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Discrete Mathematics

Module-01
Set theory
Lecture-04
Application of the principle of inclusion and exclusion

With
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In today's lecture we will study some applications of the principle of inclusion and exclusion.

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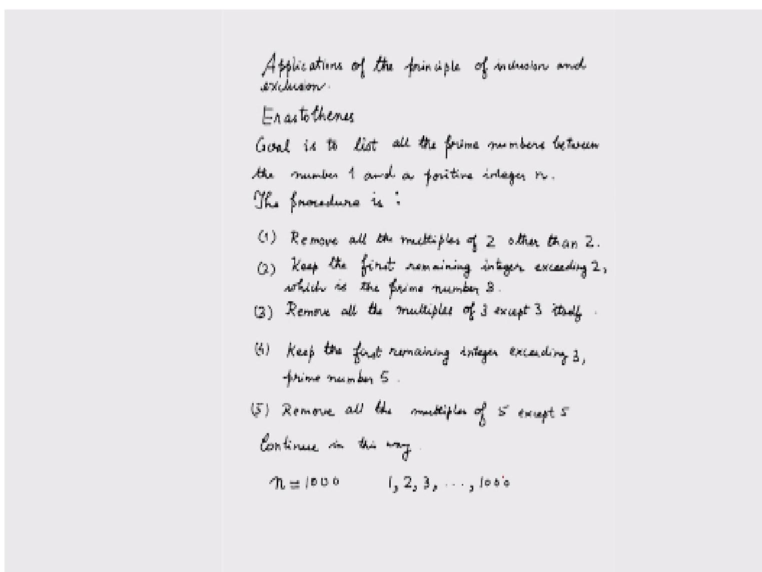

Now first let us look at this example which is based on the sieve of Erastothenes. Now the Greek mathematician Erastothenes developed a technique of listing all the prime numbers between 1 and any positive integer n. So our goal is to list all the prime numbers between the number 1 and a positive integer n. The procedure is as follows, one remove all the multiples of two other than two.

Keeps the first remaining integer exceeding two which is the prime number three? Third step remove all the multiples of three except three itself four keep the first the remaining integer exceeding three which will be the prime number. Then remove all the multiples of five except five we have to continue in this way. What happens is that if we take a positive integer n let us say n = 1000.

So we are looking at positive integers from 1 to 1000, and if we keep on repeating this process, then ultimately we will we will be left with the prime numbers between 1 to 1000. Now our problem is derived from this method which is called the sieve of Erastothenes, so let us look at the problem.

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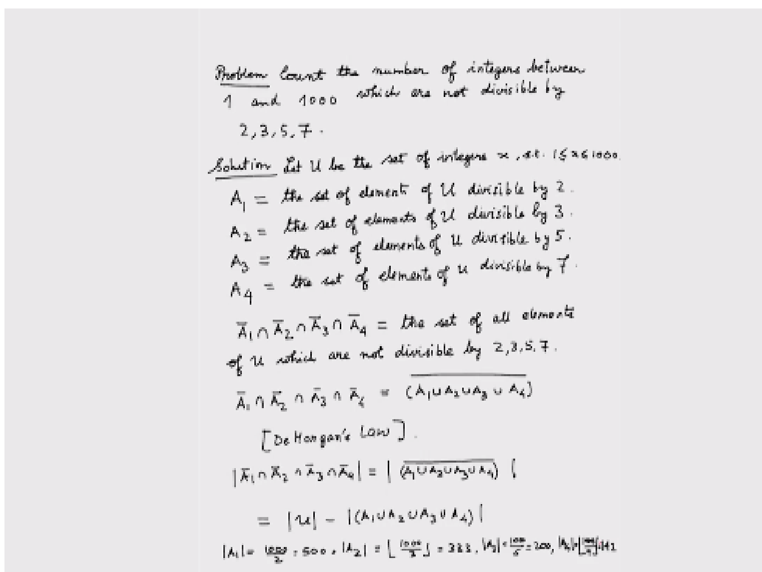

Count the number of integers between 1 and 1000 which are not divisible by 2, 3, 5, and 7. In order to solve this problem we consider certain sets, first let U be the set of integers X such that 1 less than or equal to X less than or equal to 1000. Now we define some subsets of U A1 equal to the set of elements of U divisible by 2, A2 the set of elements of U divisible by 3, A3 the set of elements of U divisible by 5, A4 the set of elements of U divisible by 7.

