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NATIONAL PROGRAMME ON TECHNOLOGY
ENHANCED LEARNING
(NPTEL)

Discrete Mathematics

Module-10
Recurrence relations

Lecture-03
Second order recurrence relation with constant coefficients (2)

With
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In the last lecture we studied the solution techniques of second-order.

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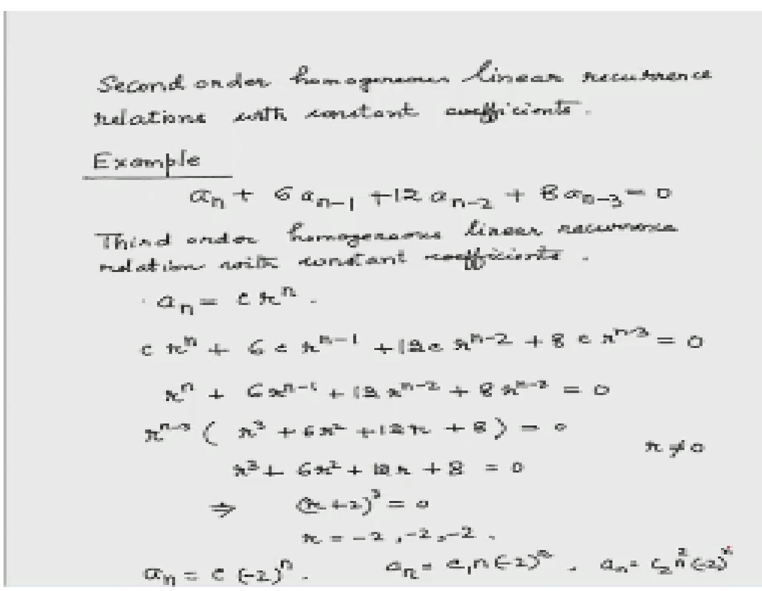

Homogeneous linear recurrence relations with constant coefficients what we will see in today's lecture is that the technique that we developed very naturally extends to higher order homogeneous linear recurrence relations with constant coefficients instead of going for a general theory and theorems that take care of any order recurrence relations we will consider some third order homogeneous linear recurrence relations with constant coefficients and the cases that we will discuss will equip us to deal with any order linear recurrence relations of course with constant coefficients and of course homogeneous.

Now to start with let us consider the third order linear recurrence relation given by $a_n + 6 \times a_{n-1} + 12 a_{n-2} + 8 a_{n-3} = 0$, now we note that this is a third order homogeneous linear reconciliation with constant coefficients third order homogeneous linear recurrence relation with constant coefficients. Now we use the technique that we have developed in the last lecture in exactly the same way we assume that the solution is of the form $n = C \times R^n$ then we substitute $n = C \times R^n$ in the recurrence relation to obtain $C R^n + 6 \times C R^{n-1} + 12 C \times R^{n-2} + 8 C \times R^{n-3} = 0$.

Now just as before we take out all the C's and since we can very well assume that C is not = 0 we get the equation $R^n + 6 \times R^{n-1} + 12 \times R^{n-2} + 8 \times R^{n-3} = 0$. Now in the next step we take out R^{n-3} which is a common factor of all the terms and write $R^{n-3} = R^3 + 6 \times R_2 + 12 \times R + 8 = 0$ and again a reasonable assumption of non-0 of our we get $R^3 = 6 R^2 + 12 R + 8 = 0$.

Now we immediately observe that this is nothing but R + 2 whole cube =0 that means we have a solution R =-2 repeated thrice thus we obtain a solution $n = C \times -2^{n.}$ Now at this point we recall that that for the second-order case we referred to a general theory which says that the general solution of a second order homogeneous linear recurrence relation will be constant times one solution and another constant times another solution which is not a multiple of the first one now in this case we see that we have obtained only one solution and the same theory is a more general one which says that if you have a k^{th} order homogeneous linear recurrence relation with constant coefficients then you will have the general solution as a linear combination of linearly independent solution whose total number is K.

