator sequentially, so delta of y 0 this can be written as y 1 minus y 0, delta of y 1 can be written as y 2 minus y 1, so likewise if you will just express delta of y n can be written as y n minus y n minus 1. So similarly we can just use for Central difference operator like delta of y r if you will just write this can be written as y of r plus half minus y of minus half here.

Similarly, if you will just use this operator in Average operator,, so this can be expressed in the form of Mu of y r this can be written in the form of half of y r plus half plus y of r minus half here. And if we are just using this operators in a Shift operator form usually E is called shift operator here, so usually this shift operator is generated in the form of E and it can be written in the form of E of f of x as f of x plus h here.

Similarly, if you will just write Differential operator so this Differential operator is signified as D here, and if it is operated on a f of x here that can be written as d by dx of f of x. So if we want to extend these operations in different forms like how we can express this Delta operator, Nabla operator or the Central different operators or this Average operators in Shift operator from then we have to expand this operators in different forms.

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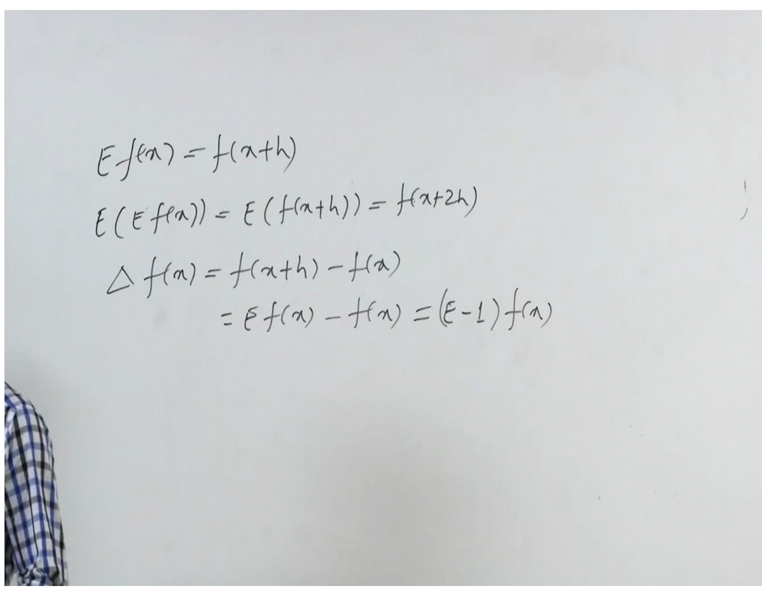

So first if you just express this Shift operators in a recursive way, so we can just write, E of f of x as f of x plus h, then E of f of x plus h as f of x plus 2 h there. So if you just express f of x plus h in a Taylor's series expansion form then we can write this Taylor's series expansion as that is basically if we are just writing E as a shift operator here it can be expressed in the form of f of x plus h here.

So then sequentially if you just use E of E of f of x here, so it can be expressed as E of f of x plus h and this can be written as f of x plus 2 h here. So similarly, if you will just use like a delta of f of x here, so it can be written in the form of f of x plus h minus f of x is this one. So it can be written as E f of x minus f of x here, and which can be written as E minus 1 into f of x.

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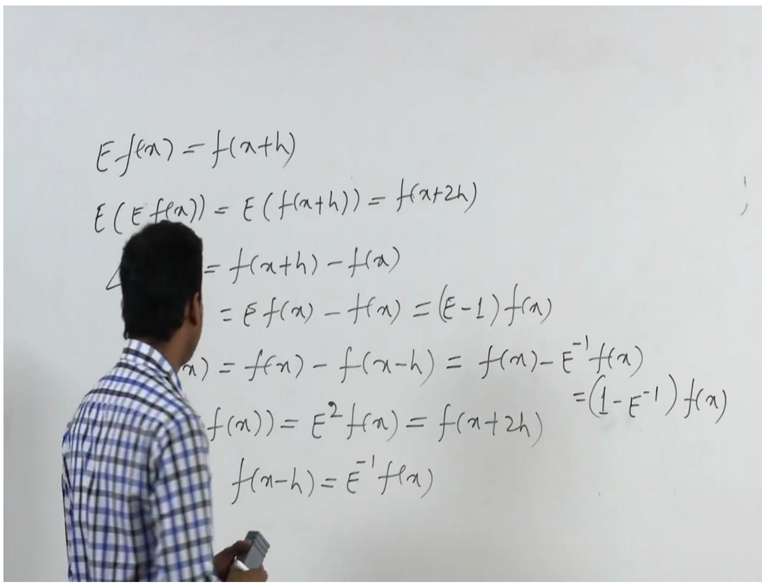

Similarly, we can express Nabla operator in the form of shift operator here, so that is basically written in the form of like Nabla of f of x here as f of x minus f of x minus h, since we are just writing E of f of x in this form here, so that is why it can be written as E of E of f of x as E raise power 2, f of x here. So this means that this can be written as f of x plus 2 h here, if we want to write f of x minus h then it can be written in the form of E of minus 1 f of x here.

So similarly we can express this expansion here as f of x minus E power of minus 1 of f of x which can be written as f of x or we can just write this as 1 minus E inverse that is the operator here operated on function f of x here. So different operators like central different operators also we can just express that is in the power of E to the power half minus E to the power minus half, since we are expressing this Central difference operator that is of delta of f of x usually it is written as f of x plus h by 2 minus f of x minus h by 2.

