Numerical Methods
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Lecture No 16
Interpolation Part 1

Welcome to the lecture series of numerical methods. In the present lecture will discuss about interpolation, in the interpolation first will just discuss what interpolation means or where we are just using this interpolation. In the 2nd step we will just discuss about the different operators like finite different operators that is a different classes it is used for this approximating this function with the polynomials and how we can just relate between this different operators so that I will just discuss in the last section of this lecture.

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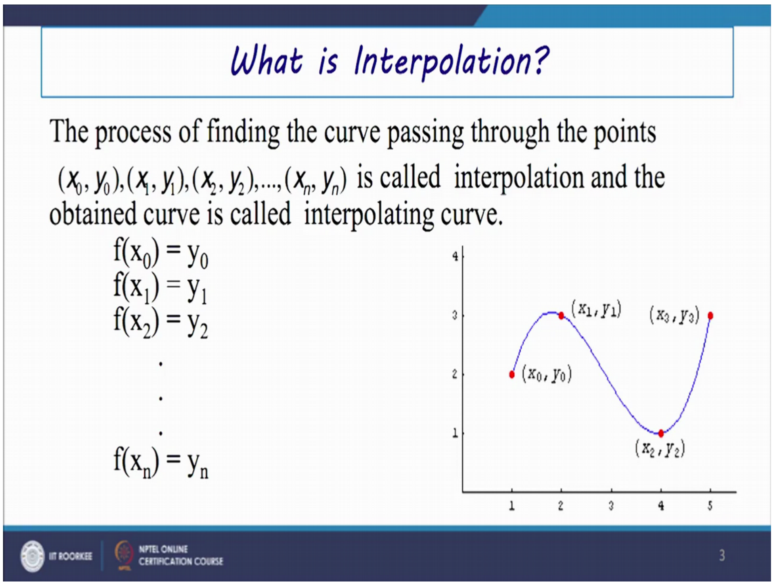

So whenever we are just going for this interpolation or we are just trying to discuss about this interpolation, interpolation is nothing but that if we will have a set of data points then if the curve is passing through this set of data points especially it is called interpolation. Suppose we will have this data points like x 0, y 0, x 1, y 1, x 2, y 2, x 3, y 3, then if you just plot a curve passing through this points then this curve is called interpolating curve.

This means that if we are just finding this curve the process of finding this curve passing this points like x 0, y 0, x 1, y 1, x 2, y 2, x 3, y 3 is called interpolation and the obtained curve is called interpolating curve. And here especially if you just see y 0 is nothing but it is just taking the value of f at the point x 0 here, similarly if you just define y 1 here that is nothing but f of x 1 here, y 2 is nothing but f of x 2 here, y 3 is nothing but f of x 3 here, this means that if we will have a function that is y equals to f of x then particularly for this function f of x if we will have the points like x 0, x 1 upto x n are the set of tabulated points or data points then at that point exactly we can just determine this functional values or the function at that point exactly.

So if we will have this functional values that is in the form of like y 0 equal to f of x 0, y 1 equal to f of x 1 and y 2 equal to f of x 2 and y n equal to F of x n then we can just put all this points and at that points various ways we can just connect this points through the curves and the best way to fit this curve with this point is called interpolation here. So basically before going to interpolation or how we can just approximate this tabular values by a curve or cord here, so we will just discuss brief history about this interpolation here.

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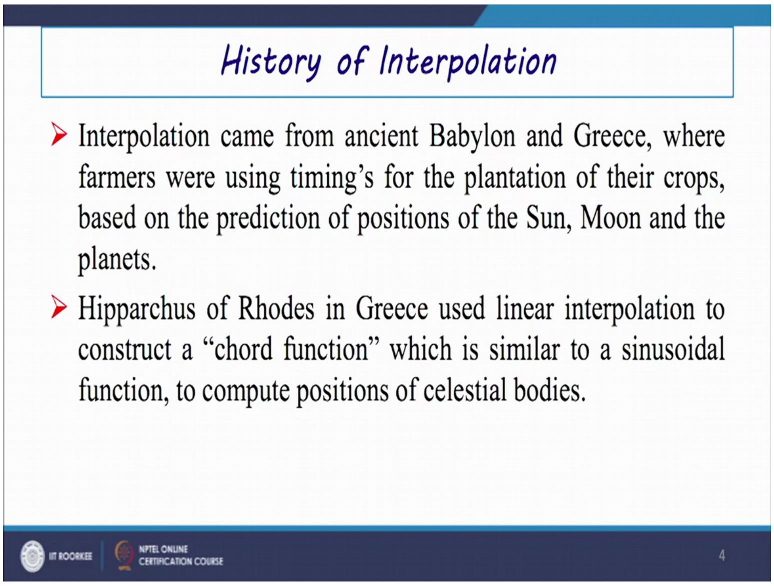

So interpolation came from ancient Babylon and Greece, where farmers are using basically further plantation of their crops based on the prediction of position of Sun, Moon and the planets. Specifically if you just see in India also we are just visualising that if this is a in month of like monsoon month it is just coming like June or July, so farmers started harvesting their crops basically.

So this hypothesis basically it came from ancient Greece only and how this prediction like movement of Sun or the movement of planets if it is just observe in a different lines or curves that at a particular point if we are just expecting that heavy rain will come and if at a particular point we can just expect that this Sun ray will be more and we will have a summer.

So based on that only it can be predicted that how we can just get the best output of the crops then if you just see the ancient history again then Hipparchus of Rhodes in Greece used linear interpolation to construct a cord function which is similar to sinusoidal function especially your sine function or cos function especially if you will just see.

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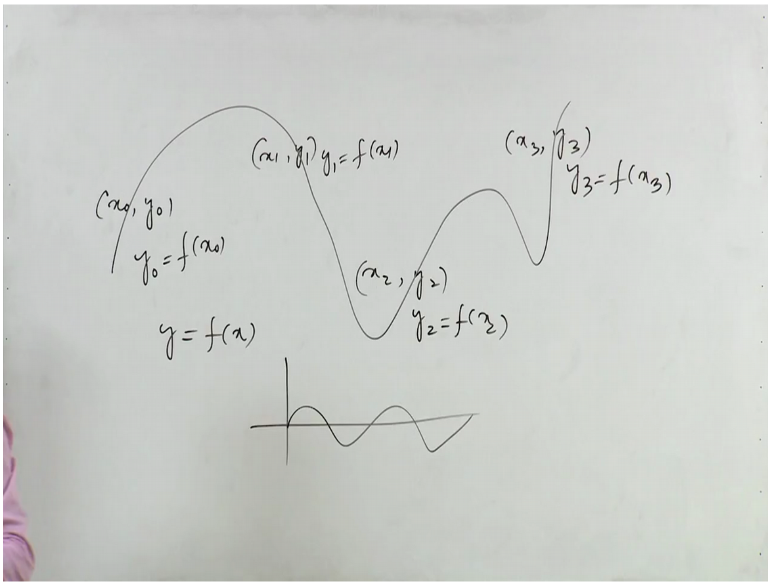

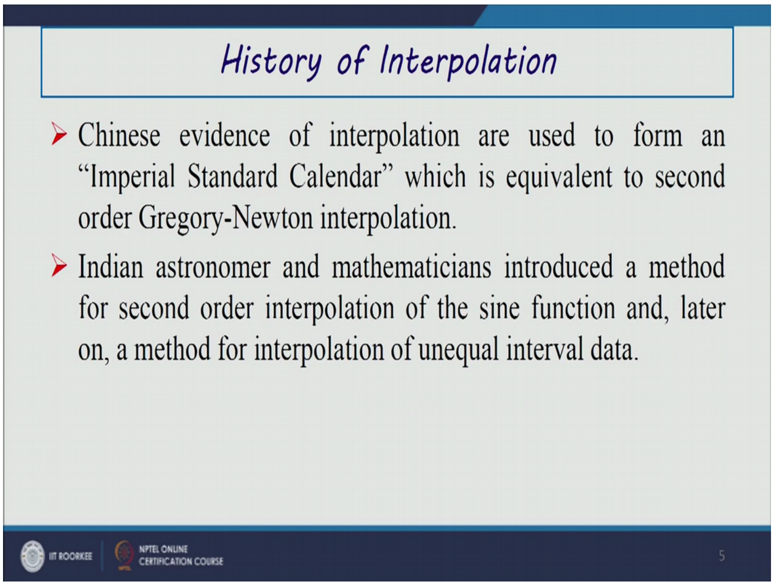

This means that if suppose sometimes some insects are also moving in a different parts this means that they are just following some functions also at that points also. This means that if we are just relating like different creatures to the nature then we can just find that interpolation existing at each phase of this nature also. So there especially use this sinusoidal function to compute the position of a celestial bodies and especially Chinese people also used this interpolation method to formulate their standard calendar especially this is nothing but in the present numerical analysis it is called Gregory newton interpolation.