Therefore in the case of third order equations we'll we will have three linearly independent solutions whose linear combination will generate all the solutions that is the general solution will be some constant times one of one solution + constant times another and a constant times a third one where each of them are linearly independent of the other two this essentially means that we won't be able to find a linear combination that is a constant times one solution + another constant times another solution becoming =the third one for all values of n.

Now in order to get the other solutions we use the same strategy as before and the final result is very easy to remember that is the second solution will be of the type n =C times well it is something different so C 1 times n -2^n and the third one will be n =C 2 times n^2 -2^n.

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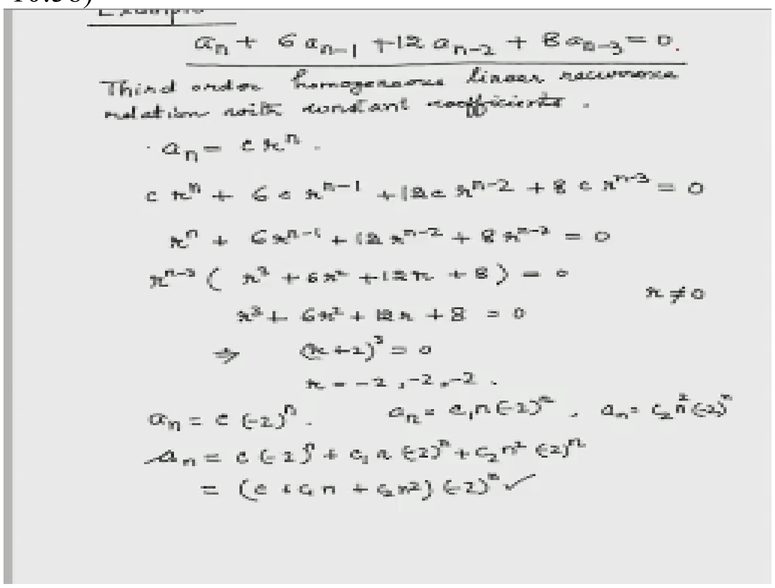

Therefore the general solution will be of the form a n =a constant times -2^n + another constant x n x -2^n + the third constant times n^2 x -2^n therefore we can club all the constant terms together all the terms involving constant together and take the common factor -2^n out to get C + C 1 n + C 2 n^2 within parentheses and into 2 -2^n, so this is a general solution of the third order homogeneous linear recurrence relation that we started with.

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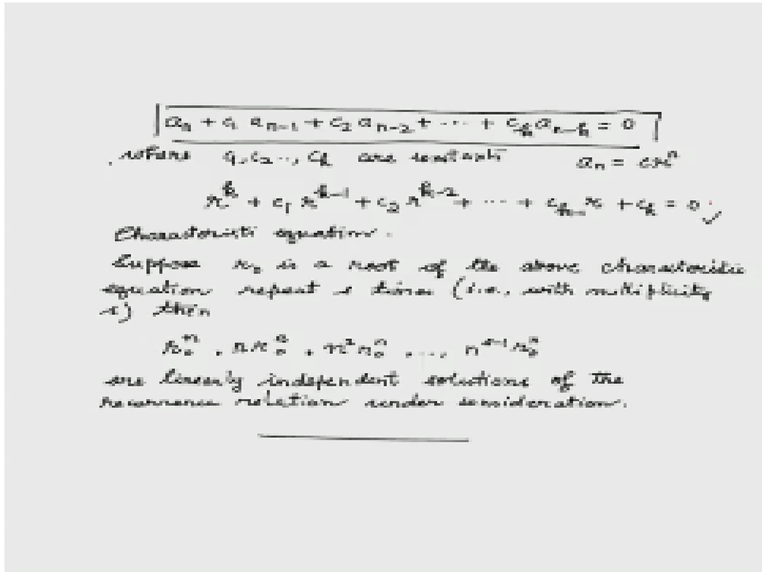