Now we are looking at the set of integers between 1 and 1000 which are not divisible by 2, 3, 5, and 7. We have in the beginning constructed four sets which are in fact subsets of U, the integers between 1 and 1000 namely A1, A2, A3, A4, where A1 consists of all the elements which are divisible by 2, A2 the set of elements divisible by 3, A3 elements divisible by 5, and A4 elements divisible by 7.

Now if we consider the set A1 complement this is the set of all the elements in U which are not divisible by 2. A2 complement is a set of all elements of U not divisible by 3. A3 complement is a set of all elements of U not divisible by 5 and A4 complement set of all elements of U not divisible by 7. Now that means that our set under consideration is intersection of all these compliments and this gives me the set of all elements in U which are not divisible by 2, 3, 5, 7.

Now we can process this a little further by considering this is in fact A1UA2UA3UA4 and the complement this is by using Demorgan's law. So the cardinality of A1 compliment ∩A2 compliment ∩A3 compliment ∩A4 compliment is the cardinality of the compliment of A1UA2UA3UA4 which in turn is equal to the cardinality of U which is the universal set minus the cardinality of A1UA2A3A4.

Now we will quickly calculate the cardinalities of A1, A2, A3, A4 and the cardinalities of AI's, intersections of AI's taken 2 at a time 3 at a time and all at a time, and then use principle of inclusion and exclusion to get the cardinality of the union of A1, A2, A3, A4. So we start our process by checking the cardinality of A1 which is 1,000/2 500, A2 which is floor of 1000/3=333, by the way floor of a real number is the largest integer less than that real number.

Then A3 1000/5 which is 200, and A4 which is the floor of 1000/7 which gives us 142. Then we take intersections of AI's for distinct I's they can two at a time.

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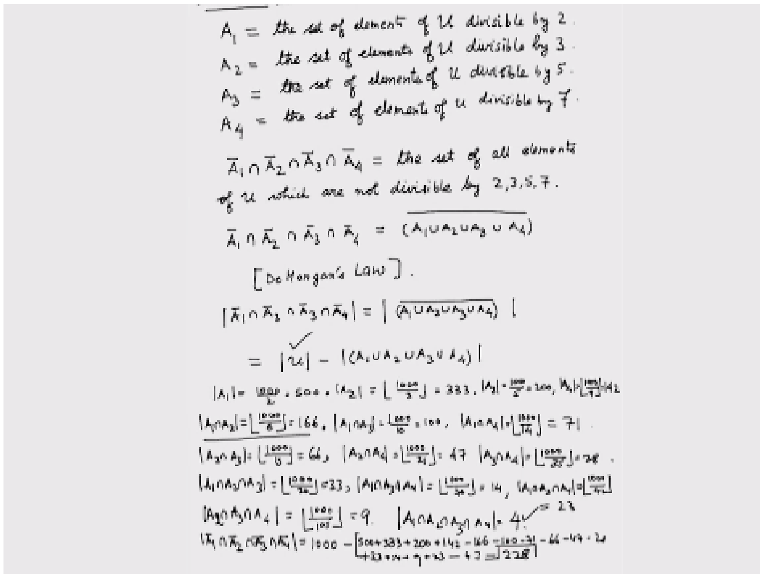

A_1 = the set of element of $\mathcal{U}$ divisible by 2.

A_2 = the set of elements of $\mathcal{U}$ divisible by 3.

A_3 = the set of elements of $\mathcal{U}$ divisible by 5.

A_4 = the set of elements of $\mathcal{U}$ divisible by 7.

$\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4$ = the set of all elements of $\mathcal{U}$ which are not divisible by 2, 3, 5, 7.

$\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4 = \overline{(A_1 \cup A_2 \cup A_3 \cup A_4)}$

[De Morgan's Law].

$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| = |\overline{(A_1 \cup A_2 \cup A_3 \cup A_4)}|$

$= |\mathcal{U}| - |(A_1 \cup A_2 \cup A_3 \cup A_4)|$

$|A_1| = \frac{1000}{2} = 500$, $|A_2| = \lfloor \frac{1000}{3} \rfloor = 333$, $|A_3| = \frac{1000}{5} = 200$, $|A_4| = \lfloor \frac{1000}{7} \rfloor = 142$

$|A_1 \cap A_2| = \lfloor \frac{1000}{6} \rfloor = 166$, $|A_1 \cap A_3| = \frac{1000}{10} = 100$, $|A_1 \cap A_4| = \lfloor \frac{1000}{14} \rfloor = 71$

$|A_2 \cap A_3| = \lfloor \frac{1000}{15} \rfloor = 66$, $|A_2 \cap A_4| = \lfloor \frac{1000}{21} \rfloor = 47$, $|A_3 \cap A_4| = \lfloor \frac{1000}{35} \rfloor = 28$.