So Gregory newton interpolation means either it is just running from the bringing of the table or it can start from the end of the table to computed the data throughout the whole table. This means that if we will just start this year at the beginning of the table then if you just use this interpolation then we can just compute that when this month will end and when the next month will start or when the year will end also.

Sometimes also if you just start this computation at the end of this year also then if you will just take this backward calculation of all this tabular values then we can just predict that when this Sunday will come and when this Monday will and which month it will just fall it out, so that we can just predict out.

Indian astronomer and mathematician if you will just see they have also introduced this method of like 2^{nd} order interpolation of sine function and later on a method for interpolation of unequal interval data. Especially Brahmagupta introduced this one and HH they have used also this like sine function and cos function for the 2^{nd} order interpolation functions there itself to visualise that how this like moon and this structural motion of this sound it is just coming in the space also.

So then we will just go for this basic introduction of interpolation, suppose if a function f of x is known to us then we can just put all this tabular values in a particular form. This means if we will have this set of data points or this particular points like x 0 and x 1, x 2 upto x 3 it is known to us also this function y equal to f of x is known to us then especially we can just determine this values like y 0 equal to f of x 0, y 1 equal to f of x 1, y 2 equal to f of x 2, at particular points.

Sometimes if this function is not known to us in explicit form then how we can just assume that this car will move in that form or if we will have this set of data points like x 0, y 0, x 1, y 1, x 2, y 2, x 3, y 3, it is known to us than at a particular point how we can determine this function, that especially we can just made it out if you will just use interpolation here.

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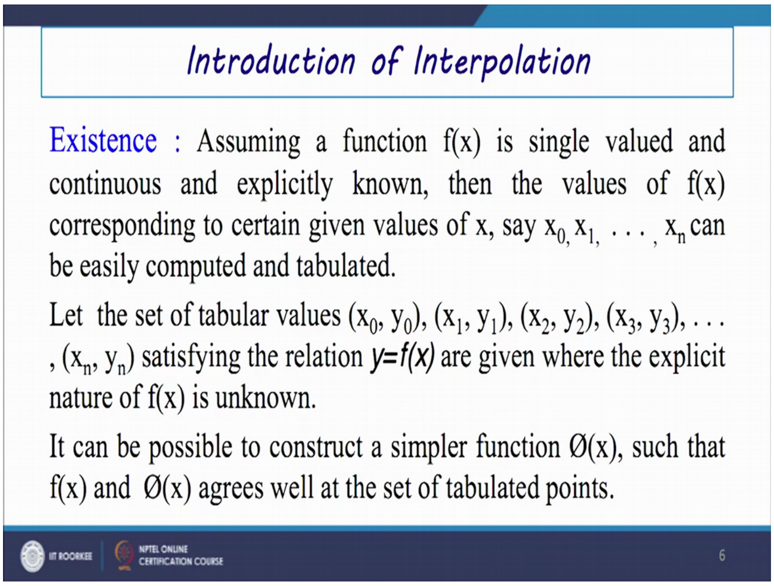

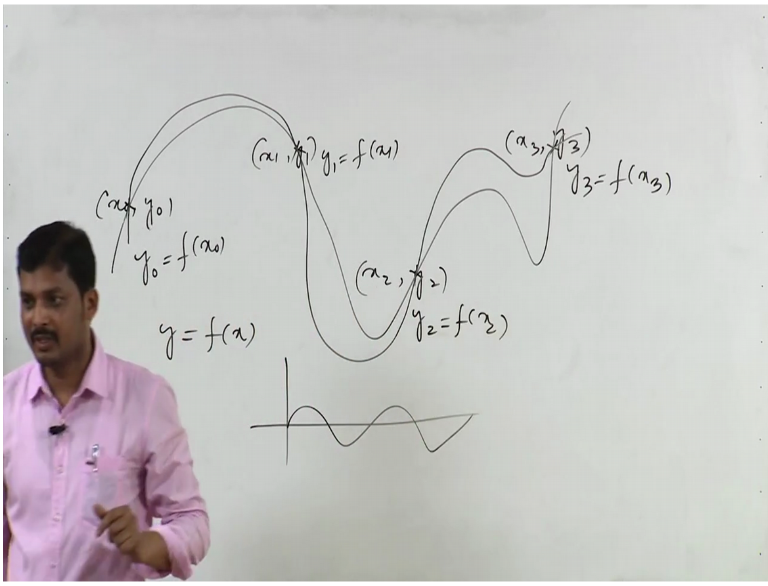

Suppose the set of data points if you will have like x 0, y 0, x 1, y 1, x 2, y 2 upto x n, y n, satisfying this condition y equal to f of x are given where this explicit nature of f of x is not known to us then it is possible to construct a simpler function suppose phi of x such that f of x and phi of x agrees well at this set of tabular points. This means if we will have this say set of data points like x 0, y 0, x 1, y 1, x 2, y 2, x 3, y 3, here then if the function is not known to us then we can just construct a simpler function suppose this is the function we can construct as phi of x here which can just pass through this points. Then we can just say that phi of x is a interpolating polynomial with this function f of x at that points.

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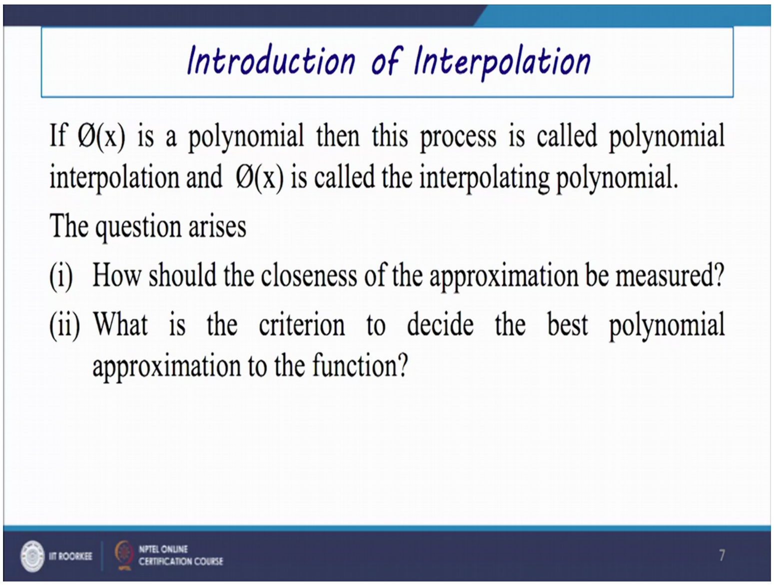

And the question now arises that how should the closeness of the approximation be measured? And what is the criteria to decide the best polynomial approximation to the function? This means that if suppose phi of x is passing through this point here then we want to find the best feet of this polynomial with this function here this means that this is a difference is existing for the functional values with this polynomial here. So how we can just choose that this arrow should be minimised here and also afterwards at all the particular points even if curve is passing at that points for this polynomial phi of X but we can just find the differences at each of the level there.