Now we can from here very easily guess general pattern suppose I have a recurrence relation of the type $a_n + C_1 a\ n-1 + C_2 a\ n-2$ + and so on up to see K a_n -k =0 where $C_1\ C_2 C_K$ are constants then the characteristic equation will be of the form $R^k + C1\ R^{k} - 1 + C_2$ R to the power K -2 + and so on the last part one term is CK -1 R + CK = 0 this is the characteristic equation and suppose there is a route are not which is repeated let us say s times let me write here suppose R0 is a route of the above characteristic equation with s repetitions repeated s times that is technically with multiplicity s then $r0^n$ n $r0^n$ x n n 2 R 0^n and so on up to n^{s-1} R 0^n are all solutions of the original recurrence relation and all of them are linearly independent.

So we may write these are linearly independent solutions of a recurrence elation under consideration, so in general what we do is when we have a recurrence relation of this type we consider a trial solution of the form a n = C R^n and obtain the characteristic equation we then solve the characteristic equation which is after all a an algebraic equation of certain degree with single variable and after solving in general we will get each solution with some multiplicity. If the multiplicity is 1 then corresponding to that solution we will get a single solution of the recurrence relation if the multiplicity is 2 corresponding to that solution we will get two solutions if multi-city is 3 corresponding to that we will get three solutions.
And in general if the multiplicity is s we will get s solutions we will sum all of them taking by multiplying each term by an arbitrary constant and like that we will get a general solution of this recurrence relation. Now we will explain what I have just told by using another example.

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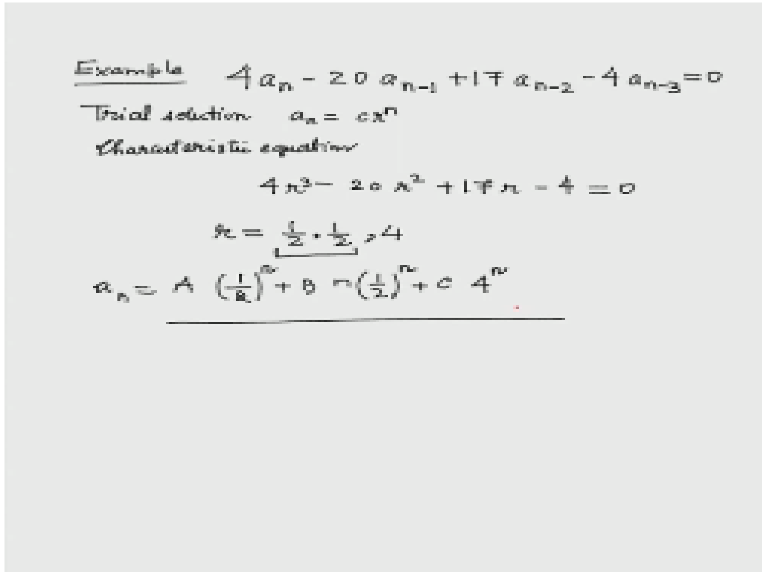

Let us now consider a recurrence relation of the form 4 a n -20 a n -1 + 17 a n -2 - 4 a n -3 = 0, now like before I will take a trial solution as a n = C R ^{n} and substitute n =R ^{n} and other values of other values a and -1 and n -2 and if we do that we will get the characteristic equation for R^3 -20 R ^{2} + 17 R -4 = 0. And now if we solve this equation we will see that R is =1/2 repeated twice that is 1/2 is the is a root of multiplicity 2 and 4, now we have to build the general solution since the value ½ is repeated twice corresponding to that we will have two solutions which are linearly independent each other one is of course ½^n.

The other one is n times ½^n whereas the loan solution for which is not repeated will give us another solution 4^n, now it is not difficult to see that no solution among these three solutions can be written as a linear combination of the other two solutions for all values of n therefore we can build the general solution of the homogeneous equation as a n =a constant let us say capital a times ½^n + B times n times ½^n and then see times 4^{n,} if we remember these examples then we will be able to solve any linear homogeneous equation with constant coefficients. Next we discuss the situation when we have non homogeneous recurrence relations.