$|A_1 \cap A_2 \cap A_3| = \lfloor \frac{1000}{30} \rfloor = 33$, $|A_1 \cap A_3 \cap A_4| = \lfloor \frac{1000}{70} \rfloor = 14$, $|A_1 \cap A_2 \cap A_4| = \lfloor \frac{1000}{42} \rfloor = 23$

$|A_2 \cap A_3 \cap A_4| = \lfloor \frac{1000}{105} \rfloor = 9$. $|A_1 \cap A_2 \cap A_3 \cap A_4| = 4$

$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| = 1000 - [500 + 333 + 200 + 142 - 166 - 100 - 71 - 66 - 47 - 28 + 33 + 14 + 9 + 23 - 4] = 228$

So we get A1∩A2 is equal to floor of 1000/6 because if a positive integer is divisible by both 2 and 3, then of course it is divisible by 6 and the converse. Therefore we will have 166 A1∩A3 this gives me 1000/10 which is 100 and A1∩A4 for which is floor of 1000/14 = 71, then A2∩A3, A2∩A4 and lastly A3∩A4 which is floor of 1000/35 = 28.

Then we have to take the intersections taking three at a time so I will have A1∩A2∩A3, so these are precisely the elements which are divisible by 2, 3, and 5. Therefore, divisible by 30 so therefore it will be 1000/30 floor of that which is 33, then I have got A1, A3, A4 which is 1000/70=14 and we have A1, A2, A4 which is 1000/42=23, and finally A1 this will be A2, A3 and A4 which is floor of 1000/105, so it is 9.

And the last one taking 4 at a time is and we can check that this is just the number 4. Now if we remember all these things then we can see that the cardinality of A1 complement ∩A2 complement ∩A3 complement ∩A4 complement is cardinality of U this one which is of course 1000 minus cardinality of A1 500 plus cardinality of A2 333 + 200 + 142 these are the cardinalities of A1, A2, A3, A4.

Then subtract from this one the cardinality of A1∩A2 which is 166, - 100, - 71, - 66, - 47, - 28, and then we start adding we add 33 + 14 + 9 so and + 23. And then again subtract the last expression that is 4, if I do this then the number that I get is 222. And this is the number of integers between 1 and 1000 which are not divisible by 2, 3, 5, and 7. Thus in this example we see how we are using the principle of inclusion and exclusion to count some number of some things. We move on to more serious examples, and this example involves Euler's ϕ function.

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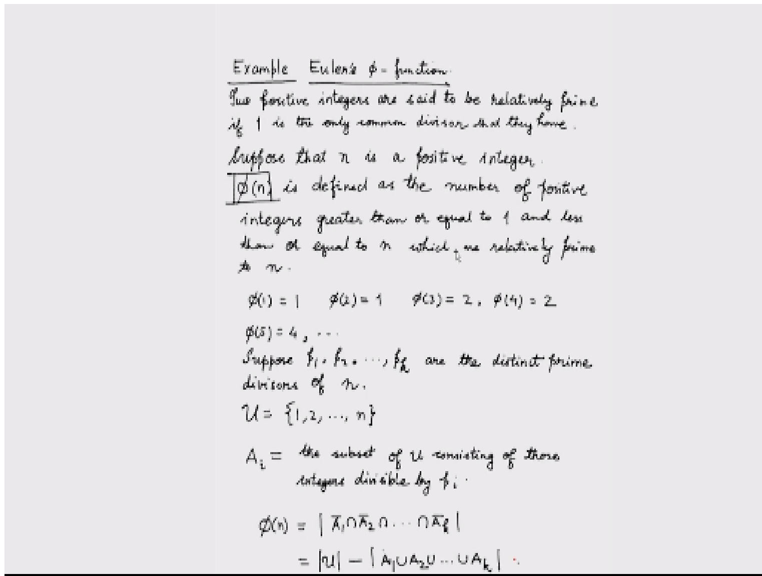

Now the first question is what is Euler's ϕ function for that, first of all we have to know what do we mean when we say that two positive integers are relatively prime to one another. Let me write the definition first two positive integers are said to be relatively prime if the number one is the only common divisor that they have. Now suppose n is a positive integer ϕn is defined as the number of positive integers greater than or equal to 1 and less than or equal to n which hard relatively prime to n.