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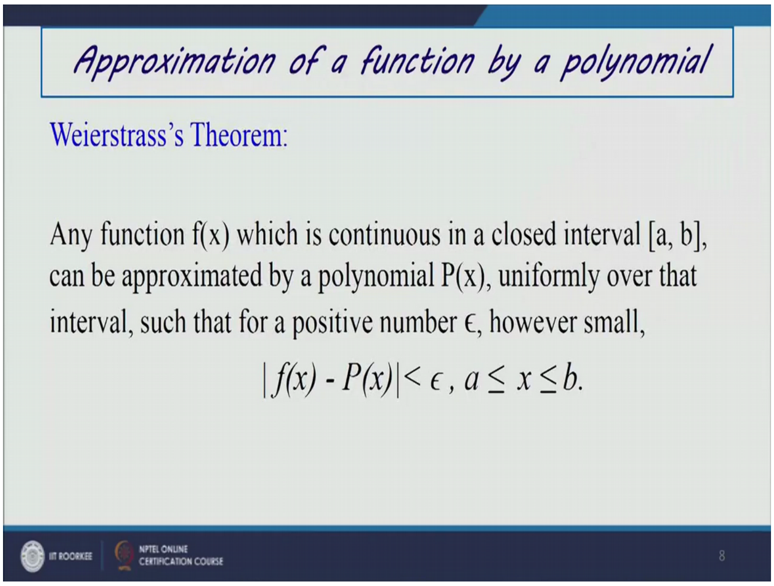

So if you just consider this points are very close to each other then this arrow can be minimised this is the first condition we can assume here. And for that if we will just go for a theorem here that is approximation of a function by a polynomial that is basically called Weierstrass's theorem this means that for using of this interpolation we have to consider this continuous function here neither we cannot say anything.

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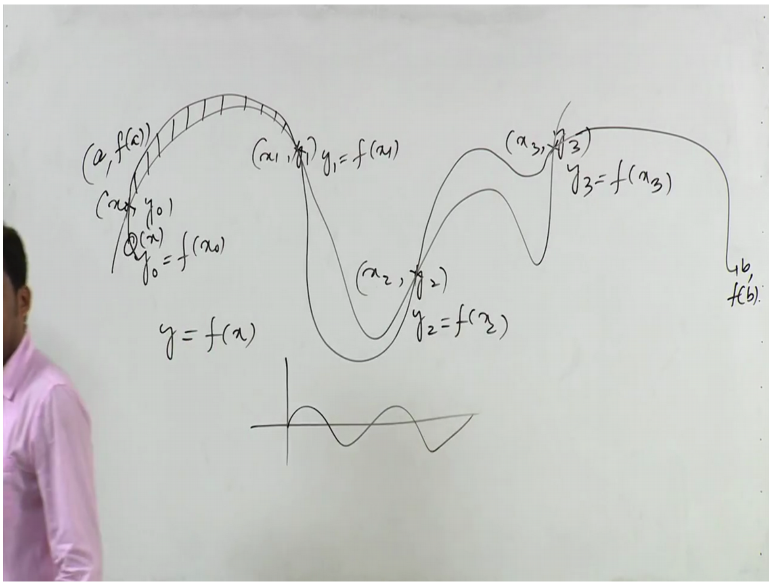

So based on Weierstrass's theorem if we will just consider that any function f of x which is a continuous function within this close interval suppose your beginning point is a here and the end point if you just consider here that is like b here suppose. And it is for associated functional values like f of a here and your associated functional value is f of b here then we can just see that f of x is a continuous function within this closed interval starting from the point a and ending at the point b there.

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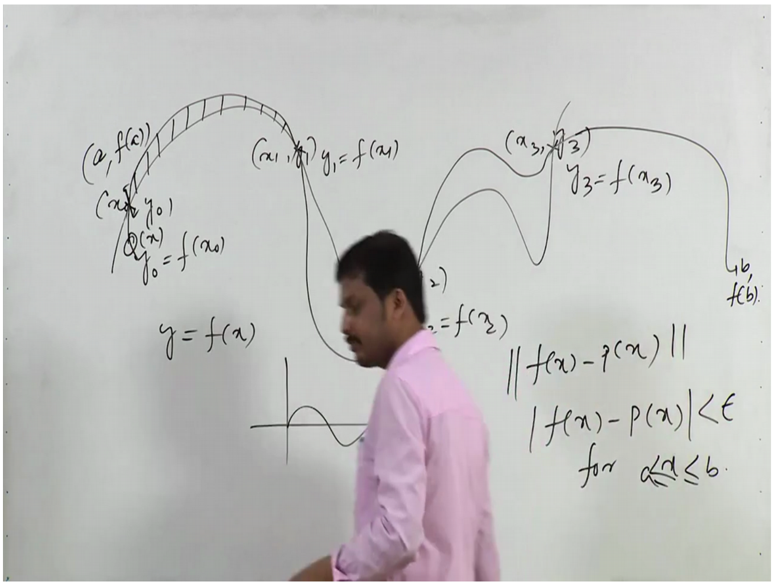

Then we can just approximate this function by a polynomial p of x suppose uniformly over each of this intervals that is like x 0, y 0, x 1, y 1, x 2, y 2, x 3, y 3 upto if the last point is considered as x n, y n there and each of this intervals this function is approximated by this polynomial p of x then we can just say that for a positive number epsilon we can just write norm of f of x minus p of x or we can just see that absolute value of f of x minus p of x they should be less than epsilon for excelize between a to b.

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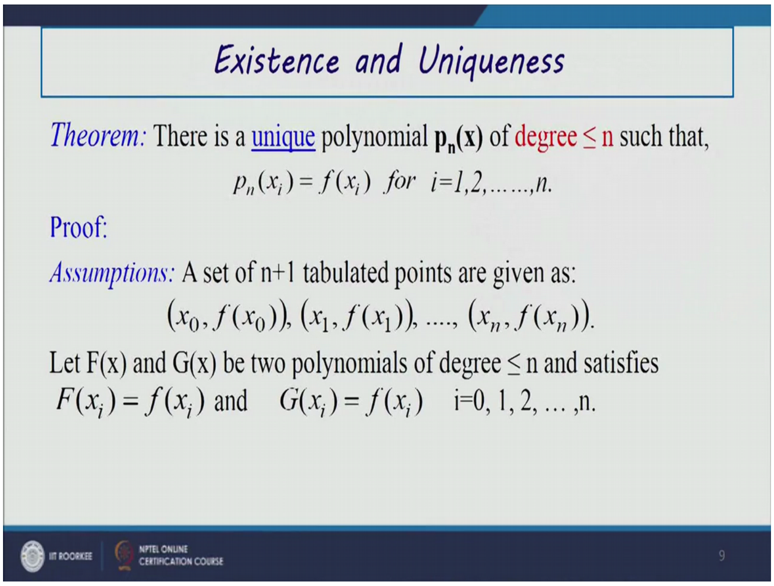

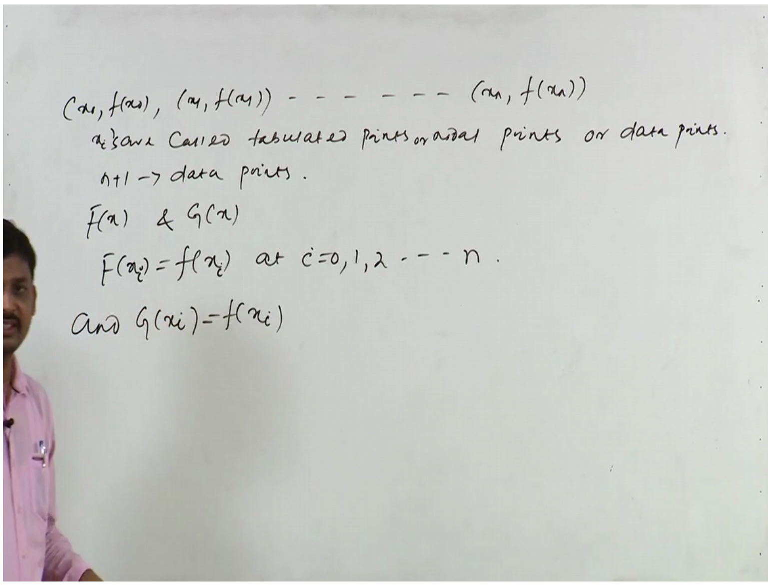

And to justify this theorem here we will just go for this existent and uniqueness theorem, the statement of this existence and uniqueness theorem states that, there is a unique polynomial pn x of degrees less than or equal n such that pn of x i equal to f of x i for i equals to 1, 2, upto n here. So to prove this theorem here if you just consider the set of tabulated data point are as if you will have the set of data points like n plus 1 data points are in the form of like x 0, f of x 0, x 1, f of x 1 upto x n, f of x n here.