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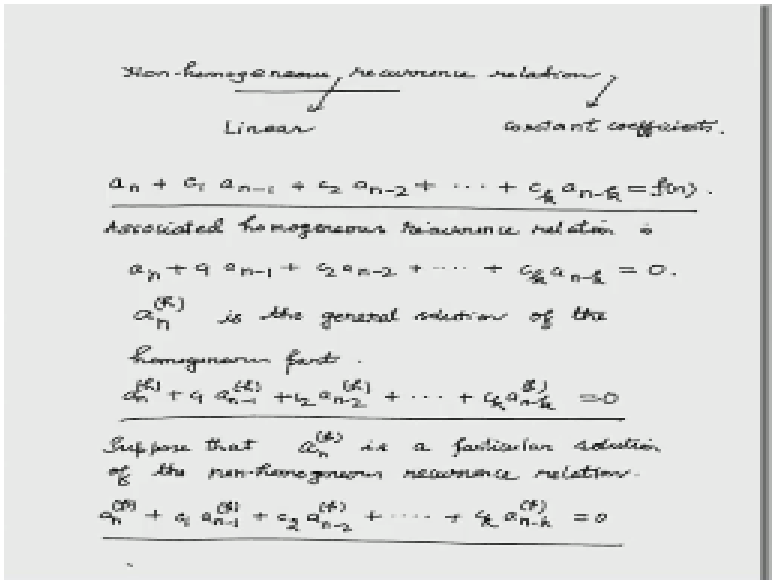

Of course in this case we will not go much away from what we have done we will be considering non-homogeneous linear recurrence relations with constant coefficients, so we attach one term here that is linear and another term over here constant coefficients. So we can write a general form of these recurrence relations as a n + C 1 x a n -1 + C 2 x a n -2 + so on + CK x a n -K = a function of n. Now as I have said that this is not really very far away from what we have discussed just now, if we look at it carefully we will see that this non homogeneous recurrence relation is associated to a homogeneous recurrence relation in a very obvious way and that is why that is by putting FN =0.

So we write that relation as associated homogeneous recurrence relation associated homogeneous recurrence relation is a n + 7 n -1 + C 2 a n -2 + and so on and we go up to CK a n -k =0. Now what we say is that what if I get the general solution of the last liquor recurrence relation and write that as a in H, so a NH is the general solution of the well we can introduce this term homogeneous part, so we know that a n H + C 1 x a n -1 ^{H} + C 2 x a n -2 H like that ck a n -K H =0.

Now suppose we get a single solution of the non-homogeneous equation which we call a particular solution, suppose that a_N^P is a particular solution of the non-homogenous recurrence relation okay then of course we know that a n P c1 a n -1 P + C 2 a n -2 P + and so on up to CK a n -K P is =0. Now we consider these two expressions and add them to get en h + n P + c1 times a n -1 h + a n -1 P + c2 times a n -2 h + a n -2 P and so on up to CK times a n -K H + a n -K P =0.

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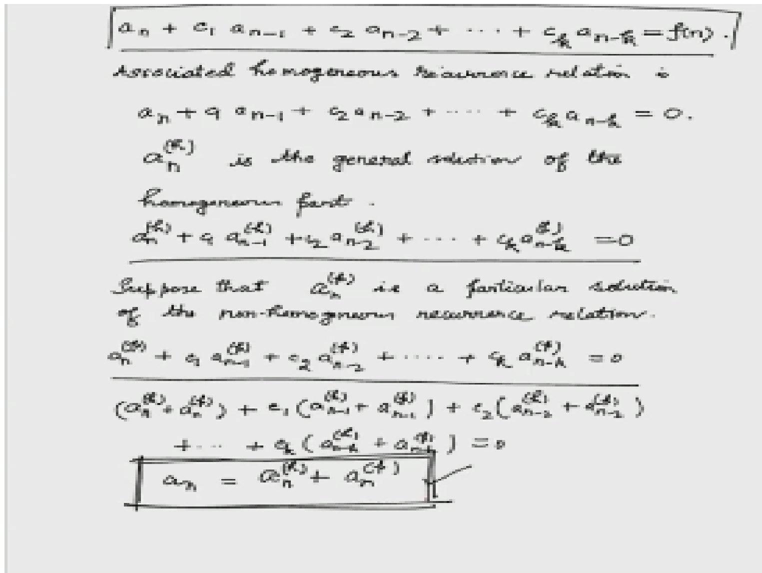