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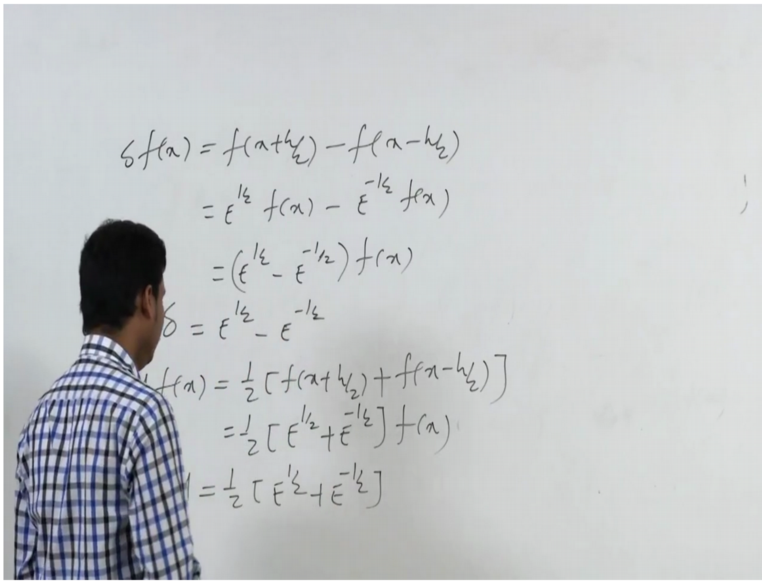

So which can be written as delta of f of x as f of x plus h by 2 minus f of x minus h by 2 and this can be written as E power of half f of x minus E power of minus half of f of x which can be written as E power half minus E power minus half f of x here. So obviously we can just write the operator as delta equal to E to the power half minus E to the power minus half here.

Similarly, the Average operator it can be expressed in the form of Mu of f of x as half of f of x plus h by 2 plus f of x minus h by 2. And this can be expressed as half of E power of half plus E power of minus half operated on f of x and hence we can just express Mu as half of E power of half here plus E power of minus half here.

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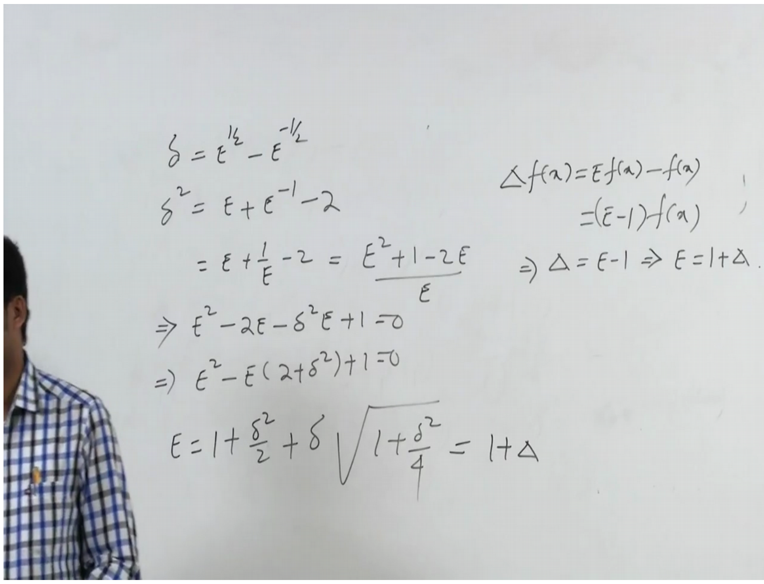

So sequentially if you apply this shift operator then we can express delta plus 1 equal to E equal to 1 plus del square by 2 plus delta square root of 1 plus del square by 4 also. So it can be obtained from this operator here since usually we are just expressing delta as E raise power half minus E raise power minus half here.

If you will take square on both the sides then we can express that as since we are just expressing delta as E power of half minus E power of minus half here, so that is why we can express del square as E plus E raised to power minus 1 minus 2 here. And which can be accessed in the form of E plus 1 by E minus 2 and which can be written as E square plus 1 minus 2 E by E here. And in a product form if you just write this can be expressed as E square minus 2 E minus del square of E plus 1 equals to 0 here. And this implies that it can be written as E square minus E into 2 plus del square plus 1 this equal to 0.

If we want to find this root here that is as E, E can be written in form of that as 1 plus del square by 2 plus Delta square root of 1 plus del square by 4. So this is formulation usually we are just obtaining here that is E in the form of Central difference operator, so if we want to express this one E as in the form of forward difference operator here, so E is usually written as 1 plus Delta also.

Since we are just writing delta of f of x as E of f of x minus f of x here which can be expressed as E minus 1 of f of x and this implies that delta can be expressed as E minus 1 this implies E equals to 1 plus Delta here. So directly if we want to find this root here we can just write this one as minus V plus or minus square root of V square minus 4 AC square root by 2 A, so here A is 1 here and the coefficient minus B means this is B coefficient is 2 plus Delta square and C is one, so that is why this is the root for this equation here.

