Integral Equations, Calculus of Variations and their Applications
Professor Dr. P. N. Agrawal
Department of Mathematics
Indian Institute of Technology Roorkee
Lecture 59
Variational problems with a movable boundary for a functional dependent on two functions

Hello friends welcome to my lecture on variational problems with a movable boundary for a functional dependent on two functions.

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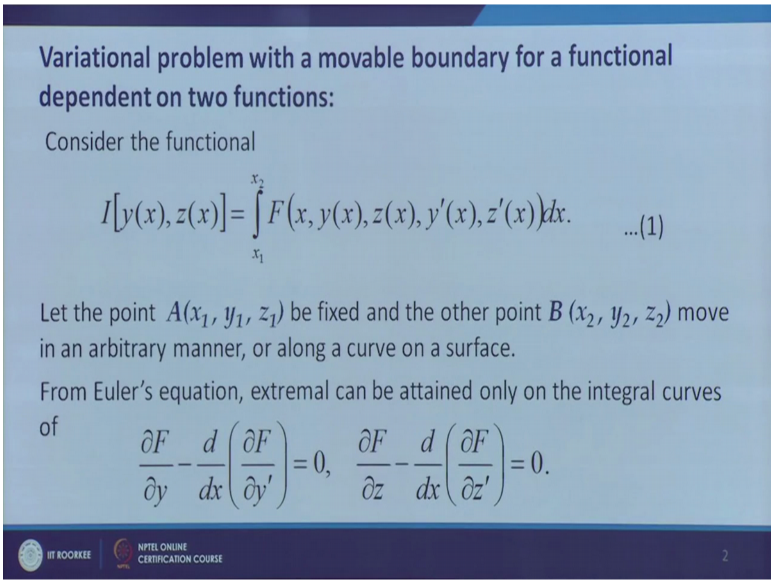

Let us consider the functional I y x, z x where y and z are two dependent variables and x is an independent variable, the functional is x integral x 1 to x 2 F x, y x, z x, y dash x, z dash x. Let us consider a point A x 1, y 1, z 1 which is fixed and the other point B x 2, y 2, z 2 move in an arbitrary manner, or it moves along a curve or it moves on a surface.

From Eulers Equations, the extremal can be attained only on the integral curves of the two equations delta F by delta y minus d over dx delta F over delta y dash equal to 0, delta F over delta z minus d over dx of delta F over delta z dash equal to 0 because we know that the necessary conditions for the extremal in the case of more than one dependent variable are the two equations given by the Eulers Equations here. So the extremal can be attained only on those curves.

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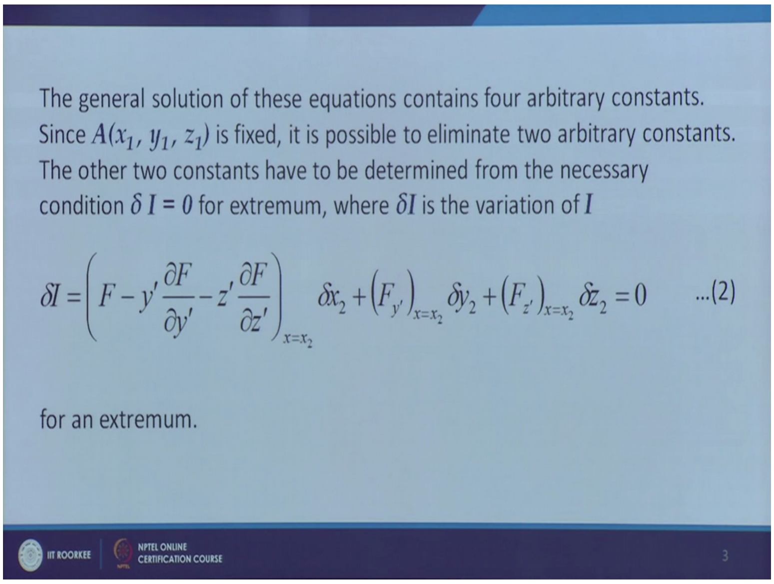

Now the general solution of these equations contains four arbitrary constants. Since x 1, y 1, z 1 is fixed it is possible to eliminate two arbitrary constants. The other two arbitrary constants have to determine from the necessary condition that is delta I equal to 0 where delta I is the variation of I. Now delta I is equal to F minus y dash delta F over delta y dash minus z dash delta F over delta z dash x equal to x 2 into delta x 2 plus F y dash x equal to x 2 into delta y 2 plus F z dash x equal to x 2 delta z 2 equal to 0 for an extremum.

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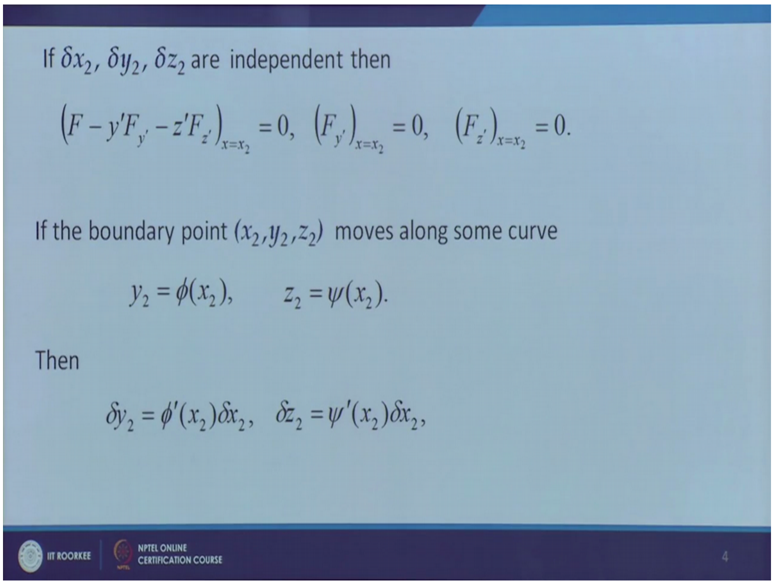

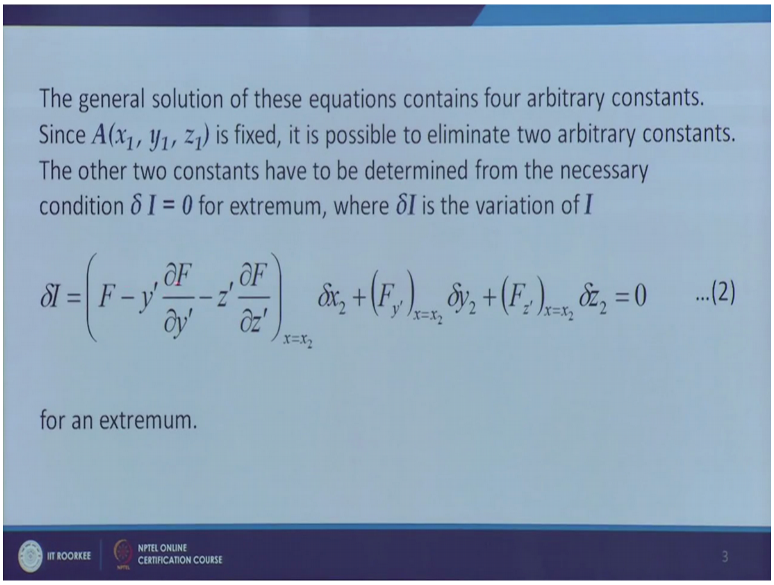

Now if delta x 2, delta y 2, delta z 2 are all independent here then their coefficients must be 0 and so F minus y dash F y dash minus z dash F z dash x equal to x 2 is 0, F y dash at x equal to x 2 is 0 and F z dash at x equal to x 2 is 0. Now if the boundary point x 2, y 2, z 2 moves along some curve and the equations of the curves are y 2 equal to Phi x 2, z 2 equal to Psi x 2 then from y 2 equal to phi x 2 we have delta y 2 equal to phi dash x 2 delta x 2 and z 2 equal Psi x 2 gives delta z 2 equal to Psi dash x 2 into delta x 2.