Now we see that the sum of two solutions that is a particular solution and the general solution gives me a solution of the recurrence relation. So this is a n-h the solution of the homogeneous part and ANP a particular solution which together gives me a solution of the non-homogenous equation. Now here we note that the general solution is not a single solution it is essentially a family of solutions that is obtained by varying the constants related to the general solution and therefore this a n =a n h + a and P is not a single solution.

But a family of solutions what is very striking is that it can be proved that any solution of the non-homogenous equation of this form can be written in this form that is the general solution with certain values of the constants + any particular solution therefore if we want to solve a non-homogeneous reconciliation of this type then we will have to first get the general solution of the homogeneous part and somehow find out a particular solution we add them together and then we put the initial values and we will get the constants and that is going to be the solution that we require, we now take up examples.

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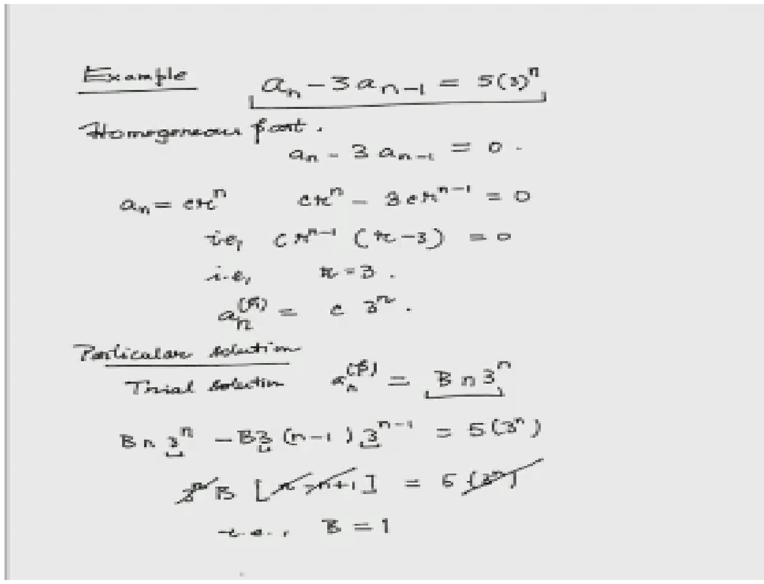

The first example is a reconciliation of Kadar one that is a first-order recurrence solution which is not homogeneous but of course linear with constant coefficients, so we have a situation like this where n -3n -1 is =5 x 3^n so when we encounter such a satanic equation we will consider first the homogeneous part which is a n -3n -1 =0 here as before we put a trial solution if we do that then we will get C R^{n-3} times C R^{n-1} = 0 if we take out C and R^{n-1} we are left with R -3 = 0 that is R =3.

Therefore the general solution of the homogeneous part is given by a n H = some constant C times $3^{n.}$ Now what about the particular solution? For the particular solution we have to do some guesswork and one can only get some perfection in the guesswork if one solves several problems from several different sources. Now let us let me let me explain the situation that we have here we see in the right hand side we have 5 x 3^n.

So somehow we know that the left hand side should be something into 3^n therefore what we do here we take a trial solution of the type a and P =be n 3^n. Now we will substitute B and C^n in the original recurrence relation will obtain B n 3^n – 3 x n – 1 3^n- 1 = 5 x 3^n. Now we see that 3^n is a common factor of the left-hand side of the of this equation therefore we can write 3^n we have missed one term over here that is B will be of course here, so we will write 3^n x B which is = which is multiplied by n -n + 1 and this is =5 times 3^n of course C to the power n gets cancelled from both sides and n gets cancelled therefore we are left with the value of B which is =1.