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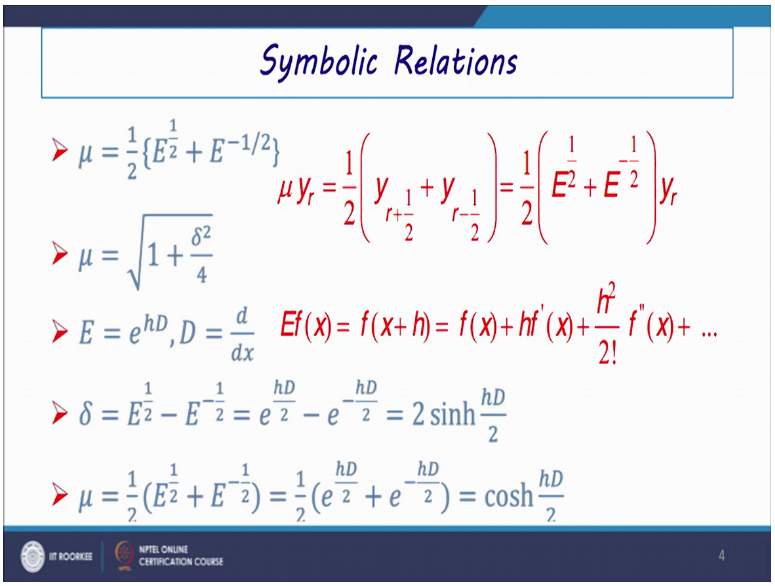

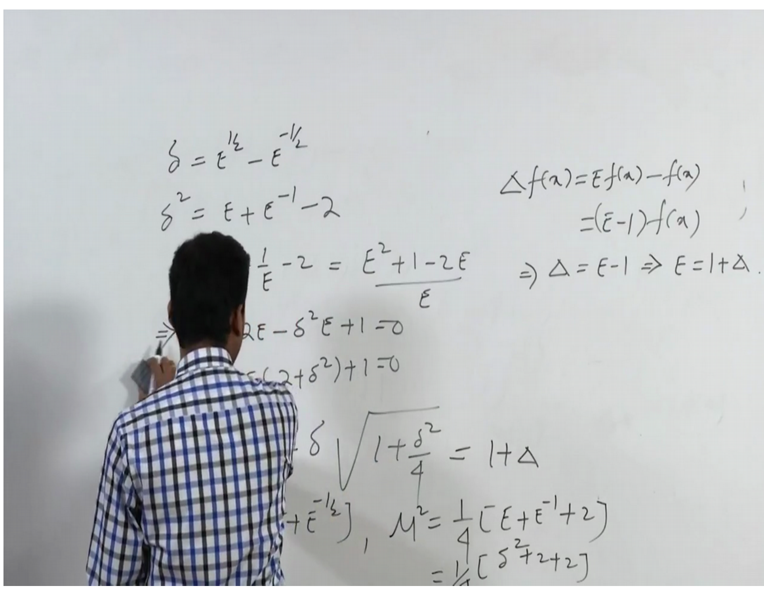

Next step we want to express this delta in the form of Average operator here. We can just express here the Average operator as Mu equal to half of E power of half plus E of minus half here. And if we take square at both the sites, so it can be express Mu square equal to 1 by 4 this can be expressed as E plus E power minus 1 plus 2 here. Since already we have known that E plus E inverse it can be written as here if you just see this can be written as delta square plus here, so can I just write this one as 1 by 4 del square plus 2 plus 2 here.

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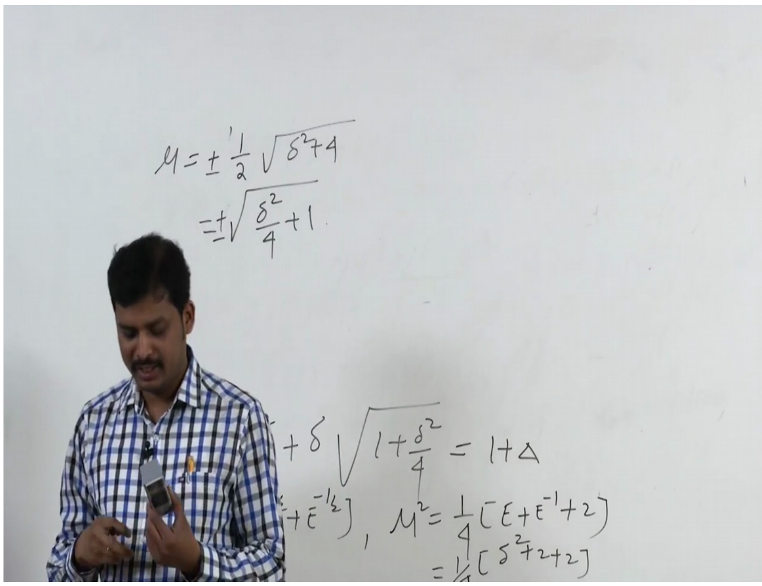

And if I want to find this square root for this Mu of function, so it can be written in the form of like Mu equals to that is expressed here also that is in the form of like plus or minus 1 by 2 square root of del square plus 4 here. So usually it is just written in the form like if you will just write inside this function here that can be written in the form of del square by 4 plus or minus here, so plus 1 here so this is the expression that is written in the form of Mu as in the form of delta here.