So in simple words we take a positive integer N and we count the number of positive integers between 1 and n which are relatively prime to n and this number is called the ϕn. Now what we are interested here is to get a get an expression of ϕn which does not seem to be very easy. If we start checking some small examples and we see that ϕ1 is of course 1, ϕ2 is also 1, ϕ3 is 2, ϕ4 is also 2, because the positive integers less than 4 is 1, 2, 3, and 4 here one is of course relatively prime to 4, 2 is not relatively prime to 4, and 3 is relatively prime to 4 and of course 4 is not relatively prime to 4 so we have got we say ϕ4 is 2, then ϕ5 is 4 and so o.

So as such there is no direct pattern that that is obvious. So we have to find out an expression of ϕ if at all it exists incidentally this function is called the Euler's ϕ function. Now suppose P1, P2 up to Pk are the distinct prime divisors of M. We consider the universal set 1, 2, up to n and denoted by U. We also consider a set like this A_i which is the subset of U consisting of those integers divisible by Pi.

So we are looking for integers which are in U and not divisible by any of the Pi therefore ϕn is equal to cardinality of A1 complement ∩A2 complement ∩ and continued in this way up to Ak complement. We can manipulate and get this equal to U minus cardinality of U-A1UA2U and so on up to Ak. Now again we see that the expression that we are getting is almost similar to the expression that we got in the last example. Only thing is that we have to know how to count the cardinalities of AI's and intersections of different AI's.

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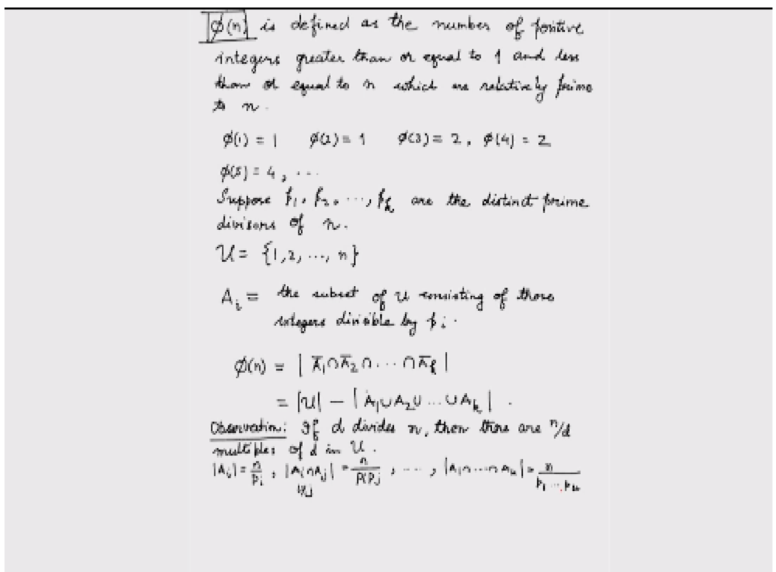

We base this on an observation if D divides n, then there are n/d multiples of D in U. This can be verified and I leave it as an exercise, but if we take it to be true which of course we can verify then we will get AI = n/Pi Ai∩Aj where I is not equal to J, equal to n/Pi Pj and proceeding in this way finally we will get A1∩ and so on up to ∩Ak = n/P1…Pk.

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$$\phi(n) = n - \left[\sum_{i=1}^{k} \frac{n}{p_i} - \sum_{\substack{i=1\\ j=1\\ i<j}}^{k,k} \frac{n}{p_i p_j} + \cdots + (-1)^{k-1} \frac{n}{p_1 \cdots p_k} \right]$$

$$\sum_{i=1}^{k} |A_i| \qquad \sum_{i<j} |A_i \cap A_j| \qquad |A_1 \cap \cdots \cap A_k|$$

$$= n - \sum_{i=1}^{k} \frac{n}{p_i} + \sum_{i<j} \frac{n}{p_i p_j} - \cdots + (-1)^{k} \frac{n}{p_1 \cdots p_k}$$

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right).$$

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

$\alpha_i \geq 1$. and p_i's are distinct prime numbers.

And therefore, considering all this we will get ϕn equal to n which is the cardinality of U and then -∑i=1 to k n/Pi this is essentially ∑i=1 to k cardinality of Ai and I put a minus over here to obtain i=1 to k n, I have to put i=1 to k and j=1 to k with a condition that I is always less than j. So this is Pi Pj and we will proceed in this way to ultimately the last expression, this is the cardinality of Ai, I am sorry A1∩ and up to a k.