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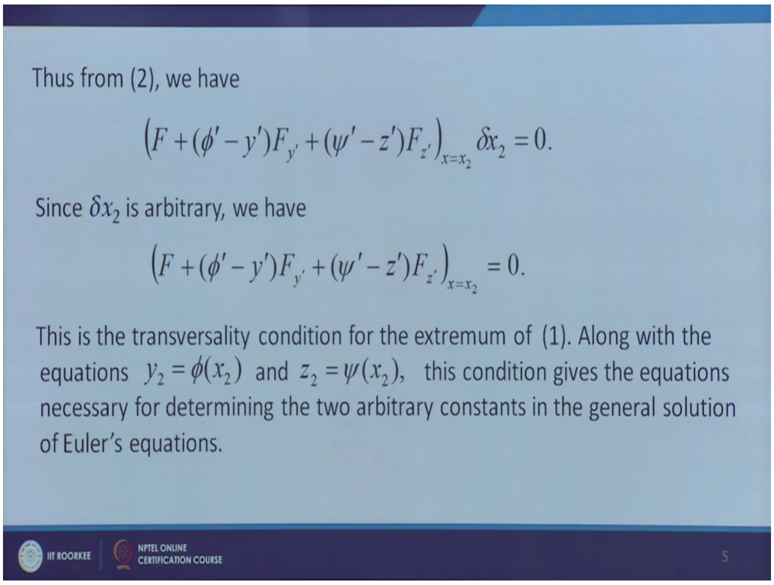

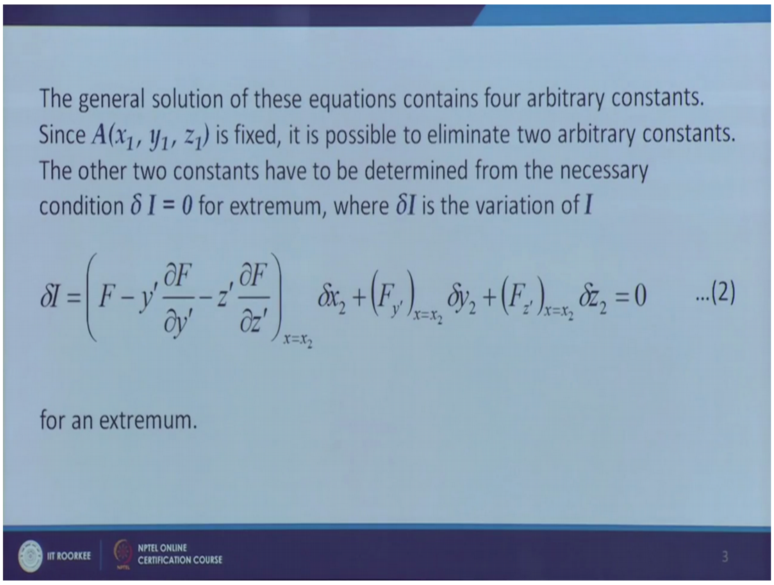

So from the equation 2 from the equation 2 means this equation.

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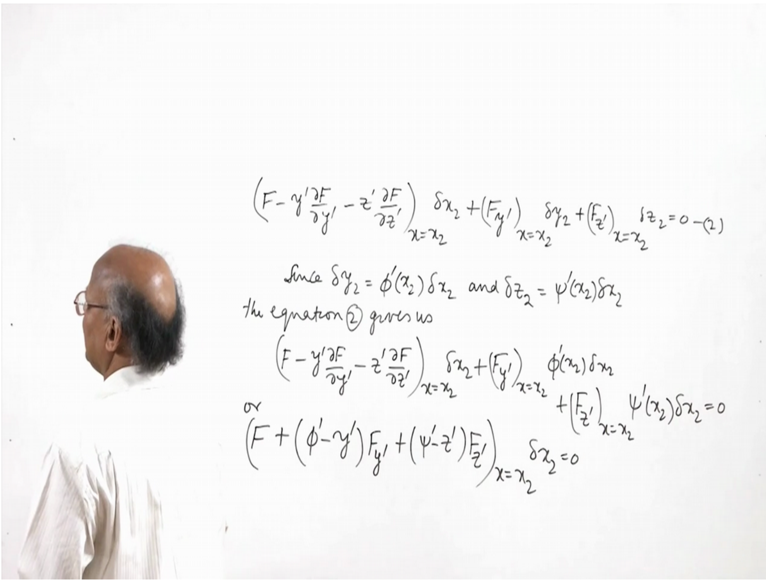

What we will have is F minus delta F by delta y dash minus z dash, delta F by delta z dash into delta x 2 plus F y dash delta y 2 plus F z dash delta z 2 equal to 0. Now we have delta y equal delta y 2 since delta y 2 equal to phi dash x 2 delta x 2 and delta z 2 equal to Psi dash x 2 delta x 2 this equation okay this equation we have called as 2 the equation 2 gives us F minus y dash F y dash delta y 2 is Phi dash x 2 delta x 2 and then F z dash delta z 2 is Psi dash x 2 delta x 2 equal to 0.

So let us we can write it as then or F plus phi dash minus y dash F y dash plus Psi dash minus z dash into F z dash at x equal to x 2 delta x 2 equal to 0 because we can collect the coefficients here delta x 2, delta x 2, delta x 2 so this is F plus phi dash at x equal to x 2 is Phi dash x 2 and then phi dash x 2 F y dash at x equal to x 2 into delta x 2 so this is here and then minus y dash F y dash delta x 2 it is here and then F z dash Psi dash x 2 into delta x 2 is here and minus z dash F z dash into delta x 2 is here, so we can write like this.

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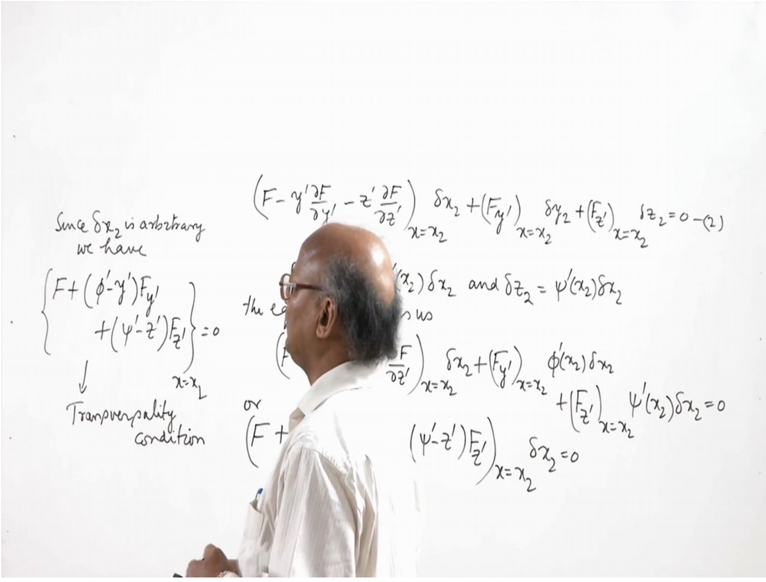

And since delta x 2 is arbitrary we have F plus phi dash minus y dash F y dash plus Psi dash minus z dash F z dash at x equal to x 2 is equal to 0, this is the transversality condition this is transversality condition. When the point x 2, y 2, z 2 moves on a curve then this is the transversality condition for the extremum of the functional one.

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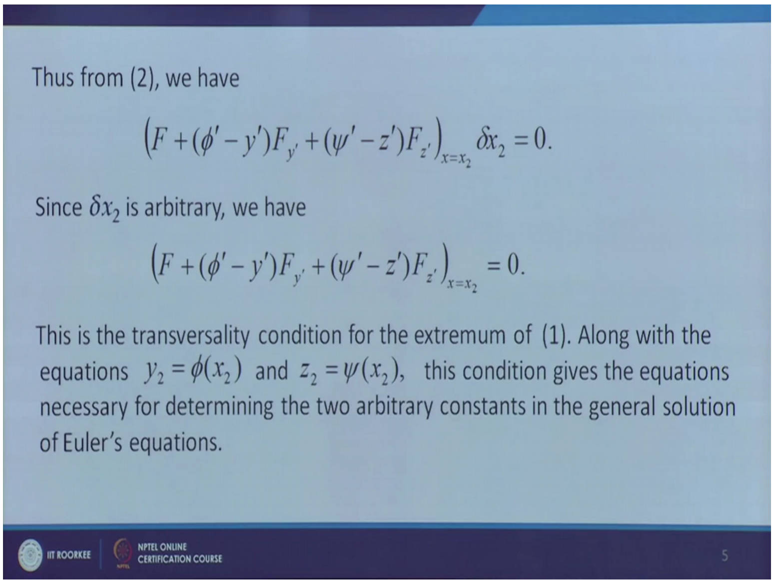

Now along with the equation y 2 equal to phi x 2 and z 2 equal Psi x 2 this transversality condition gives the equations necessary for determining the remaining two arbitrary constants in the general solution of Eulers Equations.