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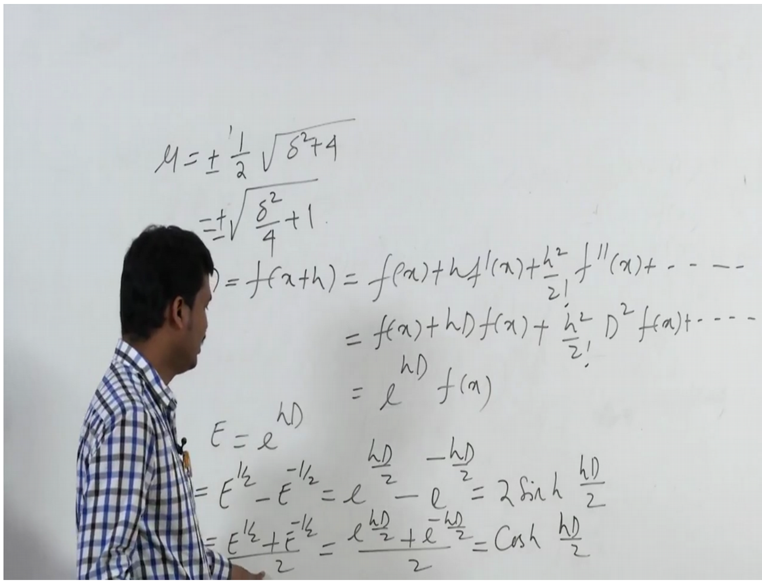

So next if we want to express this Shift operator in the form of Differential operator here, so usually E of f of x is written in the form of f of x plus h and hence if you will just expanded in Taylor series from here that can be written as f of x plus h f dash x plus h square by factorial 2, f double dash x plus this one. So we can just write this one as since f dash x is written in form of here as hD of f of x, h square by 2 factorial d square of f of x here.

So we can express this as E raise hD operated on f of x, so directly we can just write E as E to the power hD here. So specifically if you are just expressing this one this shift operator in the form of Differential operator here then we can express this Central difference operator Delta as E to the power half minus E to the power minus half which can be written as E to the power hD by 2 minus E to the power minus hD by 2 which can be written as 2 sine hyperbolic hD by 2 here.

Similarly, if you just write Average operator here Mu this can be written as E to the power of half plus E to the power minus half by 2, which can be written as E to the power hD by 2 plus E to the power minus hD by 2 and divided by 2 here and this can be written as I think cos hyperbolic hD by 2 here. Since usually it is just express in the form of half, so that is why half is coming over there so this half will cancel there, so cos hyperbolic hD by 2 here.

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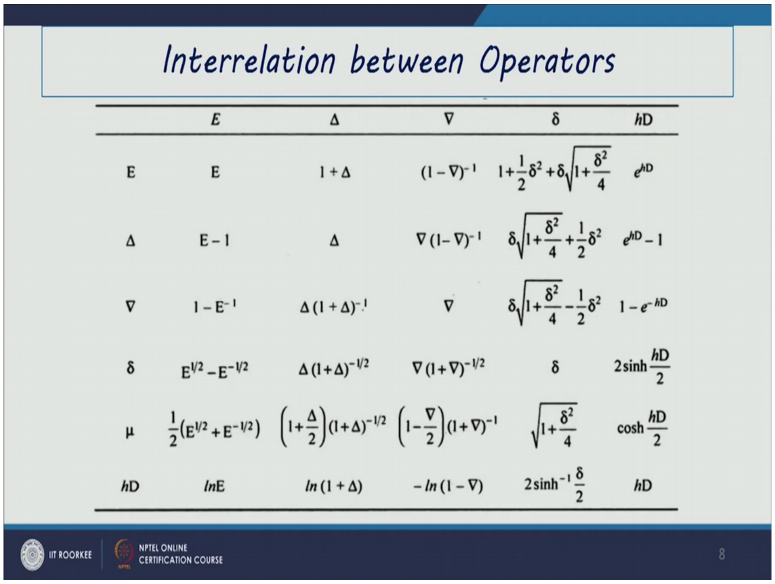

Interrelation between Operators

	E	Δ	∇	δ	hD
E	E	$1+\Delta$	$(1-\nabla)^{-1}$	$1+\frac{1}{2}\delta^2+\delta\sqrt{1+\frac{\delta^2}{4}}$	e^{hD}
Δ	$E-1$	Δ	$\nabla(1-\nabla)^{-1}$	$\delta\sqrt{1+\frac{\delta^2}{4}}+\frac{1}{2}\delta^2$	$e^{hD}-1$
∇	$1-E^{-1}$	$\Delta(1+\Delta)^{-1}$	∇	$\delta\sqrt{1+\frac{\delta^2}{4}}-\frac{1}{2}\delta^2$	$1-e^{-hD}$
δ	$E^{1/2}-E^{-1/2}$	$\Delta(1+\Delta)^{-1/2}$	$\nabla(1+\nabla)^{-1/2}$	δ	$2\sinh\frac{hD}{2}$
μ	$\frac{1}{2}(E^{1/2}+E^{-1/2})$	$\left(1+\frac{\Delta}{2}\right)(1+\Delta)^{-1/2}$	$\left(1-\frac{\nabla}{2}\right)(1+\nabla)^{-1}$	$\sqrt{1+\frac{\delta^2}{4}}$	$\cosh\frac{hD}{2}$
hD	$\ln E$	$\ln(1+\Delta)$	$-\ln(1-\nabla)$	$2\sinh^{-1}\frac{\delta}{2}$	hD

So next we will just go for this interrelation between difference different operators here, so that is E as expressed as in the form of Delta in the form of Nabla, in the form of Central difference operator Delta and then it can expressed in the form of Differential operators. So all of this operators we want to express in different forms that is E, Delta, Nabla and the Central difference operator small Delta then Mu is the Average operator then the Differential operators here.