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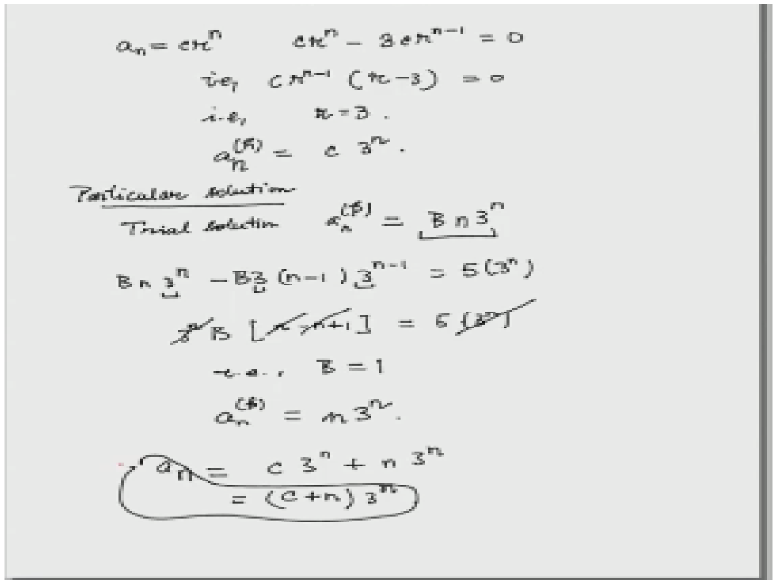

Therefore we see that a in particular a and P is = n times 3n, and therefore a general solution of the original recurrence relation is given by a n = a constant times 3 ^{n} + n times 3 ^{n} which in turn becomes see the constant + n times 3 ^{n}. Now if this recurrence relation arose from some real-world application where people knew that a 0 has certain value then they could have substituted the value of a 0 in the general solution to obtain the value of C and then they could have gotten a particular solution that agrees with the problem or model or a natural phenomena that they were observing as a last part of this talk we will solve another non-homogeneous recurrence relation but this time of second-order.

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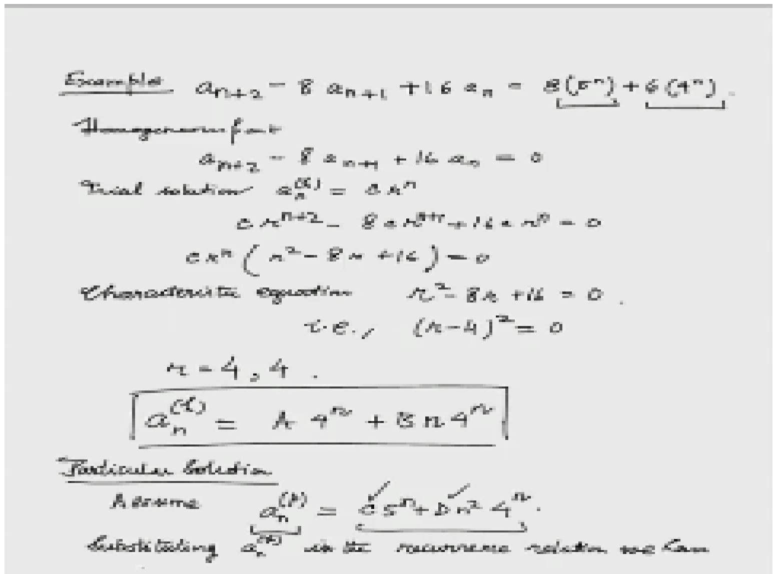

Example $a_{n+2} - 8a_{n+1} + 16a_n = 8(5^n) + 6(4^n)$.