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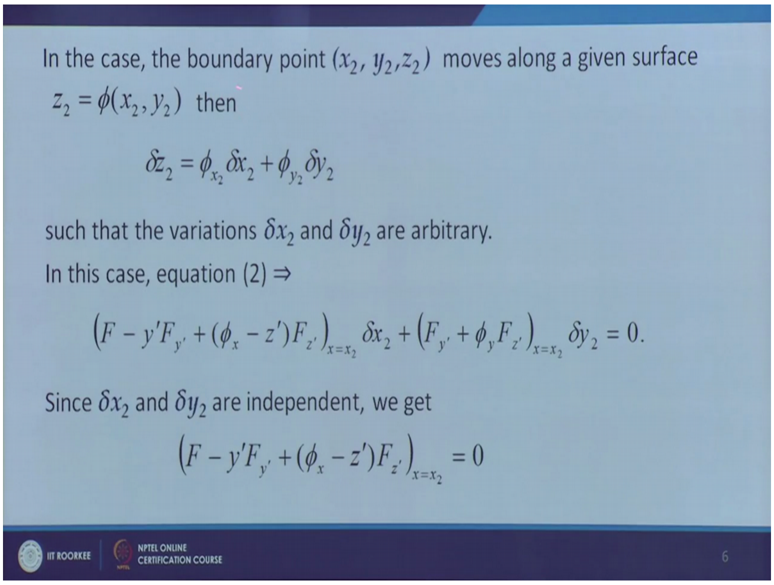

Now in case the boundary point x 2, y 2, z 2 moves along a given surface we know that the equation of a surface can be written as z equal to phi x, y. So in that case z 2 will be equal to phi x 2, y 2. Now z 2 equal to phi x 2, y 2 and z equal to phi x, y gives us the increment in z 2 that is because x 2, y 2, z 2 is the moving points so delta z 2 will be partial derivative of phi with respect to x at x equal to x 2 into delta x 2 then partial derivative of y at y 2 so phi y 2 into delta y 2. The variations delta x 2 and delta y 2 are arbitrary.

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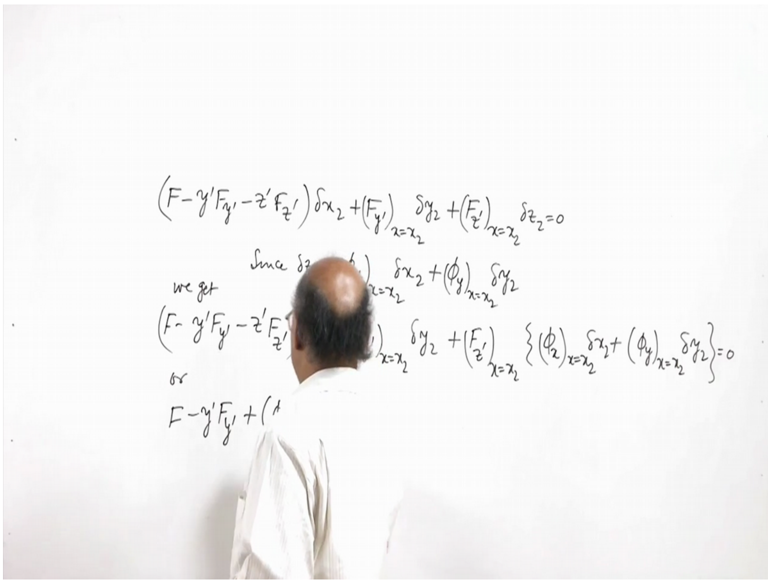

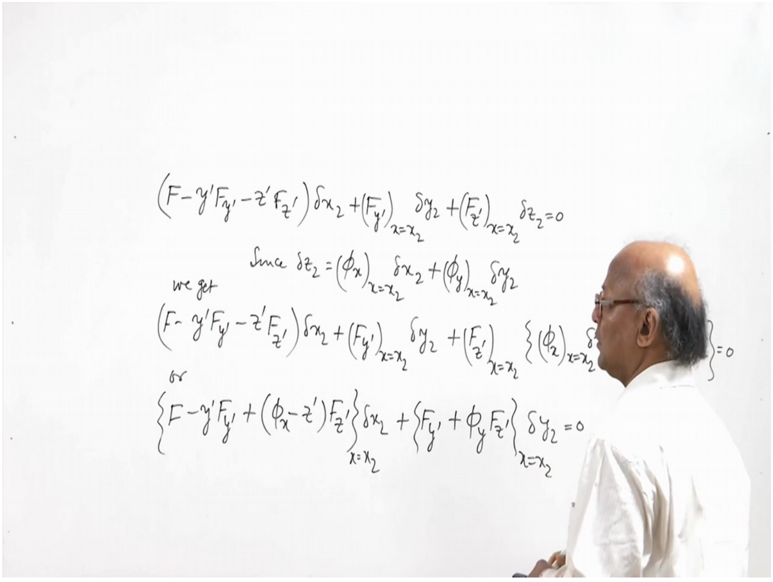

Now in this case the equation 2 will take the following form equation 2 which is F minus y dash F y dash minus z dash F z dash delta x 2 plus F y dash x equal to x 2 delta y 2 plus F z dash this equation in this case will take the form delta z 2 is so since delta z 2 is equal to phi x 2, phi x 2 means phi x at x equal to x 2, okay delta x 2 plus phi y at y equal to y 2 or x equal to x 2, then delta y 2.

So let us use this here then we can write it as F minus y dash is equal to this we get F minus y dash F y dash and then here F y dash x equal to x 2, okay minus z dash F z dash delta x 2 let us put the value first F y dash x equal to x 2 and then delta y 2 plus we have put F z dash let us put the value here delta z 2 is phi x at x equal to x 2 delta x 2 plus phi y delta y 2 equal to 0. So we can write it as or F minus y dash F y dash delta x 2 it can be combined with this term here.

So we can write phi x minus z dash F z dash into delta x 2 x equal to x 2 plus and this term can be combined with the term here so we write F y dash F y dash plus phi y F z dash at x equal to x 2 delta y 2 equal to 0. So we get F minus y dash F y dash plus phi x minus z dash F z dash x equal to x 2 delta x 2 plus F y dash plus phi y F z dash x equal to x 2 delta y 2 equal to 0.

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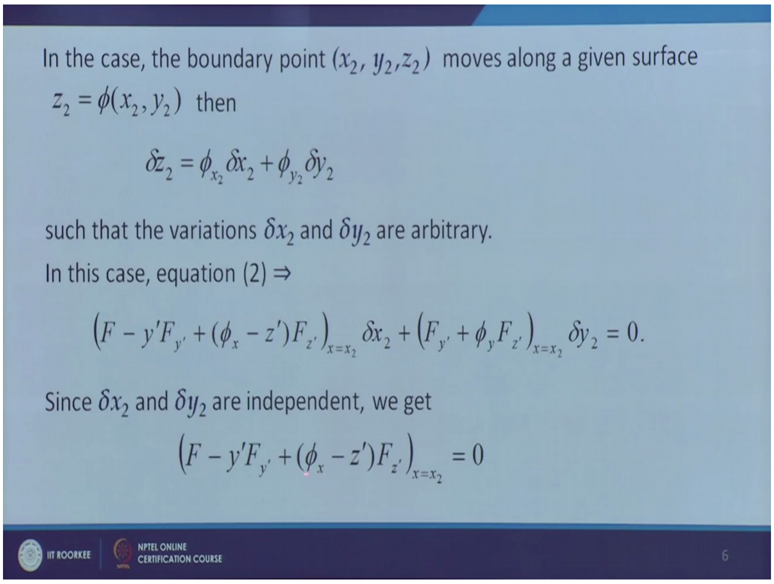

Now since delta x 2 delta, y 2 are independent we get F minus y dash F y dash plus phi x minus z dash F z dash at x equal to x 2 is 0 and also F y dash plus phi y F z dash equal to 0.