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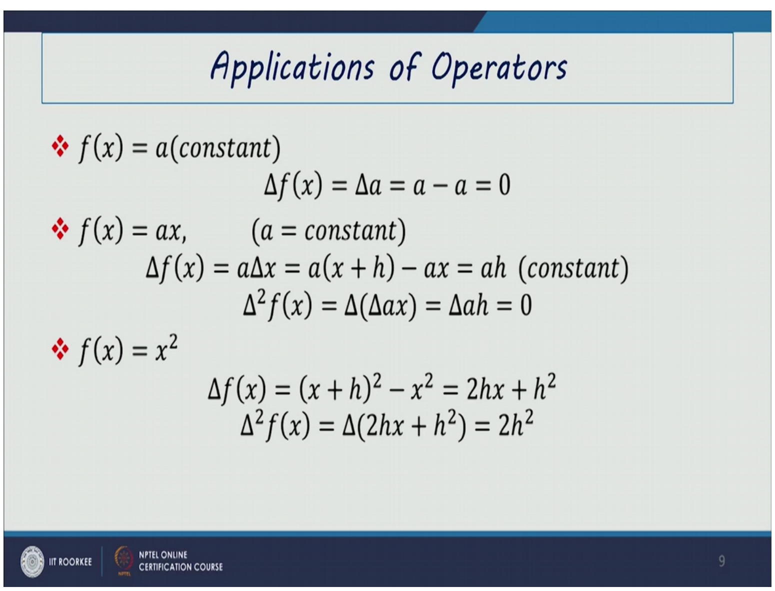

So then we will just go for this polynomial approximation in Differential operators, how we can use this Differential operators in a differential form also there. This means that how we can use this forward difference operator like your Delta, Nabla, in a differential sense for a polynomial here. So if suppose we are just considering f of x is a polynomial here, first we will assume f of x is a constant polynomial suppose.

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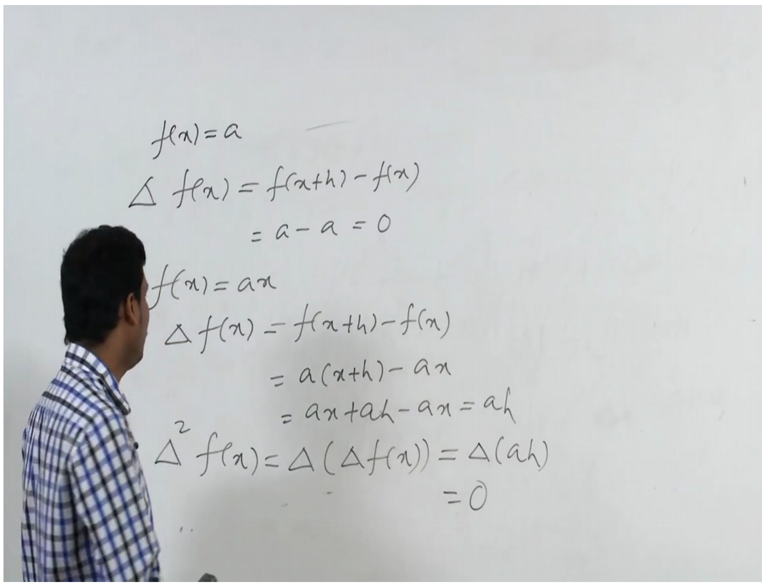

If f of x is a constant polynomial we can write f of x equals to a here. So if I am just writing this polynomial f of x as a constant polynomial here then i can apply here this forward difference operator delta of f of x can be written as f of x plus h minus f of x here, which can be written as a minus a this equals to 0 here. So next if i am just expressing f of x equal to ax that is a polynomial of degree 1 here then delta of f of x we can just write as f of x plus h minus f of x and this can be written as since f of x plus h it can be replaced as a into x plus h here minus ax, so this can be written as ax plus ah minus ax here and it can be written as ah here.

So if again will apply here Delta operator this means that del square of f of x for this function here we can just express Delta of Delta of f of x which can be written as Delta of ah and obviously we have already defined that Delta of a constant function this is just giving a zero value, so it can be expressed as 0 here.

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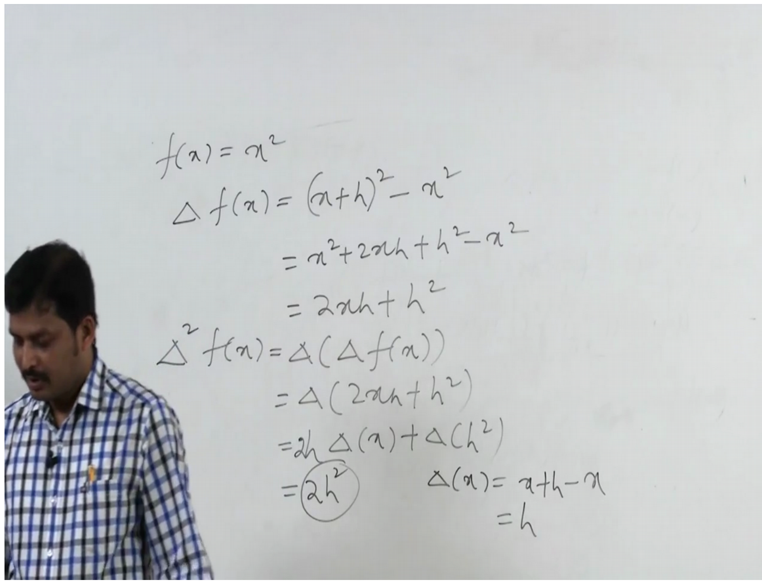

Similarly, if you just consider f of x is a 2nd degree polynomial that is f of x is equal to x square then delta of f of x this will just give you first-degree polynomial and delta square f of x that will just give you a constant polynomial there. So suppose if you are just considering f of x is equal to x square here, so if you just consider f of x equals to x square here then delta of f of x can be written as x plus h whole square minus x square. So I can just write this one as x square plus 2 xh plus h square minus x square and it can be written as 2 xh plus h square here.