And this intermediate second entry is essentially high less than J ∩Ai∩Aj cardinality. Thus we have basically used the Pj, I am sorry, we have basically used the principle of inclusion exclusion in the last part of the right hand side to plot an expression. Now we can process this further and write n -∑i=1 to k nPi + less than j nPixPj – and so on, at the end it is -1^{k} n P1 up to Pk. And the careful analysis shows that this is equal to n (1-1/P1)(1-1/P2) and so on up to n(1-1/Pk).

And thus finally we have got an expression for ϕn which is ϕn = n(1-1/P1)(1-1/P2) and so on up to 1-1/Pk, where n=$P1^{\alpha 1}$, $P2^{\alpha 2}$ and $Pk^{\alpha k}$ where αi's are greater than or equal to 1, and PI's are distinct prime numbers. Euler's ϕ function plays an important role in number theory and many other applications of number theory. This example gives us an instance where the principle of inclusion exclusion gets used in finding out a very fundamental function of number theory which is the Euler's ϕ function. Next we will talk about counting certain kind of permutations by using the principle of inclusion and exclusion.

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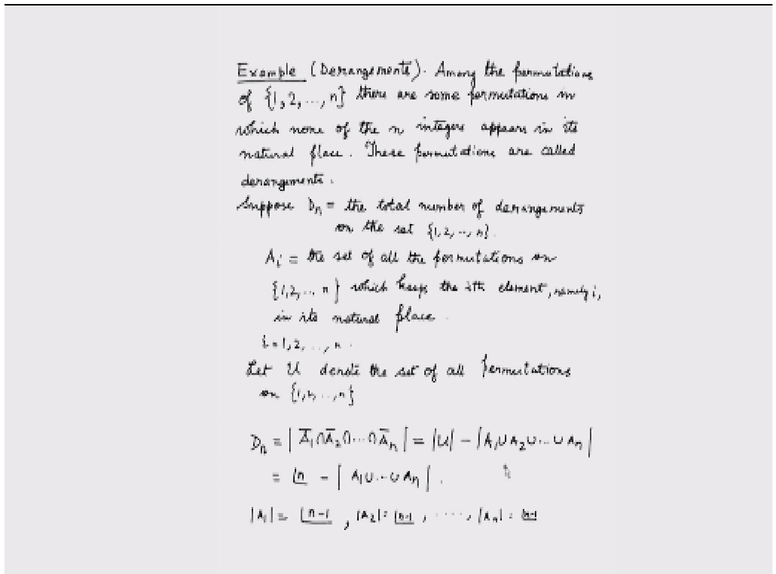

Now these permutations that we are going to study are called derangements. Let me start by defining derangements among the permutations of the numbers from 1 to n, there are some permutations in which none of the n integers appears in its natural place. Now these permutations are called derangements. Now what we would like to do is to count the number of derangement of n numbers from 1 to n suppose dn is equal to the total number of derangement on the set 1, 2, up to n.

Now just like the previous examples we are going to define some sets. So in general we define Ai equal to the set of all the permutations on 1, 2, up to n which keeps the i^{th} element namely I in its natural place. And of course I will move i from 1, 2, up to n. Let you denote the set of all permutations on 1, 2, up to n just to recall that this means that U is the set of all 1 to 1 on 2 functions from 1, 2, up to n to 1, 2, up to n.

Now from the discussions that we have done before it is now clear that dn is equal to evil complement ∩A2 complement ∩ and so on up to n complement, and which again in exactly similar way as before can be written as cardinality of U minus cardinality of A1UA2U and so on up to AN. We know that the cardinality of U is factorial n, and therefore we have to just find the cardinality of A1U and so on up to cardinality of AN.

For that we will start checking the cardinality of A1 which is factorial n - 1 the reason is that when I am counting the number of permutations or the number of arrangements that I can make out of elements from 1 to n where first element is in the first position, then I can move around the other n - 1 elements in any way I like. So I can do that in factorial n - 1 ways therefore cardinality of A1 is factorial n, and the question is that how many AI’s are there are n choose 1 many that is n many AI’s. So cardinality of A2 is going to be also n - 1 and so on up to cardinality of AN = n choose n – 1.