Specially this x 0, x 1 or x i are called tabulated points or nodal points or data points and if you will have like starting point is 0 and ending point is n here we will have n plus 1 data points and corresponding function values are called your functional value here itself. And if we will just consider supposed 2 functions like f of x and G of x two different functions which satisfies this functional values that is f of x at all of this points on the set of data points we can see that f of x is equal to f of x at i equals to 0, 1, 2, upto n here or we can just see that f of x i equal to f of x i and G of x i this equals to also f of x i at i equal to 0, 1, 2, upto n then we can just say that this is also a polynomial of degree n since we will have n plus 1 points here, this is also G of x i is also polynomial of degree n here.

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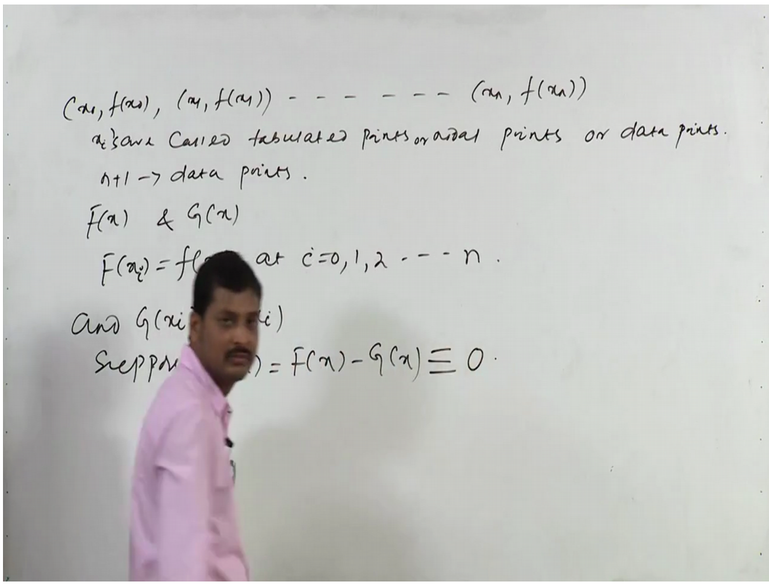

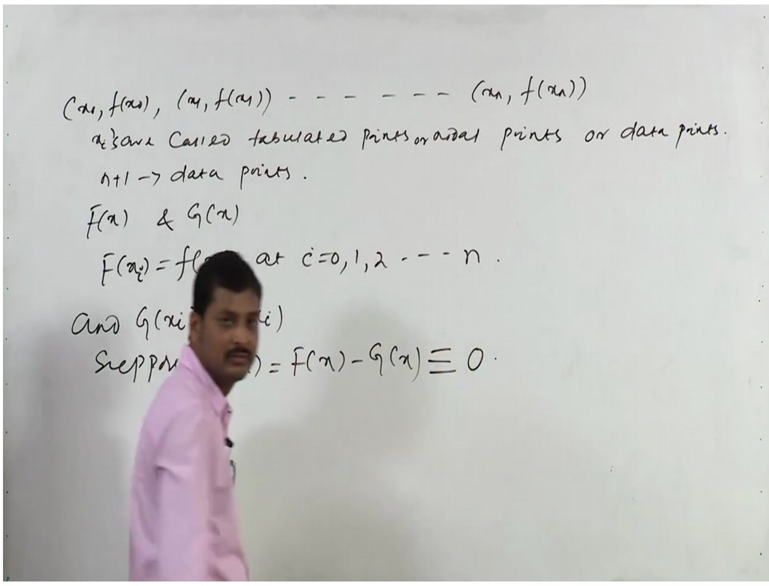

So if you just take the difference of the this 2 functions here like suppose we will have H of x is equal to f x Minus G x this will just represent a polynomial of degree less or equal to n here. Since it consist of n plus 1 points and it has zeros at n plus 1 points if you just see, but a polynomial of degree less or equal to n has exactly n has exactly n roots plus H of x is identically 0 here and especially we can say that f of x equal to or it can just see that f of x equal to G of x here this is nothing but that the interpolating polynomial is unique.

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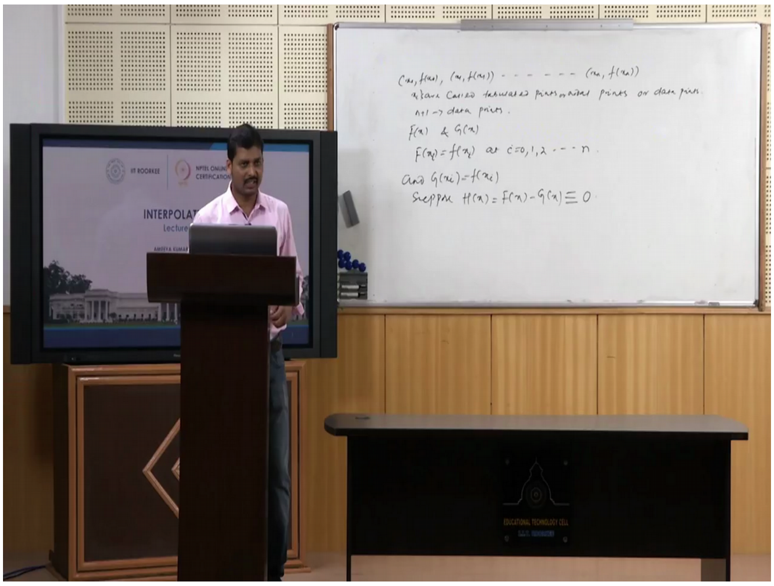

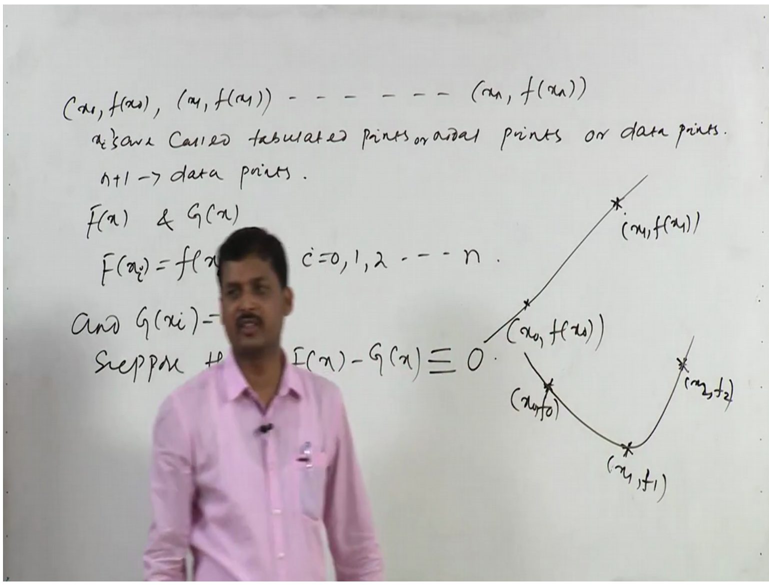

And if you just go for like examples of polynomial interpolation here, first will just go for linear interpolation here suppose. Linear interpolation means we will have like 2 points and within that if you just approximate suppose x 0, f of x 0 are the first point and x 1, f of x 1 is the 2nd point here then the best way to feed this polynomial is that if you just put the straight line on this 2 points here and it is just represent a polynomial of degree or order it should be less or equal to one here.

And if you just take three-point suppose, so if you just take 3 points then we can just represent this, this and this one as the three-point here x 1 f 1, x 2 f 2, here and if we can just join by a curve here this can just represented in this form here. And it can just represent the polygon of order less or equal to 2 there that passes through this 3 points here. So we obviously we can just say that if you will have 2 points here that is just representing a linear interpolating polynomial and if you will have just 3 points then you can see that it is just representing a quadratic polynomial here.