So repeatedly if I will just apply one more operator here this means that Delta square f of x, I can just write Delta of Delta of f of x which can be written as Delta 2 xh plus h square here. Obviously already we have defined this one I can just write h of Delta of sorry 2 h delta of x here plus Delta of h square. So already we have obtained that delta operated on a constant function that is just giving you a zero value then we can just write this one as 2 h here sorry 2 h square here, since I have just written here delta of x here, 2 h square. since delta of x can be written as x plus h minus x this one so that is why it can be written as h here, so the final function is coming as 2 h square here.

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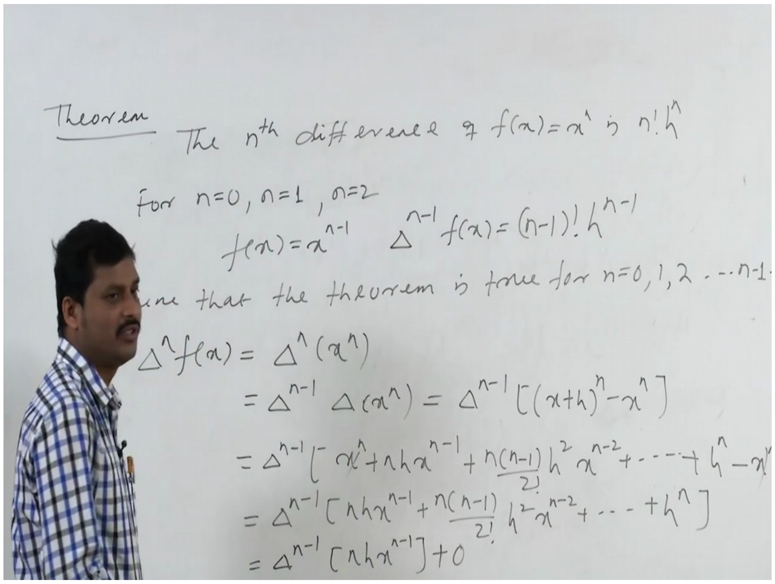

So if we are just applying this operators on a polynomial, so it is just operated on like 0, 1, 2 degree polynomial's, in a generalised form if you want to express this operators so there is a theorem for this and theorem statement states that, the n-th difference of f of x equal to x to the power n is n factorial h to the power n here. This means that the state when we can just write it off as the n-th difference of f of x is equal to X to the power n is n factorial h to the power n here.

So this we can just prove this theorem in the form of induction here, since for n equals to 0 we have already satisfied that one, for n equals to 1 we are satisfied, n equals to 2 in the previous case already we have shown that one. So if we will just assume that this theorem is true upto n minus 1 this means that 0, 1, 2 upto n minus 1 here we can just write if f of x equals to x to the power n minus 1 then it can be written as delta to the power n minus 1, f of x can be written as n minus 1 factorial, h the power n and minus 1 here.

So first we will assume that the theorem is a true for n equals to 0, 1, 2, upto n minus 1 suppose then each of this functions like f of x is equal to x it can be expressed as your function that is one there that is delta of f of x. if we are just expressing f of x equals to x square then delta square of f of x can be expressed as 2 there, so likewise it is just satisfied that this theorem is true for n equals to n minus 1 also there.

So if we will just assume that this theorem is true for 0, 1, 2 upto n minus 1 here then for del to the power n of f of x we can just write del to the power n of x to the power n here. So we can just express this one as del to the power minus 1 delta of x to the power n here. And this can be written as del to the power n minus 1 of x plus h whole power n minus x to the power n here.

And we can just expand this x plus h whole power n in a binomial sense and if we will just expand this one in binomial sense here we can just write del to the power n minus 1 and x to the power n plus n h x to the power n minus 1 plus n into n minus 1 by factorial 2, h square x to the power n minus 2, so likewise the final is h to the power n minus x to the power n here.

So x to the power n, x to the power and it can be cancel it out and i can just write del to the power n minus 1, n h x to the power n minus 1 plus and into n minus 1 by factorial 2, h square x to the power n minus 2 plus upto h to the power n. Since already we have already assumed that this theorem is true for all the values that is 0, 1, 2, upto n minus 1 if you just see here except this term all other terms will take 0 values here since all of these polynomial are degree less than n minus 1 here. So the final term we can just write as del to the power n minus 1 n h x to the power n minus 1 here, plus all other terms are assumed to be 0.

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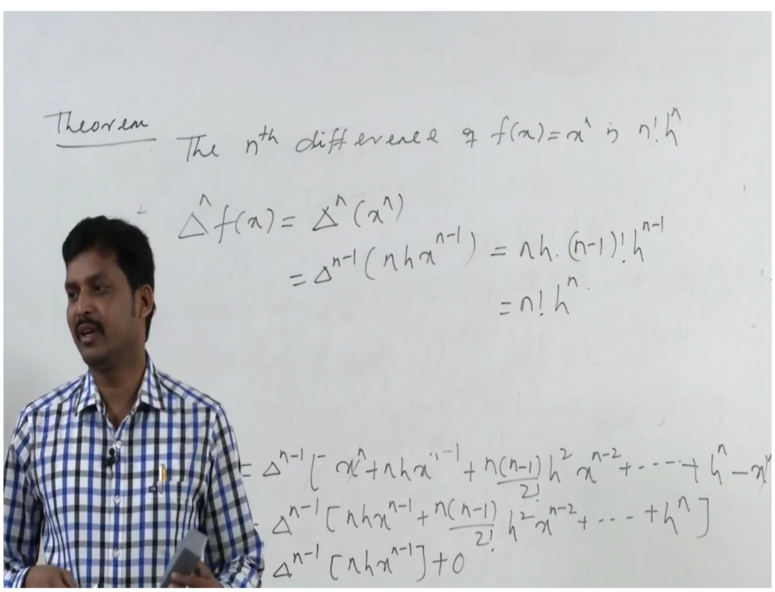