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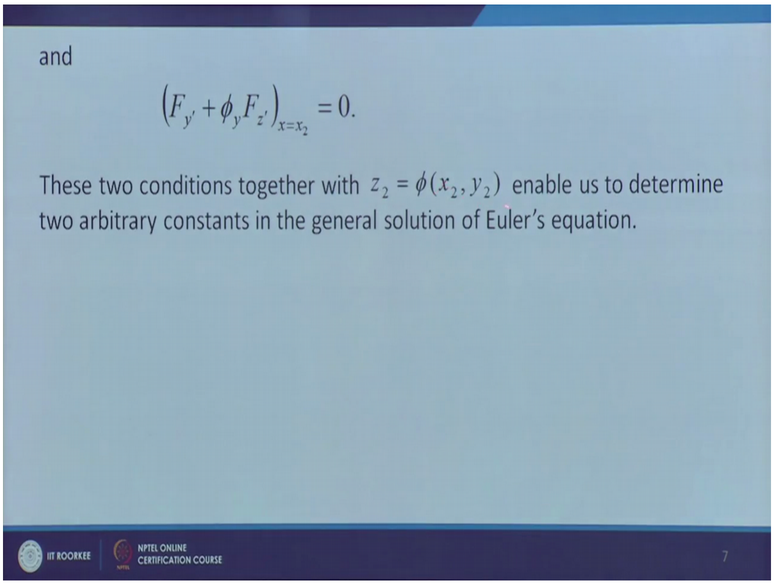

So these two equations along with the fact that x 2, y 2, z 2 lies on z equal to phi x, y we get z 2 equal to phi x 2, y 2 so these equations enable us to get the remaining two arbitrary constants in the general solution of the Eulers Equation.

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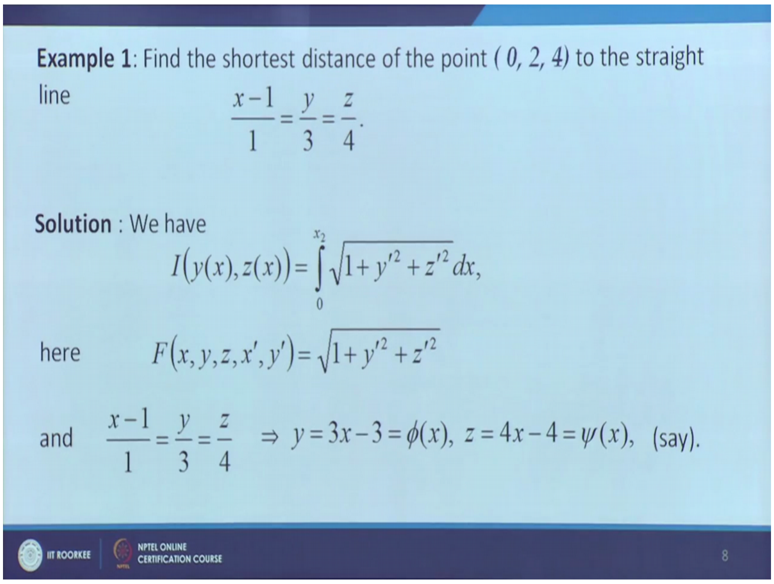

Now let us take the example on this article let us find the shortest distance of the point 0 to 4 to the straight line given by x minus 1 over 1, y over 3, z over 4 these equations of the line are given in the symmetric form. So we know that I y x, z x, okay is integral because the x 1 point is 0, this is x 1, y 1, z 1 which is a fixed point so x 1 is 0 here, x 2, y 2, z 2 is a variable point which will lie on the line given here so and then we want to find the shortest distance we know the formula dx square plus dy square plus dz square under root that is ds.

So when we write it as we write ds over dx into dx, ds over dx can be written as square root 1 plus dy over dx whole square plus dz over dx whole square, so we have written it as under root 1 plus y dash square plus z dash square into dx and the limits of integration are x 1 equal to 0 and x 2 is a variable, so x 1 0 goes 0 to x 2. So here F x, y, z, y dash, z dash is equal to square root 1 plus y dash square plus z dash square.

Now the equations of the lines x minus 1 over 1 equal to y over 3 gives us y equal to 3x minus 3, so we can write it as a function of x say phi x and x minus 1 over 1 equal to z over 4 gives us the other equation z equal to 4x minus 4, we can denote it by Psi x. So y is a function of x and z is a function of x.

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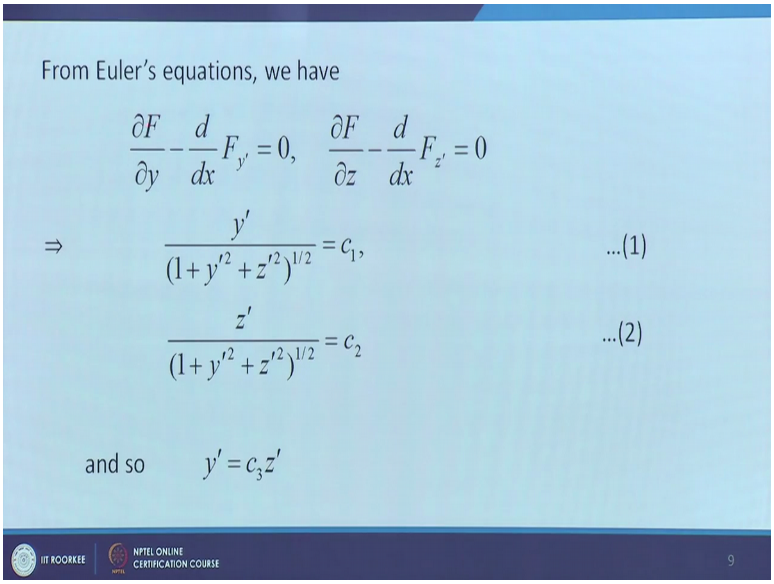

Now the Eulers Equation are for the extremal the Eulers Equation are F y minus d over dx F y dash equal to 0, F z minus d over dx of F z dash equal to 0.

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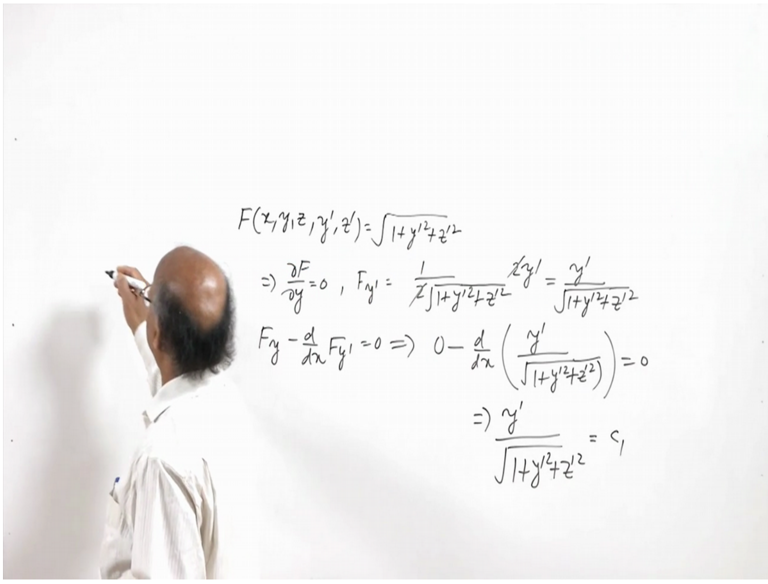

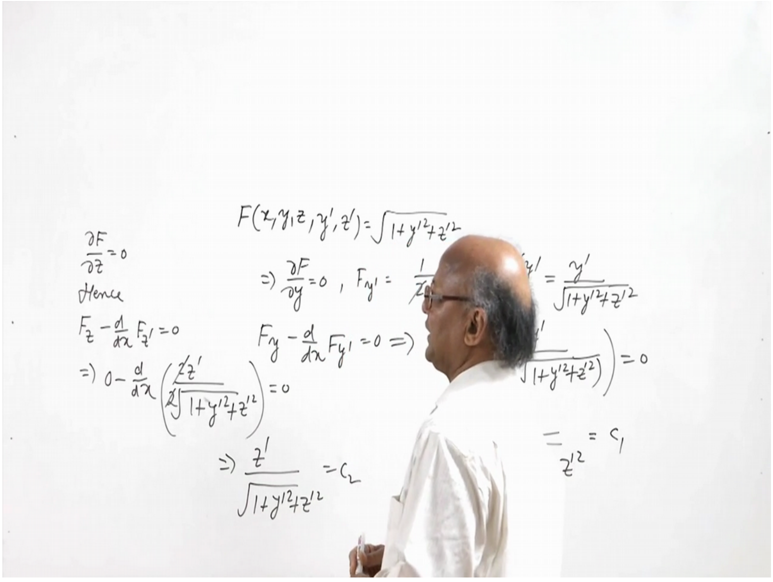

So since our extremal is since our F x, y, z, y dash, z dash is equal to square root 1 plus y dash square plus z dash square F is independent of y. So its partial derivative with respect to y 0 and therefore F y minus d over dx of F y dash is equal to 0 gives us 0 minus d over dx of F y dash. Now here F y dash if you find then F y dash is 1 over 2 square root 1 plus y dash square plus z dash square into 2 y dash. So this is y dash divided by under root 1 plus y dash square plus z dash square. Let us put it here, so y dash divided by under root 1 plus y dash square plus z dash square, so this is equal to 0.