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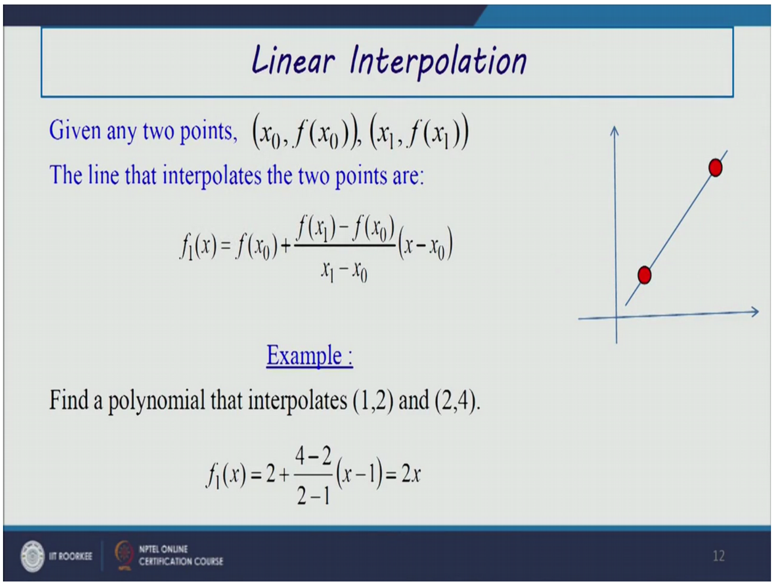

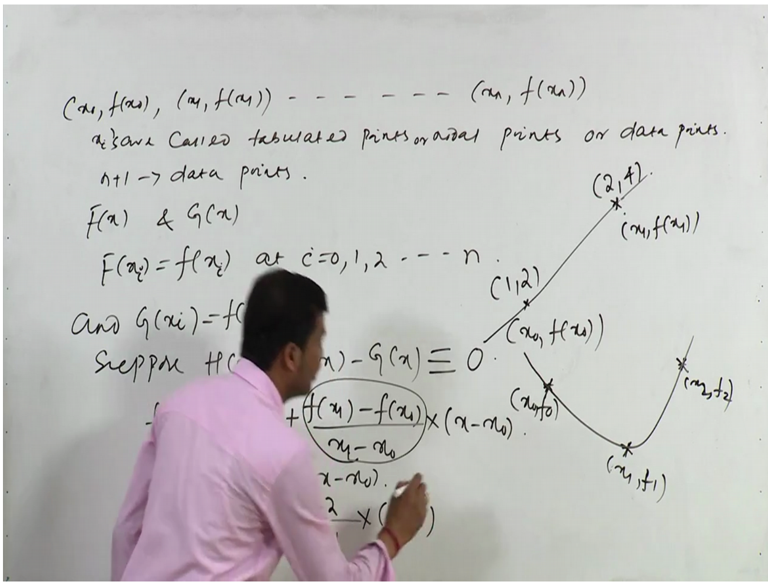

So first if you will just go for this linear interpolation here we will like to points x 0, f of x 0 and x 1, f of x 1 the lying interpolating with this 2 points are in the form of like if you just write that can be presented in the form of f 1 x this equal to like our intercepting form we can just write as f of x 0 plus you are slow that is nothing but f of x 1 minus f of x 0 divided by x 1 minus x 0 into your values that is in the form of like x minus x 0 here. Especially we used to write this one as y minus y 0 this as m into x minus x 0 here, m is nothing but the slope that is nothing but the dy by dx sometime we are just writing, sometimes if the functional value is known to us then we can just represent as this one here.

So this is a first representation of this linear interpolation polynomial. For example if you just consider find a polynomial that interpolates suppose the point 1, 2 and 2, 4, than directly if this point is written as 1, 2 here and the 2nd point if it is written as 2, 4 here, so directly we can just put this y 0 as 2 here and this x 0 is 1 here then like x 1 value we can just put as 2 here and f of x 1 can be represented as 4 here.

And if we just compare this one, so f 1 x can written in the form of first value f of x 0 we can just write that is nothing but 2 here and then plus f of x 1, f of x 1 is nothing but 4 minus we can just say f of x 0 is 2 here divided by x 1 minus x 0, so x 1 is considered as 2 here, 2 minus 1 into x minus 1 here and obviously this will just represent the value as 2 x here.

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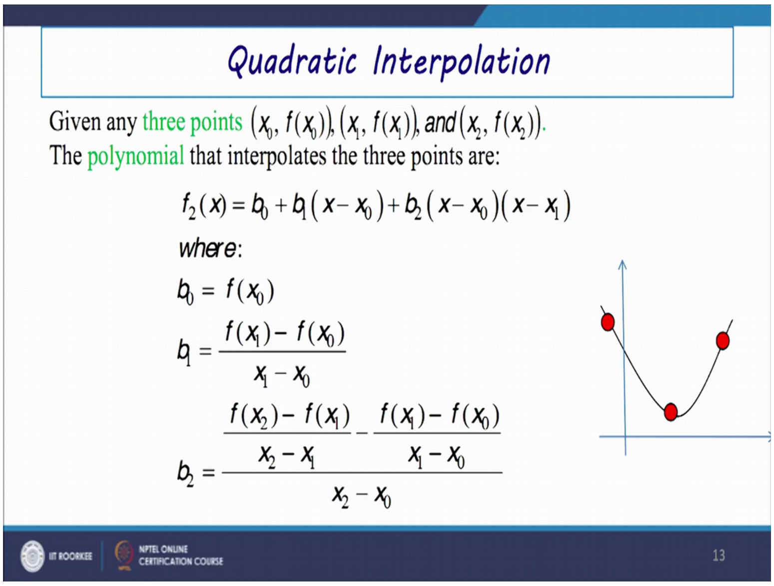

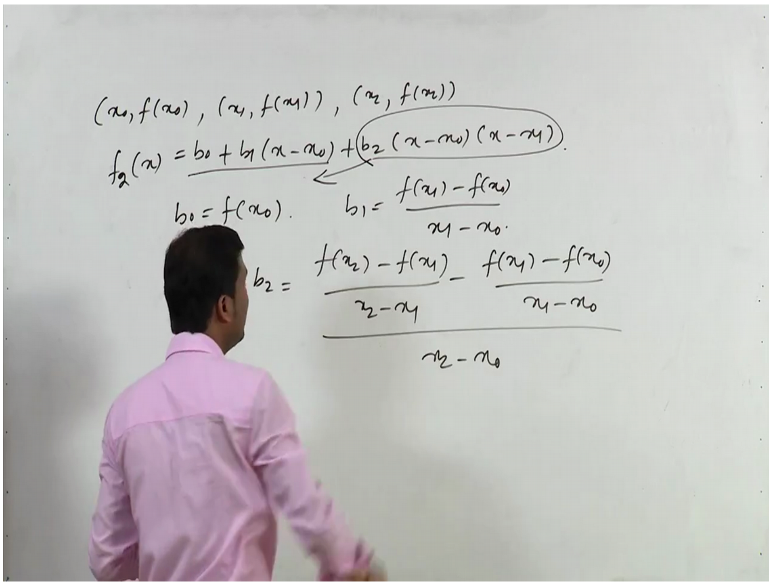

Similarly, if you just go for a quadratic interpolation here. So the quadratic interpolation representation can be written as the quadratic interpolation can be constructed by considering like three-point here that is x 0, f of x0, x 1, f of x 1, x 2, f of x 2 here. If you just consider this 3 points then we can just write this formulation that is interpolating or quadratic interpolating polynomial as f 2 x this equal is to b 0 plus b 1 x minus x 0 plus b 2 x minus x 0, x minus x 1 here. Where this coefficient b 0 can be defined as f of x 0 here and if you just write b 1 here that is nothing but f of x 1 minus f of x 0 divided by x 1 minus x 0 here.