So in the final form we can just write delta of f of x sorry del to the power n f of x that is del to the power n, x to the power n, it can be written as del to the power n minus 1 n h x to the power n minus 1 here. And which can be written as n h into n minus 1 factorial, h to the power n minus 1, so it can be written as n factorial h to the power n. Finally we are just obtaining if f of x is a polynomial of degree n then we can write del to the power n, x to the power n as n factorial h to the power n here.

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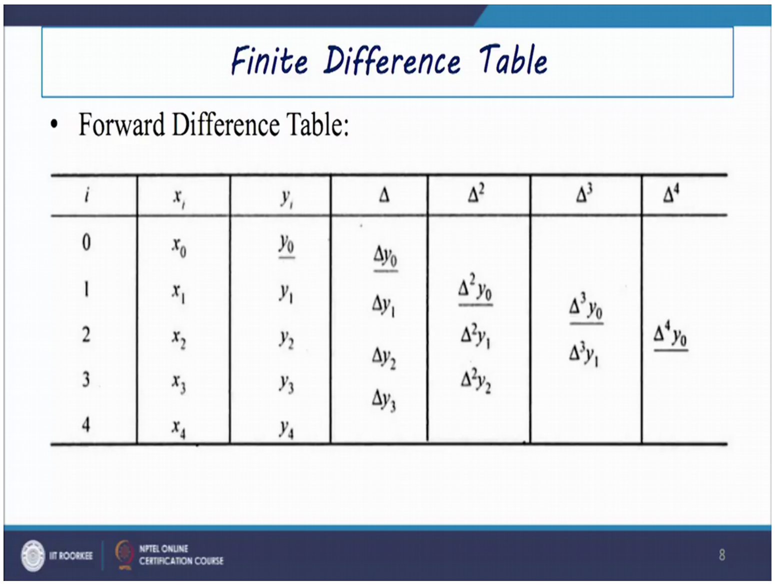

- Forward Difference Table:

i	x_i	y_i	Δ	Δ^2	Δ^3	Δ^4
0	x_0	y_0	Δy_0			
1	x_1	y_1	Δy_1	$\Delta^2 y_0$	$\Delta^3 y_0$	
2	x_2	y_2	Δy_2	$\Delta^2 y_1$	$\Delta^3 y_1$	$\Delta^4 y_0$
3	x_3	y_3	Δy_3	$\Delta^2 y_2$		
4	x_4	y_4				

So next we will just discuss about how we can just use in a tabular form of the forward difference operators, backward difference operator, Central operators and Average operators here. So if you will just write these operators in a tabular form in the forward difference table if you will just see you just moving in the end of the table to the beginning value of the function in a forward difference form. In a backward difference form we will just move to the value which is existing at the end of the table there.

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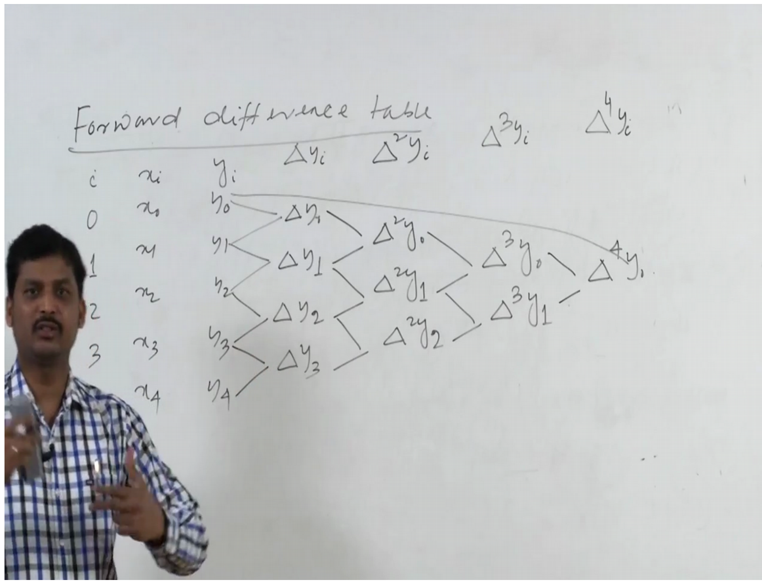

So if I am just writing this forward difference table, so this forward difference table can be written as, if i am just writing here i, xi, y i here, then i equals to 0, 1, 2, 3, 4, suppose, then xi values are x 0, x 1, x 2, x 3, x 4 are the values associate this functional values will be y 0, y 1, y 2, y 3, y 4, here. Then if you just use this forward difference operator then delta of y i, so first 2 differences I can write Delta of y 0 here, then 2nd one I can just write Delta of y 1 here, 3rd one I can just write delta of y 2 and last one I can just write delta of y 3 here, since difference of these 2 is just giving you delta y 0 here.