So first equation gives us y dash divided by since derivative of y dash divided by under root 1 plus d y dash square, z dash square to the power half is 0, this gives y dash divided by under root 1 plus y dash square plus z dash square is equal to some constant let us take this constant as c 1 and similarly let us note that the F x, y, z, y dash, z dash independent is independent of z.

So partial derivative of F with respect to z is also 0, hence F z minus d over dx of F z dash equal to 0 gives us 0 minus d over dx of derivative of F with respect to z dash is 2 z dash divided by square root 1 plus y dash square plus z dash square 2 times so this 2 cancels out with this and we get z dash divided by square root 1 plus y dash square plus z dash square is equal to 0, d over dx of this d over dx of this is 0 so this is some constant let us say c 2.

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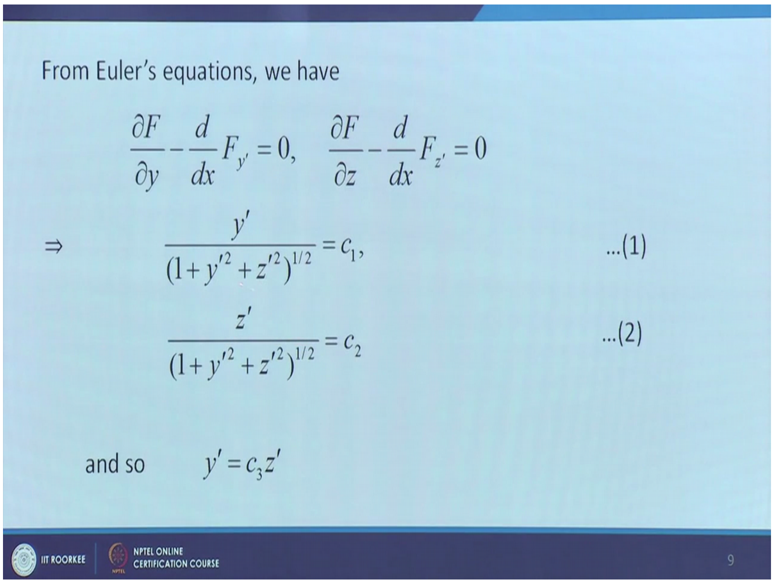

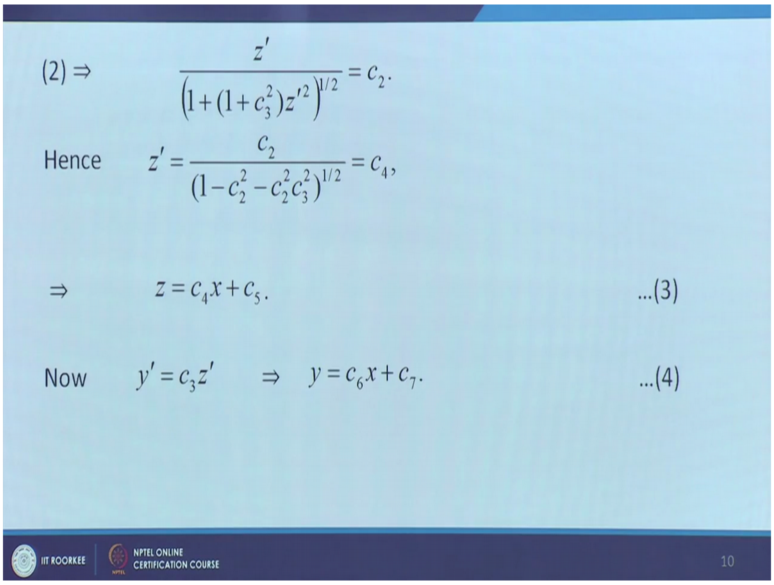

So we get these two equations y dash over under root 1 plus y dash square plus z dash square is equal to c 1, z dash over under root 1 plus y dash square plus z square is equal to c 2. So let us divide these two equations then we get y dash over z dash equal to c 1 over c 2 and c 1 over c 2 is another arbitrary constant we can write it as c 3. So y dash is equal to c 3 z dash.

Now let us put the value of y dash equal to c 3 z dash in this equation z dash equal to 1 plus y dash square plus z dash square to the power half we will get z dash divided by 1 plus 1 plus c 3 square z dash square to the power half equal to c 2 or we can solve it for z dash z dash is equal to c 2 upon 1 minus c 2 square minus c 2 square c 3 square to the power half. So this is let us call it as c 4, in some arbitrary constant we will call it as c 4. Now this is dz over dx equal to c 4.

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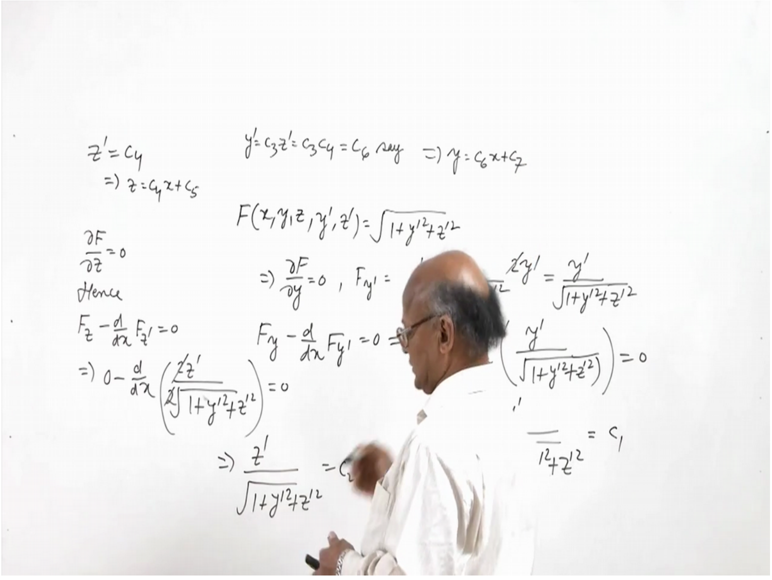

So z equal to c 4 x plus some constant c 5.

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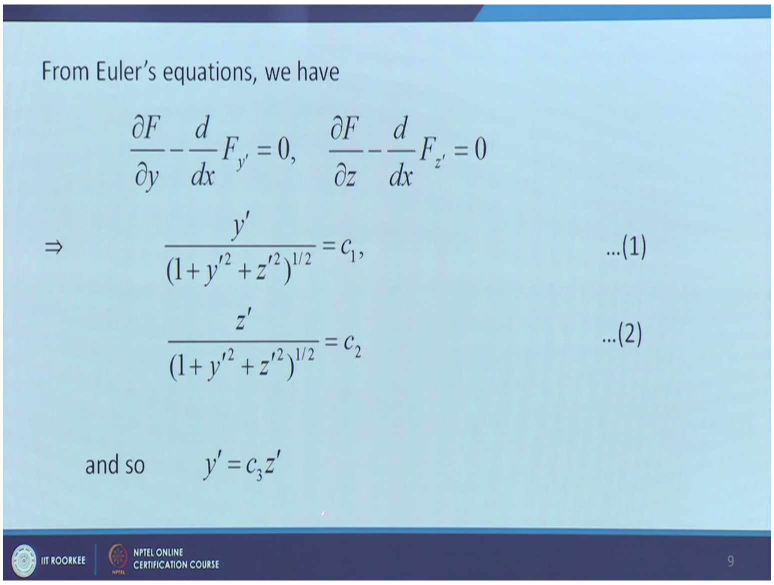

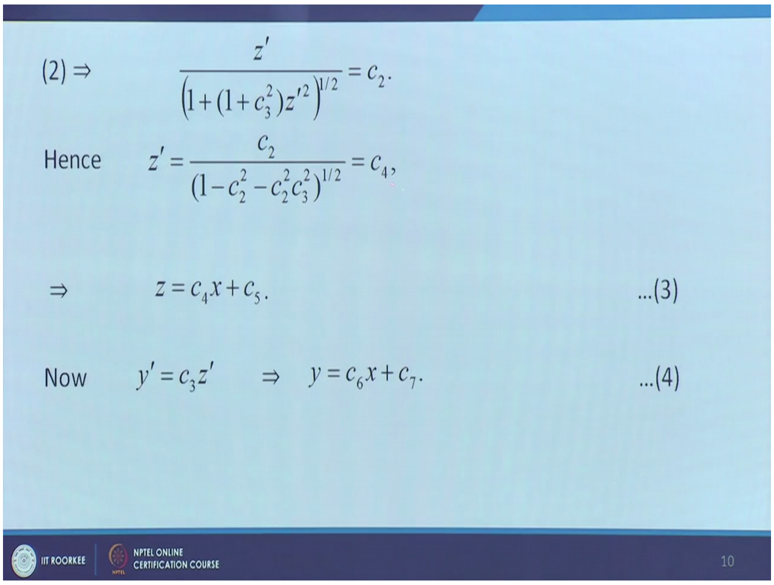

And y dash is equal to c 3 z dash we have y dash equal to c 3 z here so y dash equal to c 3 z dash gives us z dash is some constant c 4, so c 3 into c 4 y dash is equal to c 3, c 4 we can so this gives us let us say this is c 6, so then y is equal to c 6 x plus c 7. So we get the expressions for y and z in terms of x.