Homogeneous part

$$a_{n+2} - 8a_{n+1} + 16a_n = 0$$

Trial solution $a_n^{(h)} = c r^n$

$$c r^{n+2} - 8 c r^{n+1} + 16 c r^n = 0$$

$$c r^n (r^2 - 8r + 16) = 0$$

Characteristic equation $r^2 - 8r + 16 = 0$.

i.e., $(r-4)^2 = 0$

$r = 4, 4$.

$$\boxed{a_n^{(h)} = A 4^n + B n 4^n}$$

Particular Solution

Assume $a_n^{(p)} = C 5^n + D n^2 4^n$.

Substituting $a_n^{(p)}$ in the recurrence relation we have

Now let us start with the equation this is a n + 2 - 8 a n + 1 + 16 a n = 8 x 5 ^{n} + 6 x 4 ^{n}. Now as usual we are going to split this relation into the homogeneous part and the original non homogeneous part, so homogeneous part is a n + 2 -8 a n + 1 + 16 n = 0 we consider the trial solution a n H = a constant times R to the power N and as usual we substitute this in the homogeneous equation to obtain a constant C times R ^{n} + 1 -a times constant C R ^{n} + 1 + 16 times C R ^{n} = 0.

And if we take out C and R ^{n} then we are left with R 2 -8 R + 16 = 0 that is we obtain the characteristic equation R^2 -8 R + 16 = 0, of course we see that this is R - 4^2 = 0 so we have 4 as repeated roots of the characteristic equation, therefore from what we have discussed already we can write a in homogeneous is = sum' let us say a times 4 ^{n} + B x n x 4 ^{n}. Now we move on to the problem of finding out particular solutions, so particular solution, now we go for an assumption.

So we are assuming that the particular solution a N particular is going to be a constant times C going to be a constant C times five ^{n} + another constant D times n ^{2} x 4 ^{n} of course there is a question that how do I arrive at this one there is no rule as such but one can expect that something like this will happen because we see the right-hand side is a linear combination of 5^n and 4 ^{n} and then we can if we like take 5 ^{n} and 4 ^{n} and take some coefficients dependent on n and easiest may be n n ^{2} n ^{3} and like that and we can hit and try to check which gives me a particular solution.

In this case we know that it will be of this form or right now we assume and we will find out that we are able to obtain the values of C and D by substituting this particular solution in the reconciliation, if we do that then we have an expression which is somewhat long, so let me write it down. So substituting a_n particular in the recurrence elation we have $C \times 5^{n+2} + D \times n+2^{2} 4^{n+2} - 8 \times c5^{n+1} + D \times n+1^{2} 4^{n+1} + 16 c5^{n} + 16 D n^{2} 4^{n} = 8 \times 5^{n} + 6 \times 4^{n}$.

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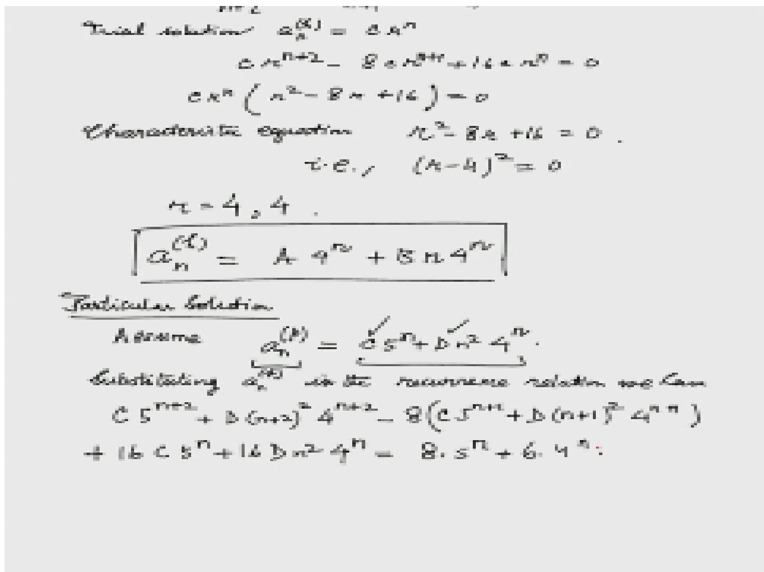

This we obtain directly by substituting in the original equation, now what we do here now what we do after this is collect all the coefficients of 4^{n} and 5^{n} if we do that we will obtain well something like this.