So then difference y 2 minus y 1 is just providing delta of y 1 , then difference of y 3 minus y 2 is providing delta of y 2, then difference of y 4 minus y 3 it is just providing the value of y 3 here. Similarly, if you just go for this 2nd operator here that is in former difference form I can just write the difference of this to it can be written as delta square of y 0 here, difference of this two can be written as del square y 1 here, the difference of these two you can just write delta square y 2 here.

Similarly, we can just go delta cube of y i and where we can just get the difference of these two it will just give you delta cube of y 0 here, difference of these two it will just give you delta cube of y 1. Again if you take the final difference since we have here existing 5 different points, so 5 different points means lastly we can just go upto the polynomial of degree 4 here, so then I can just write the difference of these two can be written as del to the power 4 of y 0 here.

So in the final firms we are just obtaining this value that if you just see in the final form this difference is moving to y 0 point in the final form, this is that if a value is existing at the beginning of the table and if it is asked to evaluate then we can use this forward difference form to evaluate this value in a differential form there.

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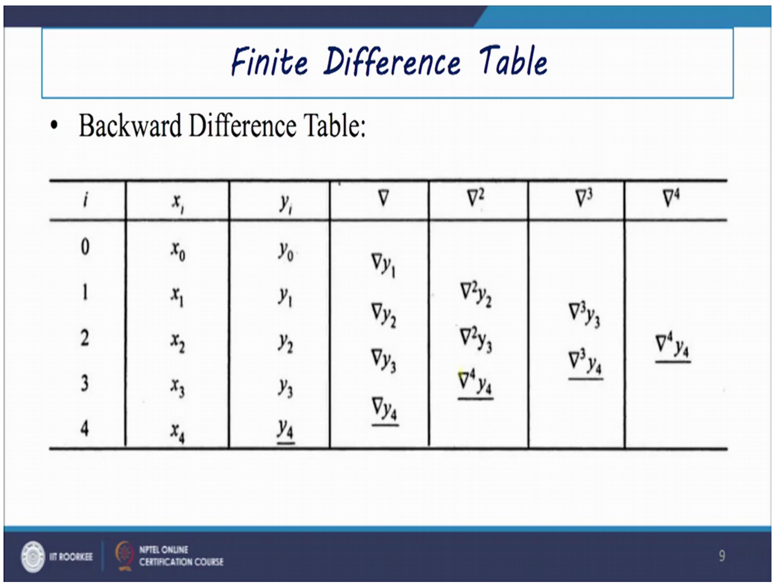

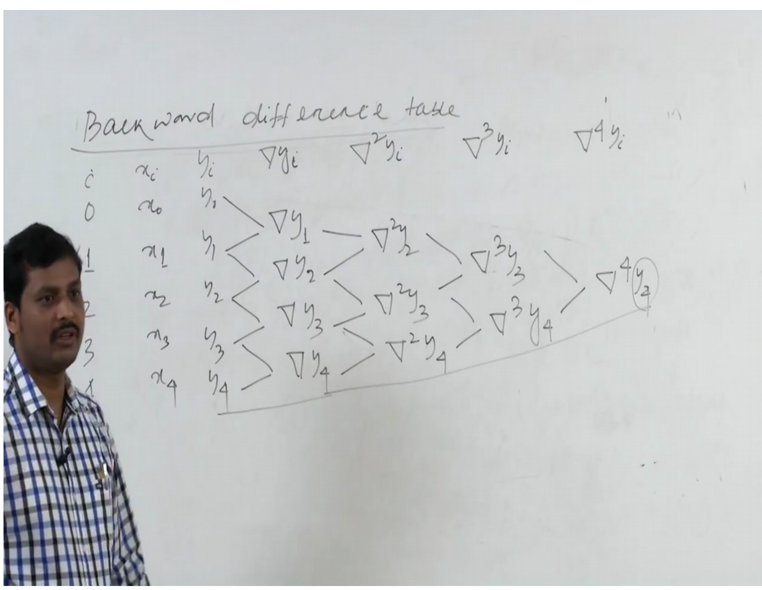

Then we will just go for the discussion of backward difference table here, so if we will just go for backward difference table the same formulation we will just use but this will take step back of the values for each of this calculated values or the calculated tabular points. So if you just use this backward difference table here that is in the form of i equal to 0, 1, 2, 3, 4 and associated tabular values are like xi is that is x 0, x 1, x 2, x 3, x 4 here, and associated variable values that is y i as y 0, y 1, y 2, y 3, y 4.

If you will just express this tabular values then we can just write this as Nabla y of i here, so Nabla y i we can just right here Nabla of y 1, then difference of y 2 minus y 1 this will just give Nabla of y 2 here, difference of y 3 minus y 2 that will just give you Nabla y 3, difference of y 4 minus y 3 this will just give you Nabla of y 4 here. Similarly, if you will take difference of Nabla's here, then Nabla square y i I can just write, so Nabla of y 2 minus y 1 so this will just give you Nabla square of y 2, then Nabla square of y 3, then difference of Nabla of y 4 minus Nabla of y 3 that will just give you Nabla square y 4 here.

If you take again difference of this 2 here that is Nabla cube of y i I can just write Nabla cube of y 3 here, then the difference of these 3 that is Nabla cube of y 4. And if I will just write Nabla 4 of y i here then the difference of these 2 that will just give you Nabla 4 of y 4. In the final form if you will just see this difference is moving towards the end of tabular value here. This means that if a function is asked to evaluate at the end of the table we can use this backward difference table to evaluate the values here.

So in the next lecture will just continue this Central difference operator and average difference operator and next onwards will just go for this Newton's forward difference formula and backward difference formula, thank you for listening this lecture.