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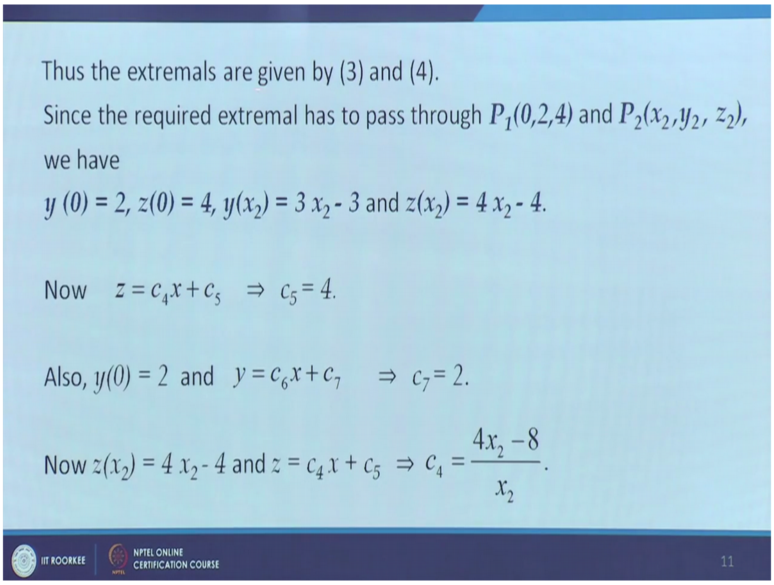

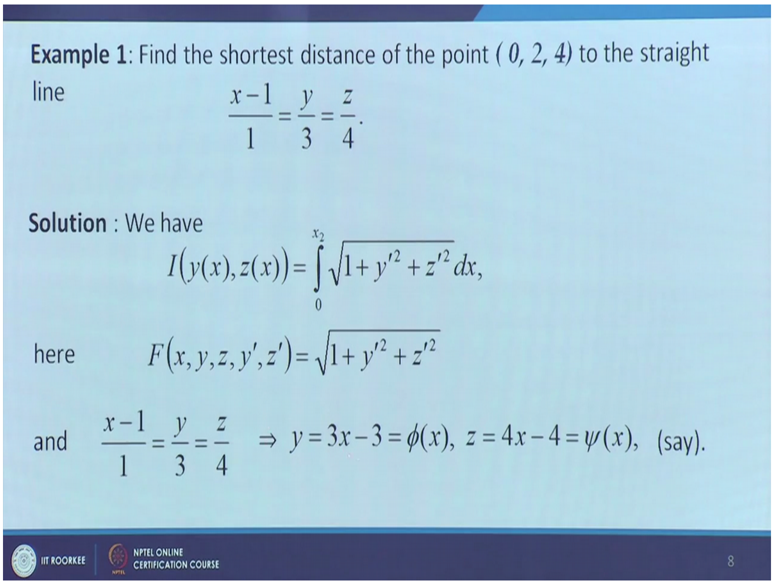

Now let us find the extremals extremals are given by equation 3 and 4, since the required extremal has to pass through 0 to 4. The line of shortest distance has to pass through the fixed point 0 to 4 and the point variable point x 2, y 2, z 2 we have y at 0 is equal to 2 and because y is a function of x and z is also a function of x. So y at 0 is equal to 2 and z at 0 is equal to 4, this is x 1, y 1, z 1, y at x 1 is y 1 and z at x 1 is z 1 and similarly y at x 2 is y 2. So y at x 2 is 3 x 2 minus 3 y x is equal to we have y equal to 3 x minus 3. So y at x 2 is equal to 3 x 2 minus 3 and z 2 equal to 4 x 2 minus 4. So we get z x 2 equal to 4 x 2 minus 4.

Now what we have z equal to c 4 x plus c 5, so in this you put x is equal to 0 x equal to 0 when we put what we get z equal to 4 so c 5 is equal to 4. And y at x equal to 0 is true so what we get is y c 7 is equal to 2. Now z x 2 is equal to 4 x 2 minus 4 and z is also equal to c 4 x plus c 5, c 4 x plus c 5. So c 4 from here what will happen z equal to c 4 x plus c 5 which means z 2 equal to c 4 x 2 plus c 5.

And so what we will get is c 4 equal to c 4 we have already found, c 4 no c 5 we have found c 5 is equal to 4, so z is equal to c 4 x plus 4 and let us this z equal to c 4 x plus c 5 passes through x 2, y 2, z 2.

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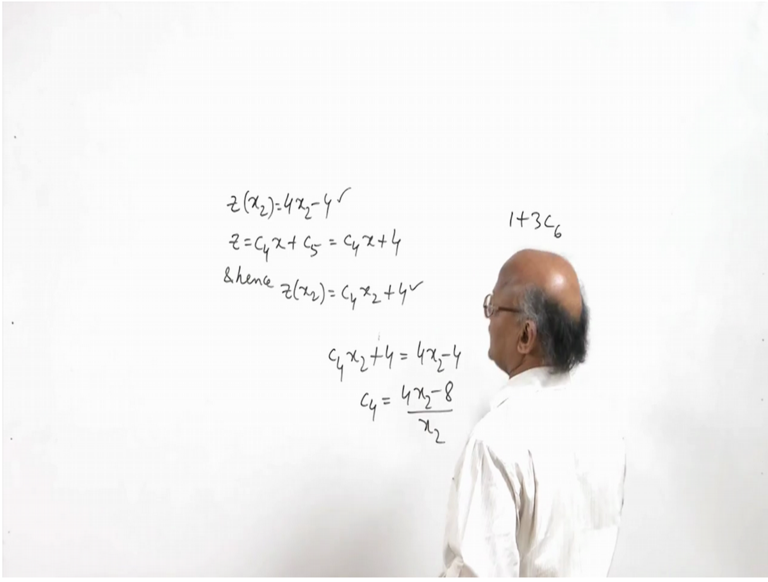

So we get z x 2 equal to z x 2 is equal to 4 x 2 minus 4 and z equal to c 4 x plus c 5 gives us c 5 is 4 and so and hence z x 2 is equal to c 4 x 2 plus 4, so what do we get from this equation and this equation c 4 x 2 plus 4 is equal to 4 x 2 minus 4. So we get 4 x 2 minus 8 c 4 is equal to 4 x 2 minus 8 divided by x 2.