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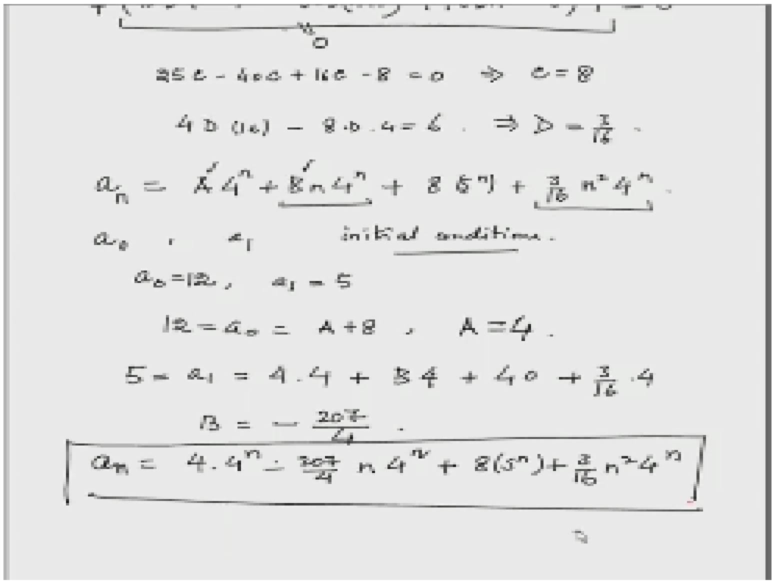

25 C - 40 c + 16 C - 8 ^{n} x 5 ^{n} and then + 16 D n + 2^2 -32 D n + 2 n + 1^2 n + 1 ^{2} and then we will have + 16 b n ^{2} -6 4 ^{n} = 0, now this means in turn that this term into 5 ^{n} is = this term into 4^n the negative of this term into 4 ^{n} for all values of n and this is possible if and only if the individual coefficients are = 0 if we do that then we obtain two equations 25 C - 40 C + 16 C - 8 = 0 which implies C = 8 and the next one involving D reduces to 4 D x 16 -8 D x 4 = 6 and if we solve it then we will get D = 3 /16.

Therefore at the end we are getting the general solution of the homogeneous sorry the general solution of the non-homogeneous equation as a n = the homogeneous solution that is $a^{4\,n}$ + B n ^{4n} and then the particular solution of the original equation which is C here C is 8 x 5 ^{n} + 3 / 16 times n ^{2} x 4 $^{n.}$ Now if somebody gives me the values of a 0 and a 1 that is initial conditions then we will again have two equations involving A and B and then we can solve those two equations and obtain a solution of the of this recurrence relation well let us try that, so suppose somebody tells me that a 0 =12 and a 1 = 5 then we will have 12 = a 0.

Now put 0 if you put 0 then you will this term will finish and so will this term and 5^n will be 1 4^n when n is = 0 will be 1 so we will have a + 8 and this will mean that a = 4 all right, now we consider a = n = one if n = 1 we are told that a 1 is 5 a 1 I already know that a = 4. so I put 4 over here and another 4 which gives me 16 + now this is B times 4 + this is 8 times 5 ^{1,} so 8 x 5 is 40 + 3 / 16 x 4.

Now if we solve this equation then we will get V = -of to 0 7 / 4, thus under this initial conditions we obtain the solution of the recurrence relation as 4 times 4^n -207 / 4 n^n n x 4^n + 8 times 5^n 3 / 16 times n 2 4^n this is what we were looking for thus in this lecture we started by.

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