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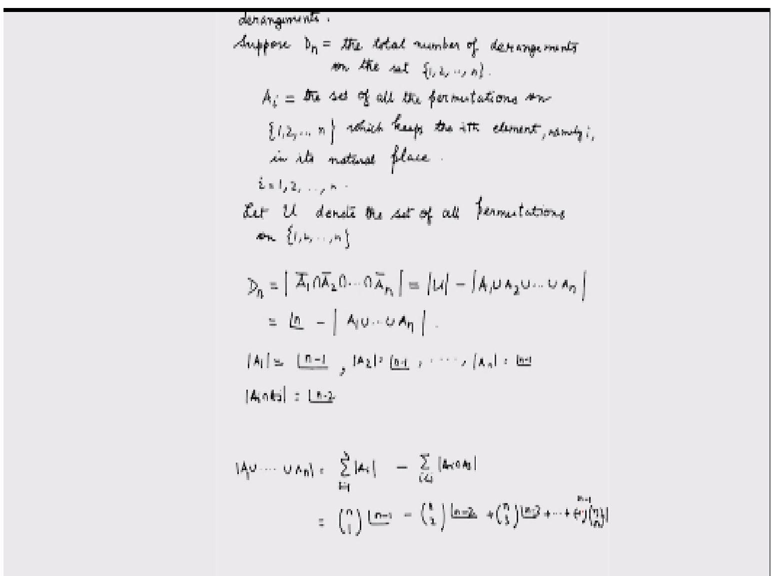

Therefore, if I am considering the cardinality of the union A1UAN the first term which is ∑i=1 to n cardinality of AI this will be n choose 1 n - 1 that is n into cardinality of n – 1, because all the AI’s have the same cardinality. The second term is going to be higher less than J, AI ∩AJ the question is that how many times I can choose these two distinct AI's from n distinct AI.

So that number of times is n choose 2 then the first question is that what is the cardinality of AI∩AJ and that happens to be n - 2 factorial, because after all I am fixing the i^{th} element to the i^{th} place and j^{th} element to the j^{th} place. So I have got n minus too many elements left which we can move around anyway we like. Therefore we will get n choose 2 into factorial n – 2, and then further on I will have n choose 3 factorial n - 3 and so on. And at the very end I am going to get -1^{n-1} n choose n of 1.

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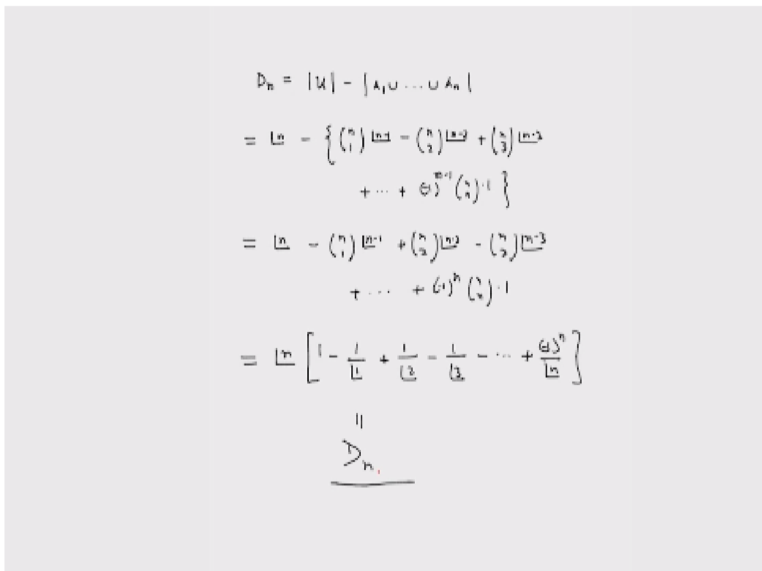

Now if we go back to the expression that we started writing of dn we wrote that dn is equal to cardinality of U minus the cardinality of A1U and so on up to AN which means that dn is factorial n - n choose 1 factorial n - 1 - n choose 2 factorial n - 2 + n choose 3 factorial n - 3 + which is equal to factorial n - n choose 1, and if you process it further we will get the final result as factorial n(1 - 1 choose 1 + 1 choose factorial 2 - 1 choose sorry 1 – 1/factorial 1 + 1/factorial 2, 1/factorial 3 and so on.

And at the end we will have – 1^n factorial n. This is the final result for the number of derangements that we have on n positive integers from 1 to n. In this lectures we have studied three examples in which principle of inclusion and exclusion has been used to solve certain counting problems. And some of these problems are very fundamental to combinatorics and number theory, we stop the lecture now thank you.

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