Similarly b 2 can be written as f of x 2 minus f of x 1 divided by x 2 minus x 1 minus f of x 1 minus f of x 0 divided by x 1 minus x 0 whole divided by x 2 minus x 0 here. Especially if you just see the first coefficient here upto this one this just represent the liner interpolating polynomial and the extra term it is already in this form in this equation here that is just representing this quadratic interpolation here.

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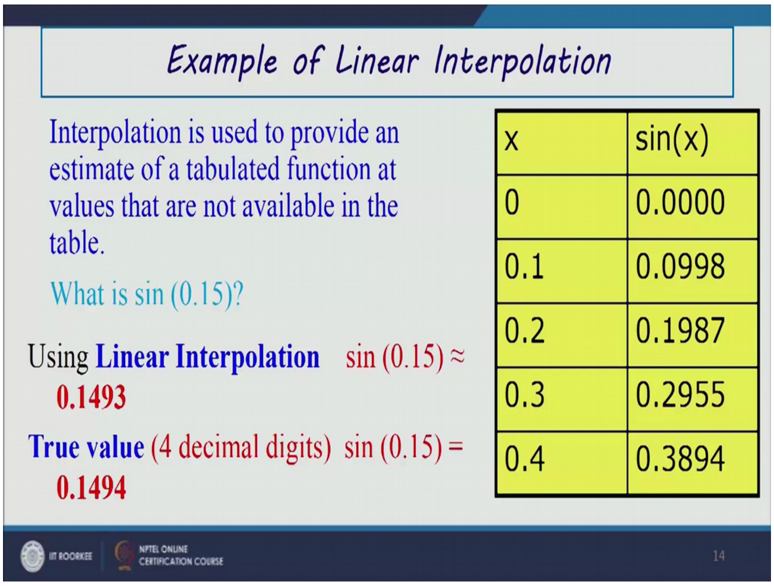

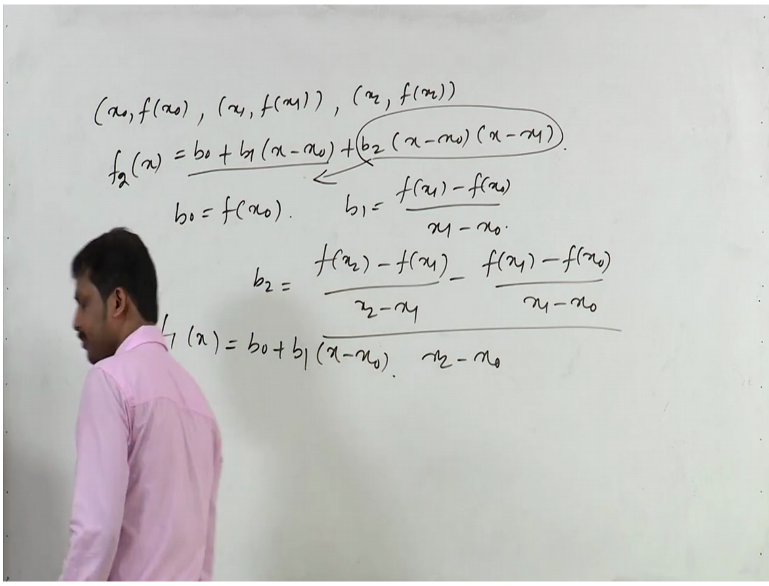

So based on this if you just go for some examples of like linear interpolation and quadratic interpolation we can just find that interpolation is used to provide an estimate of a tableted function at the values where this functional values is not known to us. Particularly if you will have a curve here and at a particular point if you want to calculate this functional values then we will just use this interpolation.

Then suppose the question is asked that if the sign is a function here and the functional value of sine x is given as 0, 0.1, 0.3, 0.4 here and it is asked to compute sine of 0.15 based on this tabular values. Then if you just used this linear interpolation that is represented in the form of like f 1 x as b 0 plus b 1 x minus x 0 formulation then we can just obtain this value as 0.1493 here but exactly the true value is 0.1494 upto 4 decimal places.

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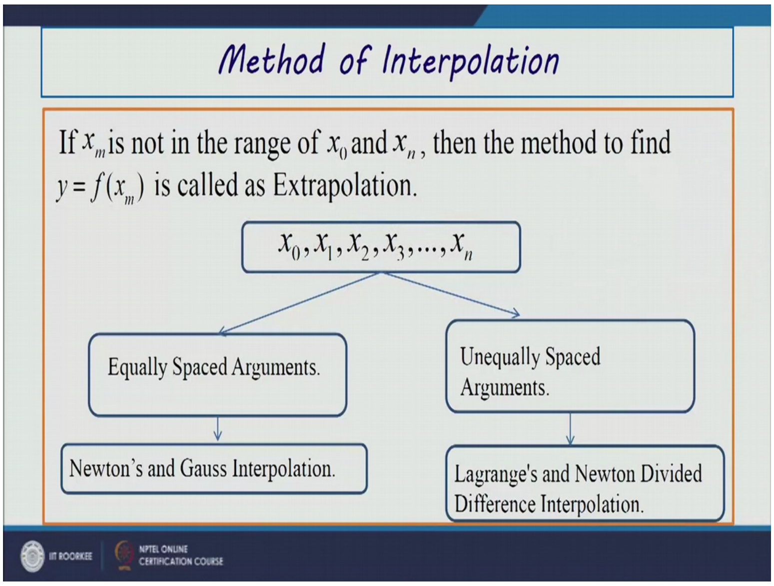

So if you will just locate it then this error is existing after like a 3rd decimal precision and next we will just go for this method of interpolation that where we can just use this interpolation. If suppose the tabular values are equally spaced or unequally spaced suppose if sometimes some points if it asked to evaluate it is not within the range of this interval where this function is continuous, then the process to find this functional values outside this interval is called extrapolation.

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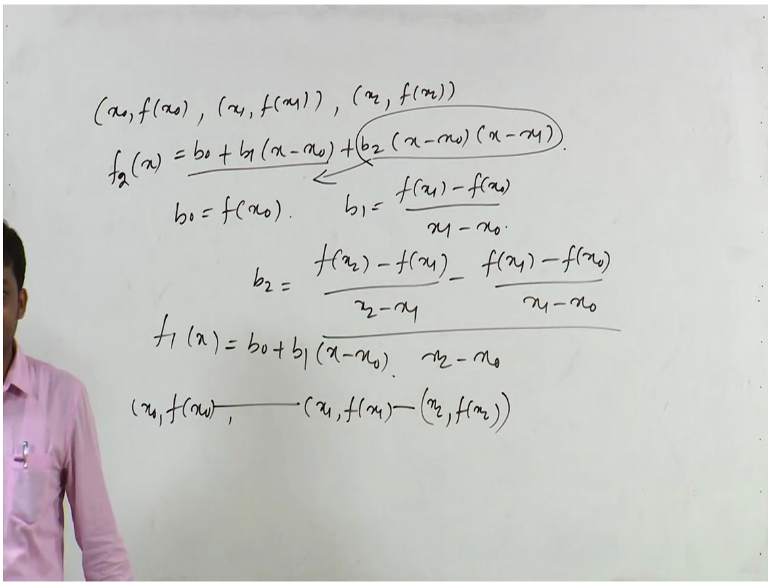

And we can just visualise that if the data points are existing like x 0, f of x 0, x 1, f of x 1, x 2, f of x 2, so this distance is not equals to this distance also sometimes. So then we will have like a 2 types of arguments are there that is equally spaced arguments and unequally spaced arguments, some methods are existing for this integration that is based on this equally spaced intervals and some methods that is basically existing for unequally spaced arguments.

Basically for equal spaced arguments we are just using Newton's and Gauss interpolation for unequal spaced intervals we are just Lagrange's and Newton's divided difference interpolation. And the basic advantage of this unequal spaced argument formulation is that it can handle both these equally spaced and unequally spaced arguments.