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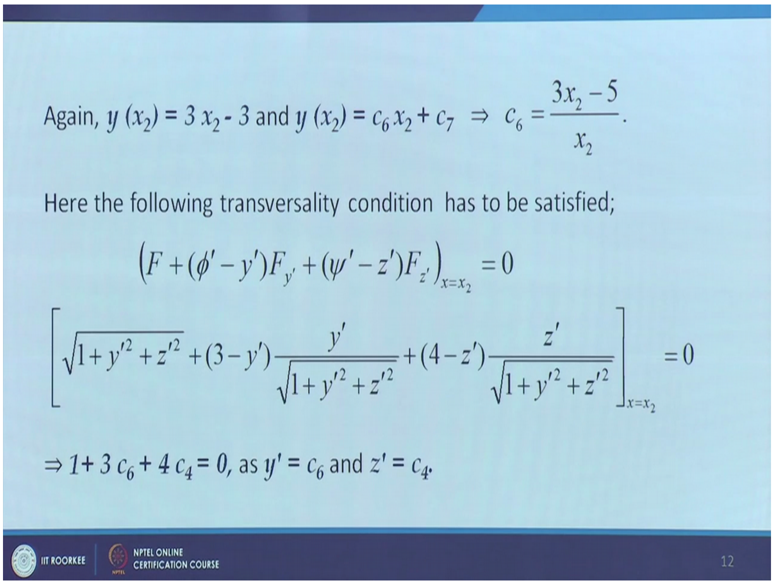

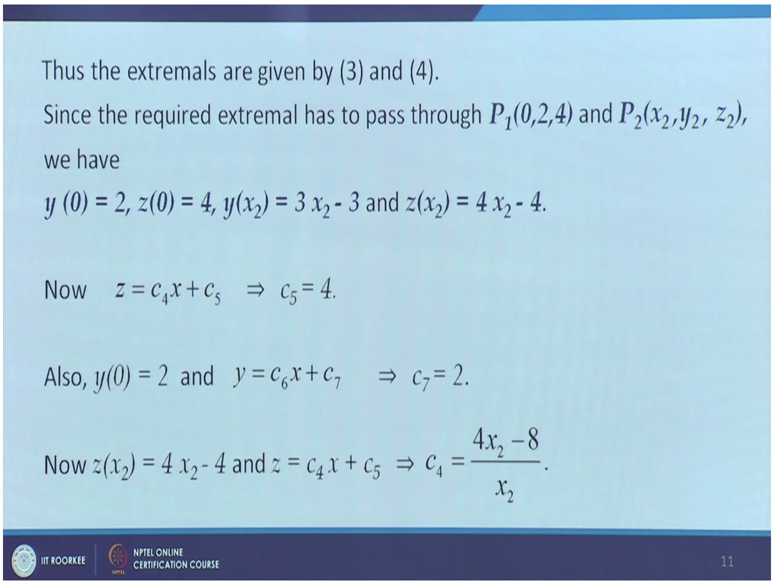

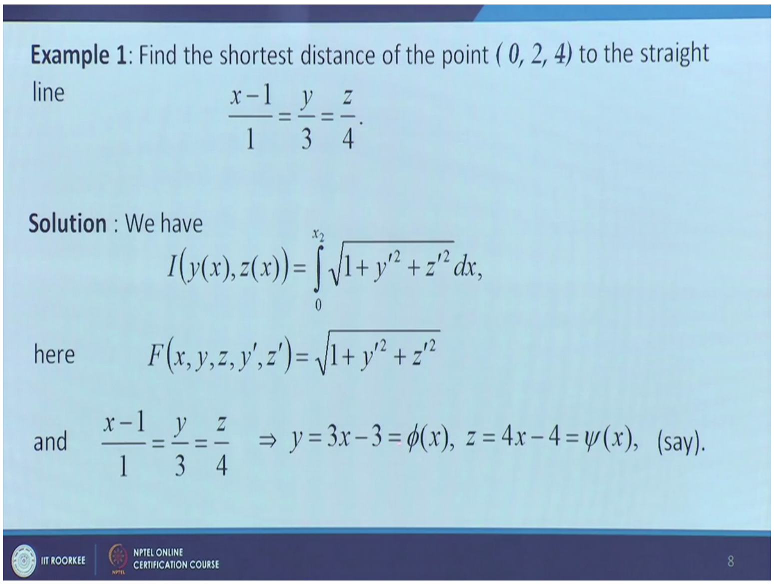

And then y x 2 is equal to 3 x 2 minus 3 and also y x 2 is equal to c 6 x 2 plus c 7 and we have already found the value of c 7, c 7 is 2. So we can put here 2 and then solve these two equations for c 6, c 6 comes out to be 3 x 2 minus 5 over x 2. Now since the point x 2, y 2, z 2 lies on the curve the following transversality condition has to be satisfied F plus phi dash minus y dash F y dash plus Psi dash minus z dash F z dash x equal to x 2 equal to 0, F is given to be under root 1 plus y dash square plus z dash square phi dash phi x is equal to 3 x minus 3 phi x is equal to 3 x minus 3, so phi dash is equal to 3 and Psi dash is equal to 4.

So let us put these values here, so we get then 3 minus y dash F y dash is y dash over under root 1 plus y dash square plus z dash square, Psi dash is 4 minus z dash F z dash is z dash upon under root 1 plus y dash square plus z dash square at x equal to x 2 is 0. Now so you take LCM here under root 1 plus y dash plus z dash square is the LCM and then the numerator becomes 1 plus y dash square plus z dash square 3 y dash minus y dash square 4 z dash minus z dash square, so y dash square z dash square will cancel and we will get 1 plus 3 y dash plus 4 z dash equal to 0 but y dash is equal to c 6.

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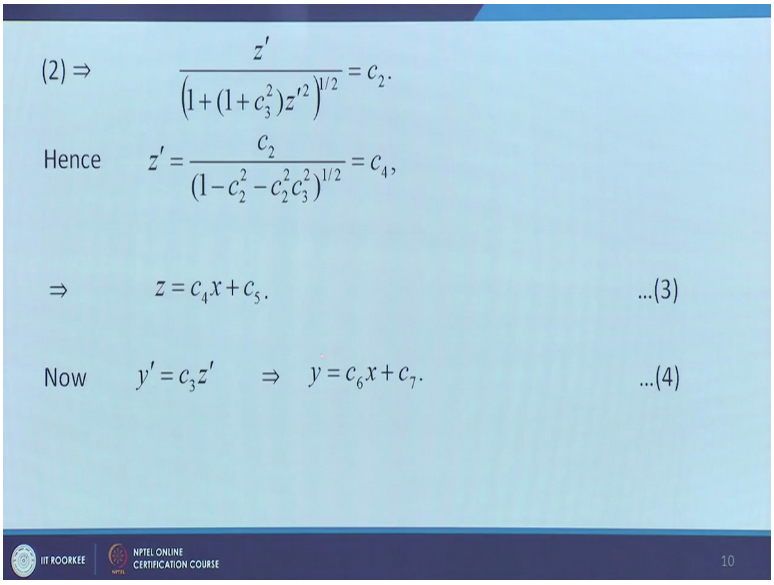

Let us see y dash we have seen to be equal to y dash is equal to c 3 into z dash and c 3 into z dash was equal to c 4, so c 3 into c 4 we defined as c 6, so y dash is c 6 and z dash is c 4.

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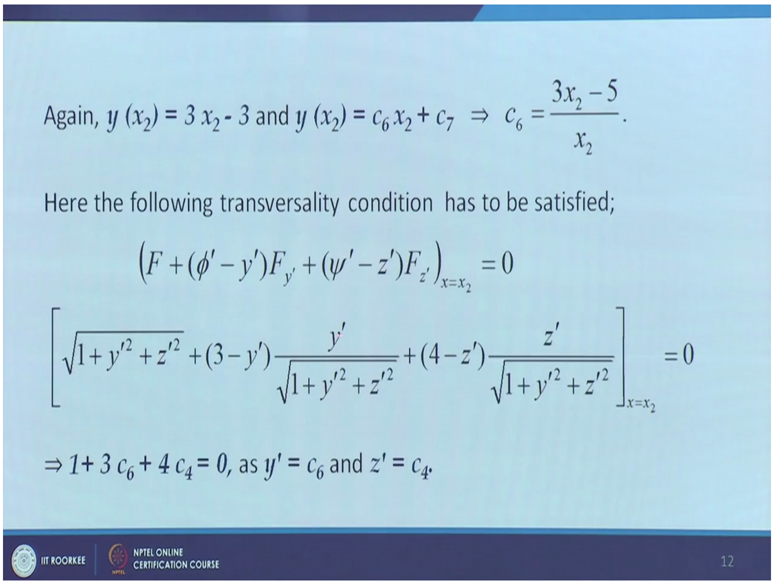

So let us replace these values there, so 1 plus 3 y dash means 1 plus 3 c 6 and 4 z dash means 4 c 4, so this is equal to 0 as y dash is c 6, z dash is c 4.