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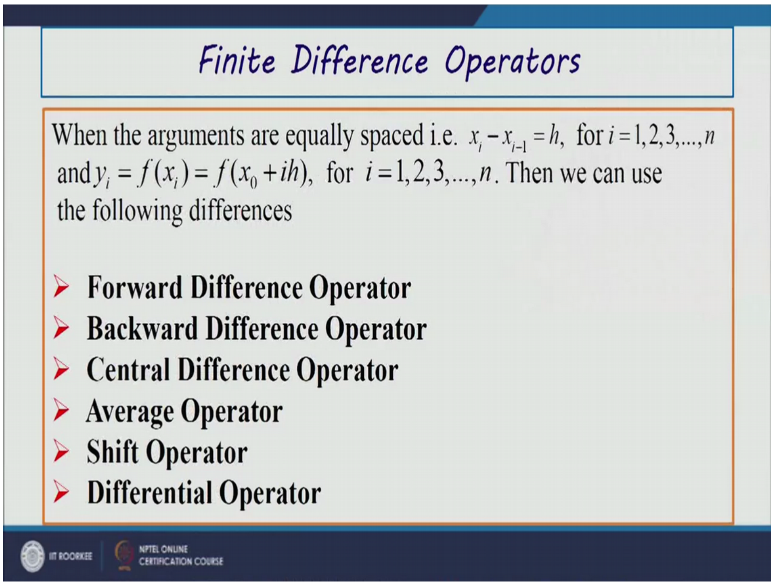

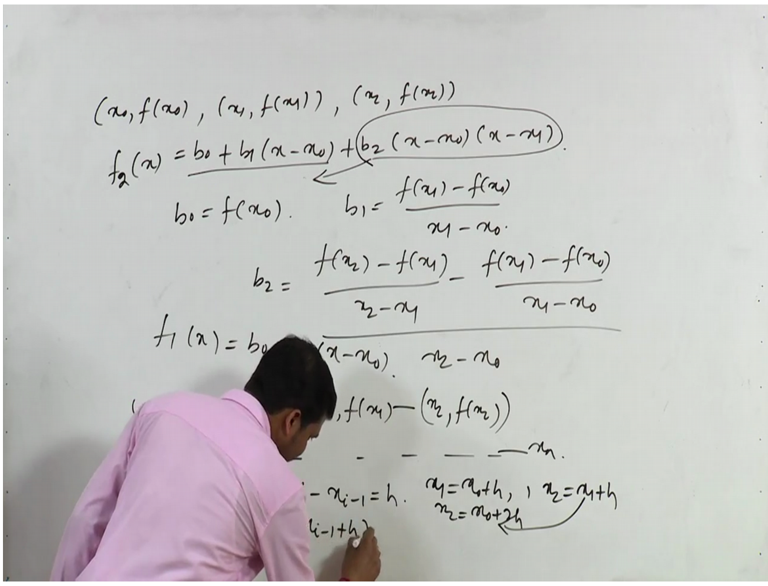

And if you just go for this like finite difference operators here, that is when the arguments are equally spaced suppose. Equally spaced means we can just consider that your x 0, x 1, x 2 upto x n all are equally spaced, means the distance between these 2 points all are equal here.

And then we can just right here x i minus 1 this equals to h here, if the space size is h here and specifically if x 0 is the starting point then we can just write x 1 equal to x 0 plus h here and x 2 can be written as x 0 plus 2 h here or we can just write x 2 equal to x 1 plus h this means that x 0 plus 2 is there itself.

And if you just use this functional values at this point specially we are just writing f of x i this is nothing but f of x i minus 1 plus h here, for like i equals to one, 2, 3 upto n there then we can just use like different finite difference operators for this formulation or to find this values using this finite difference point x 0, x 1 upto x n, for the like finite difference interpolation here or finite difference operators here.

So specifically if you just see here this finite difference operators basically it is existing like forward difference operators, backward different operators, central difference operators, average operators, shift operators and differential operators. So first we will just go for this forward difference operators and back difference operators, based on this like if all this points are equally spaced here or we can just say that the distance between all of these points are equal.

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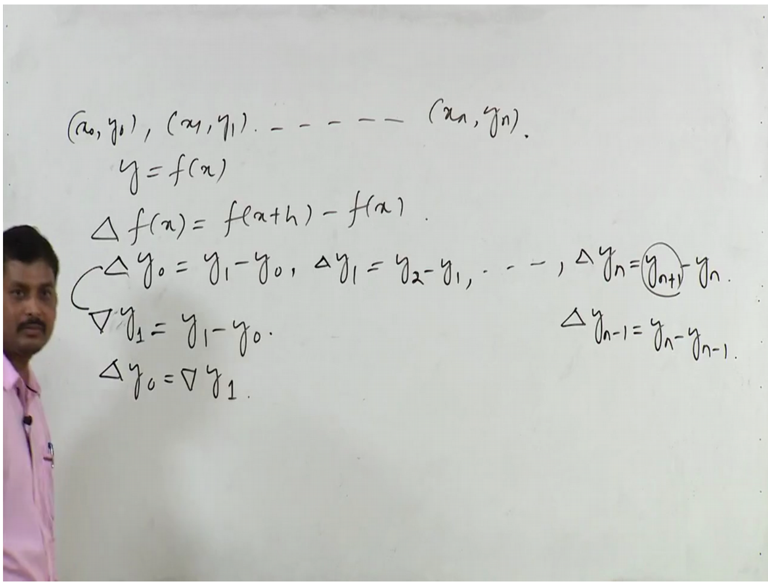

So for the forward difference operators if you will just use the standard value that is in the form of like x 0, y 0, x 1, y 1,upto x n, y n here, then we can just use this function as y equals to f of x here and delta of f of x can be written as f of x plus h minus f of x here. If h is the space size between these 2 points x 0 and x 1 here and especially if you just write in a functional formula here delta of y 0 this can be written as y 1 minus y 0.

Similarly if you just write delta of y 1 this can be written as y 2 minus y 1 and if you just continue upto delta of y n here so it can be written as y n plus 1 minus y n, if you just see y n plus 1 is not within this tabular value then we can just compute this delta operator upto y n minus 1 here, so especially it can be written as y n minus y n minus 1.

If you will just use this backward difference operators, so backward difference operators it is just moves this functional values towards the back of the point there this means that nabla is the operator which is called the backward operator and the delta is the operator which is called the forward operator here.

So if you just use this nabla operator here then we can just write this one as y 1 minus y 0, and if you just relax this operators here then we can just see that delta of y 0 this is nothing but nabla of y 1 here. And similarly all other operators values you can just define here that is nabla of y 2 that can be written as y 2 minus y 1 here and if you will just go for this upto last point here we can just write nabla of y n as y n minus y n minus 1 here.

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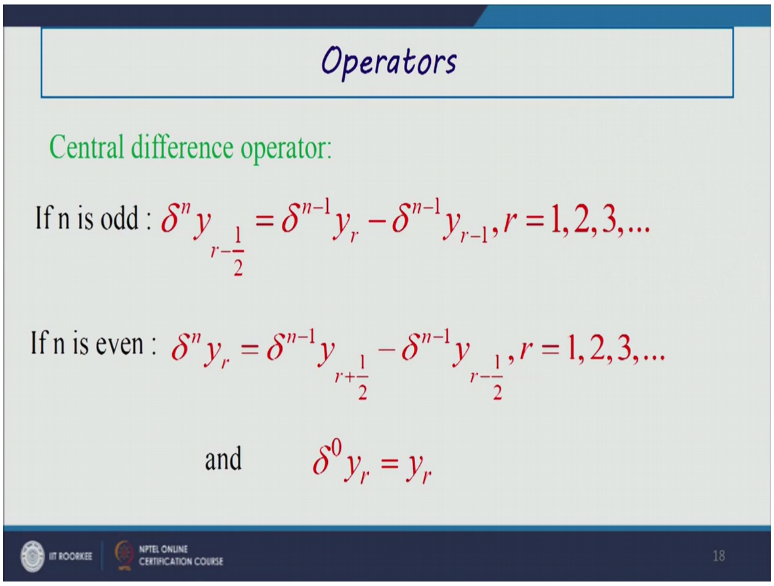

And if you just go for like central difference operators, central different operators means? We can just write this central difference operators as small delta here and if it is operated on function f of x especially this can be written in the form of like f of x plus h by 2 minus f of x minus h by 2 here. And in the complete form or in the relation of y r if it can be written, so it can be written in the form like delta to the power n of y r minus half as delta to the power n minus 1 y r minus delta to power n minus 1 y r minus 1, r equals to 1, 2, 3 here.

Especially if you just see here that if n is odd we can just write in this form, if n is even we can just write delta to the power n y r equals to delta to the power n minus 1 it y r plus half minus delta to the power n minus 1 y of r minus half here. and if you just consider delta to the power 0 of y r this is nothing but y r here.

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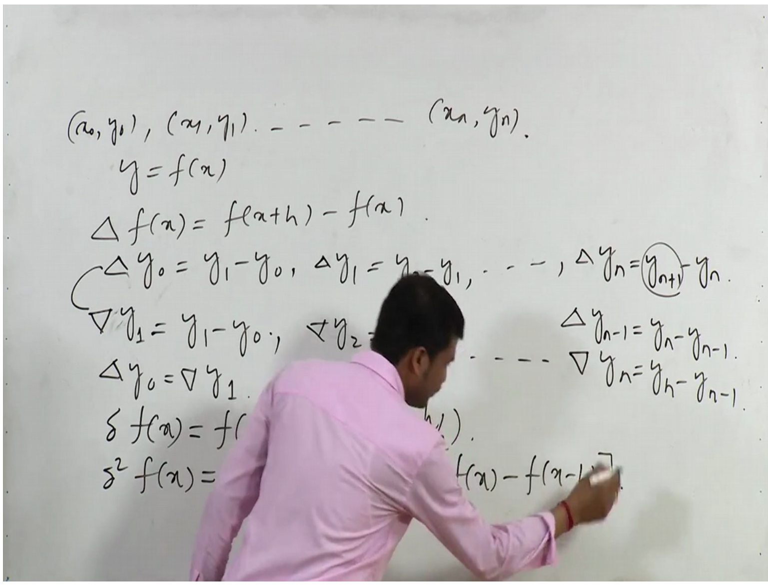

So if you just see here that if twice if you just use this operator that is nothing but like delta can be operated on f of x plus h by 2 then again one h by 2 will be added here and it can just represent as f of x plus h here and it can just give another value that is f of x plus h by 2 sorry minus h by 2 here then it can just give you f of x plus h minus f of x there.

Similarly, if you just use this delta operator once more for this function here this can just give you like f of x minus h by 2 plus H by 2 minus f of x minus h by 2 minus h by 2 here. So then this will just give you the function like f of x minus f of x minus h that will just give you a complete form of this equation if you will just use this operator twice here. This can be written as f of x plus h minus f of x minus you can just say that this is f of x minus f of x minus h.

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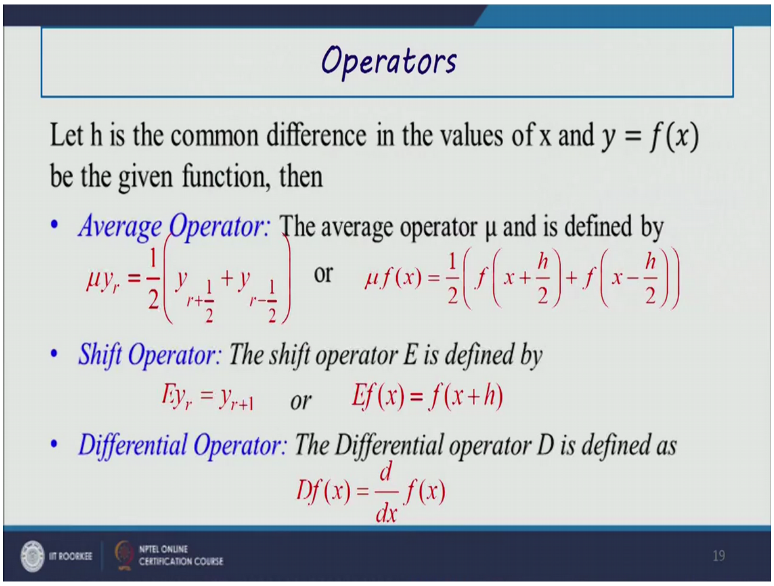

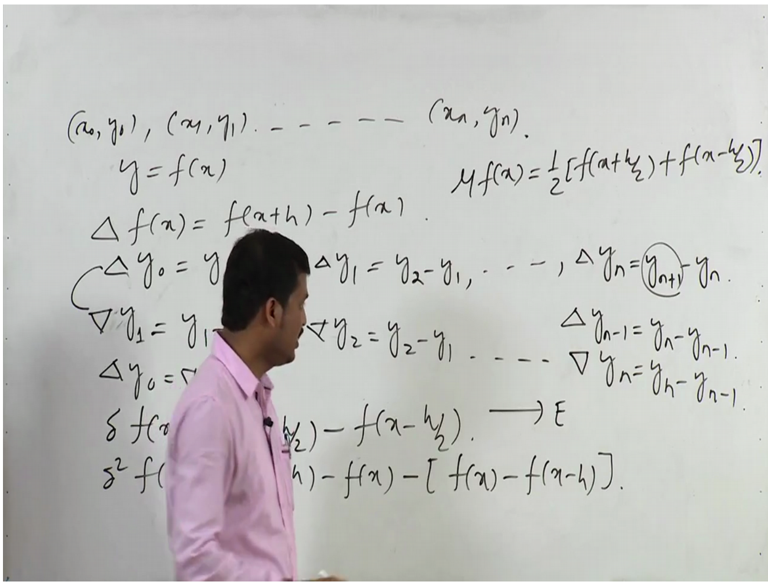

Similarly if you will just go for this average operator, so average operator means we can just write this one as in the form of Mu operator that is Mu f of x can be written as half f of x plus h by 2 plus f of x minus h by 2 here. And in the y r form if you just write this equation that can be written in the form of like Mu of y r equals to half of y r plus half plus y of r minus half here. Specifically all this operators that are given in on a new operator this is called shift operator here.

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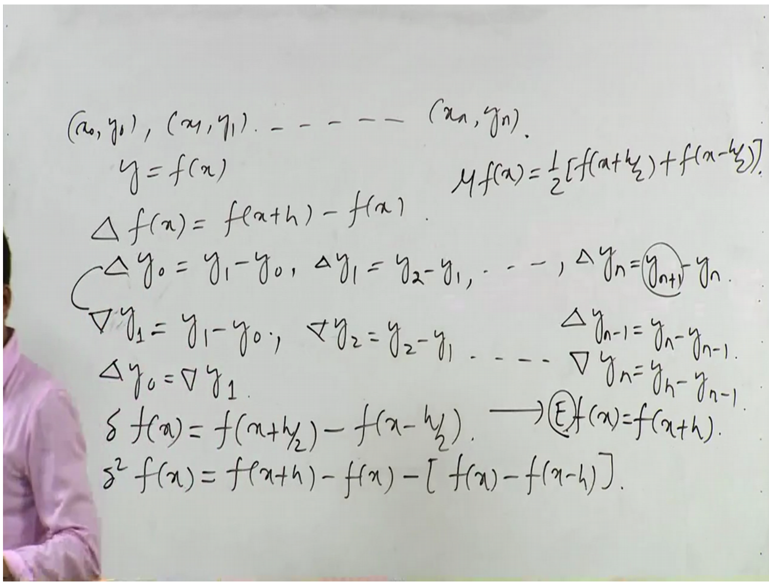

So specifically the shift operator is denoted by the capital letter E here and this is basically called your shift operator and whenever it is operated on this function f of x it is just move this function to the immediate next step here. This means that you can just write E of f of x as f of x plus h here, so all this functions whatever we have just discussed here like Newton's forward difference operator, backward difference operator, central different operator or average operator all this functions can be expressed in the form of shift operator here, that is in the form of E here.

So specifically there is operator that is called differential operators usually this differential operator is designated as d of f of x that is nothing but d by dx of f of x here. So with this I am just ending this lecture and in the next lecture I will just continue for this Newton's forward difference operator and backward difference operators or how this operator relation can be established based on this shift operator here. Thank you for listening this lecture.