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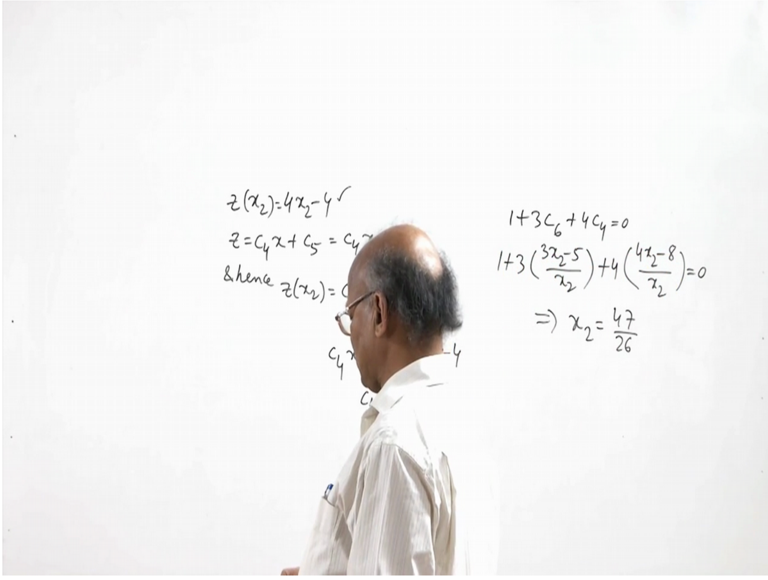

Now then okay so let us see so we have 1 plus 3 c 6, 1 plus 3 c 6 plus 4 c 4 equal to 0. In this we can put the value of c 6 as 3 x 2 minus 5 over x 2, 3 x 2 minus 5 over x 2 and then we can put the value of c 4 also, so 4 x 2 minus 8 y x 2.

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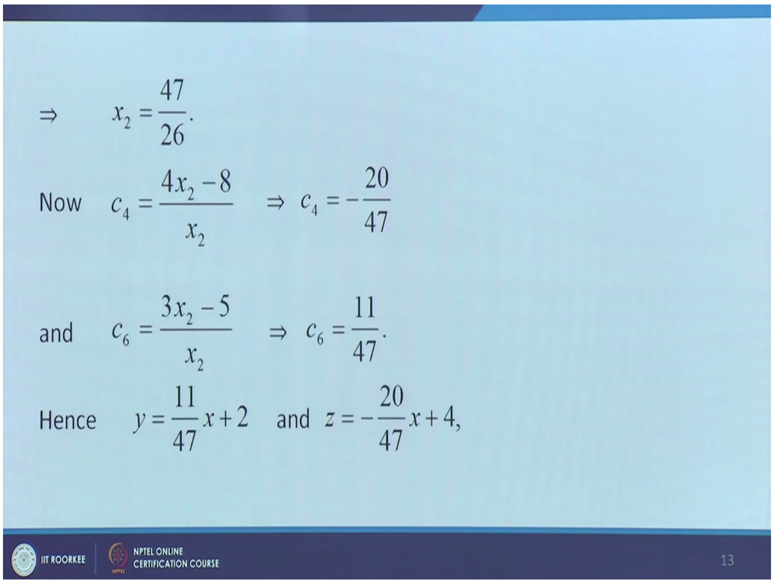

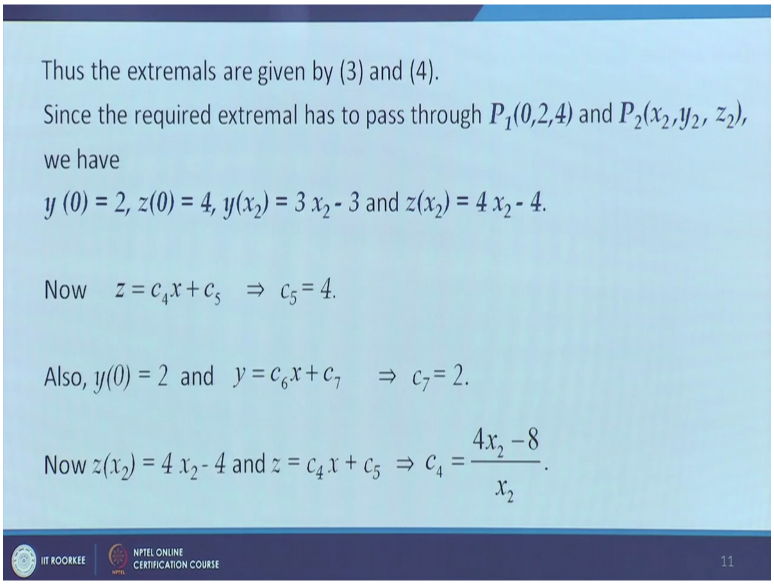

And simplify we will get the value of x 2, x 2 comes out to be (47 minus) 47 by 26, so x 2 comes out to be 47 by 26. Now c 4, c 4 is equal to c 4, here is c 4, c 4 is 4 x 2 minus 8 by x 2. So let us put the value of x 2 here and we get the value of c 4 as minus 20 by 47, c 6 is 3 x 2 minus 5 over x 2, so we get c 6 as 11 by 47 and then we get the value of y equal to y was equal to c 6 x plus c 7. So we get 11 by 47 x plus 2 and z is equal to c 4 x plus z is equal to c 4 x plus c 5, c 5 is 4 and c 4 we have already found c 4 is minus 20 by 47, so z is equal to minus 20 by 47 into x plus 44. So these these are the equations of the line of shortest distance.

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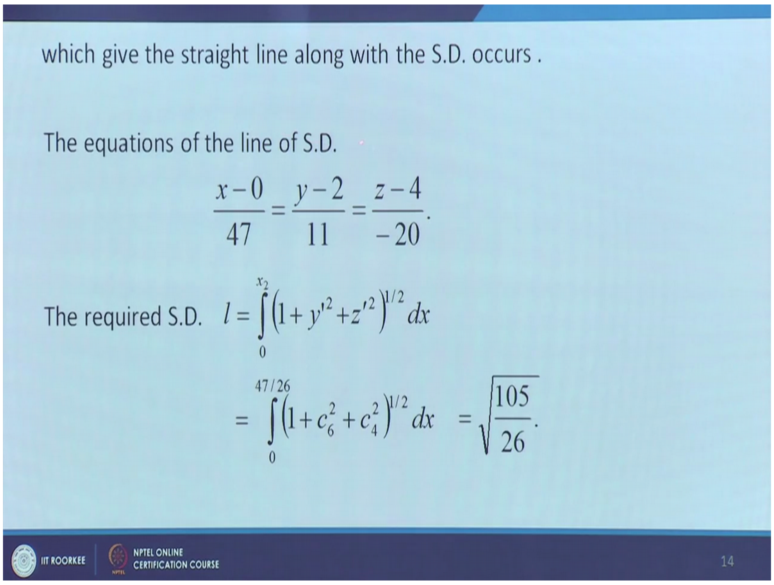

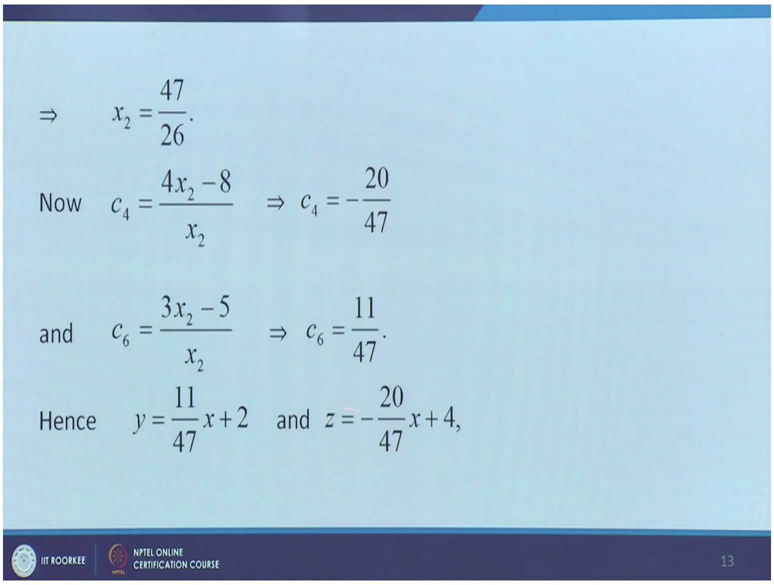

Now so this shortest distance occurs along the straight line equations of the line of shortest distance can be put in this symmetric form these equations which are the equations of the line of shortest distance we can easily put in the symmetric form and the symmetric form is x minus 0 by 47 equal to y minus 2 by 11 equal to z minus 4 by minus 20. Now the required shortest distance is integral 0 to x 2 1 plus y dash square plus z dash square to the power half dx and x 2 we have found to be 47 by 26, so 0 to 47 by 26 the value of c 6 is 11 by 47, c 4 is minus 20 by 47.

So we can put it them here take the square root and then integrate we get the value square root 105 by 26, so we get the equations of the line of shortest distance and also the required shortest distance, with this I would like to conclude my lecture, thank you very much for your attention.