INDIAN INSTITUTE OF TECHNOLOGY ROORKEE

NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)

Discrete Mathematics

Module-08
Boolean Algebra and Boolean function

Lecture-02
Boolean function (1)

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In today's lecture we will be discussing Boolean functions.

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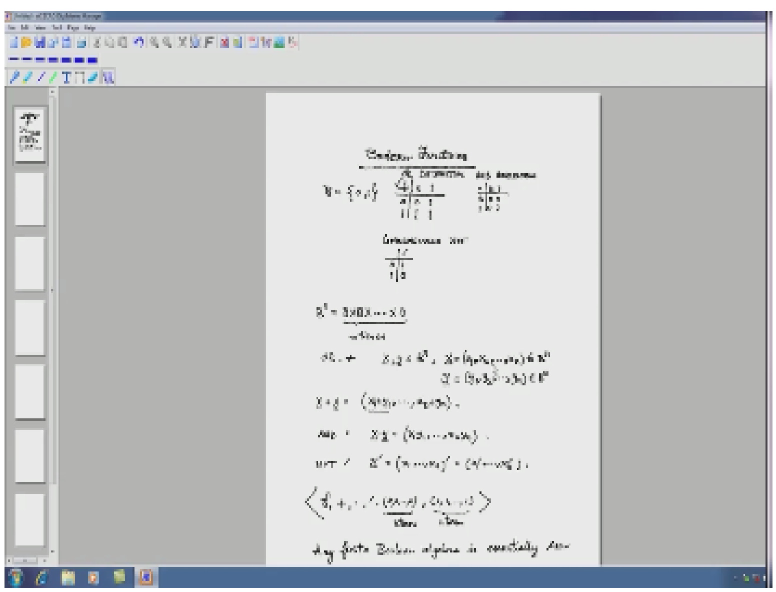

We have already discussed in previous lectures Boolean algebras but still to start with I will recall the definition of a Boolean algebra we know that a Boolean algebra is a lattice which is complemented distributive having at least two elements and also having least element and the greatest element. Apart from that it has three basic operations one called disjunction or plus and the other being the conjunction the end or product and the complementation operation.

Now the probably this simplest Boolean algebra consists of the set (0 1) with the operations defined as this one which is the or disjunction or simply we may call it plus and then another operation which is called end or conjunction which is denoted usually by a simply a dot or by juxtaposition of two elements, which is defined as this and complementation which is simply this usually denoted by A' and 0' is 1 and 1'is 0 sometimes it is also called 0.

Now as we have seen before that a Boolean function this Boolean algebra consisting of only 0 1 can be extended to B^n which is the Cartesian product of n copies of B and the operations disjunction conjunction and complementation or R and 0 are defined as this R+ consider two elements X bar and y bar inside BN therefore X bar x-bar is of this type X 1 X 2… xn belonging to BN and y bar is y 1 y 2… y n belonging to R be n X bar plus y bar is X 1 plus y 1, and so on up to xn + yn.

Now this plus operation is same as the plus operation that we have defined before over B similarly I can define and or just the dot has X bar and y bar = X 1 dot y 1 and so on up to xn yn not that is a complementation denoted by the simple symbol prime is X bar prime equal to x1 up to xn' which is again x1'… xn', it is fairly easy to check that BN along with the operations just defined which are induced from the operations defined on B is a Boolean algebra for any positive integer value of M.

So if we are very strict about that then we can write the Boolean algebra system involving BN as BN then this plus the dot the Prime and of course the least element which is the all 0 vector having n terms and the greatest element which is the all 1 vector having n terms, combining all these things we have the Boolean algebra bn. In the beginning it seems that this is a very restricted class of Boolean algebra, but we can show that any finite Boolean algebra is essentially same as BN we are not going into a theoretical proof of that or a theoretical definition of what is called isomorphism of Boolean algebras.

But we shall write the statement over here that any finite Boolean algebra is essentially same as BN for some positive integer value n.

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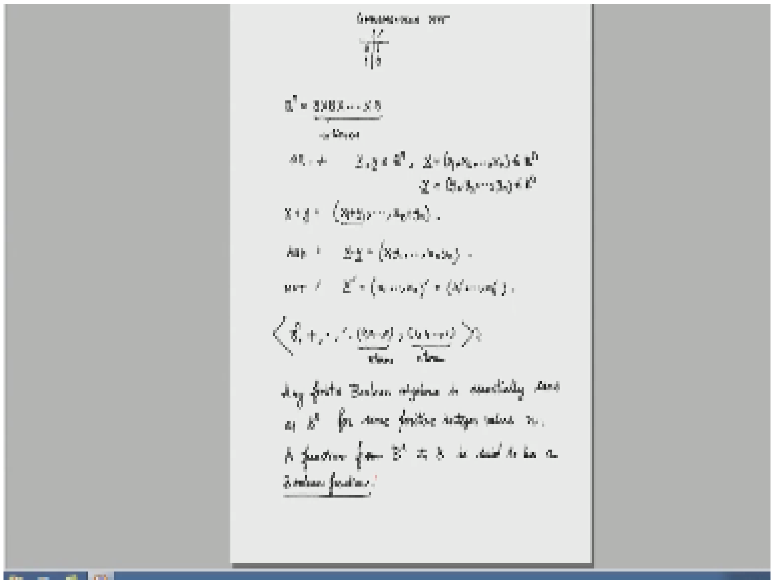

Now we want to define functions over Boolean algebras, the first type of functions that we define and which are called Boolean functions or functions from BN to be a function from BN to B is said to be Boolean function and that is what we are going to study today. Now let us start with examples of Boolean functions, to start with Boolean functions are very simple.

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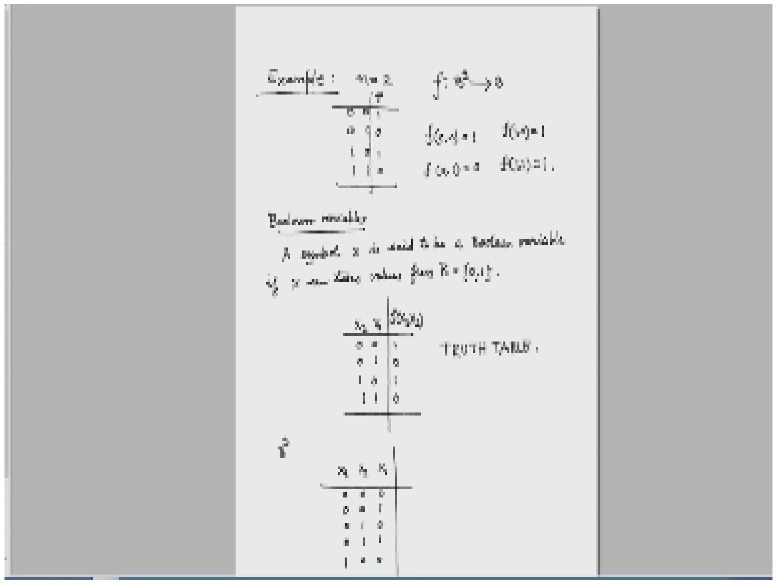

So let us consider possibly one of the most simple examples of Boolean functions here we consider n = 2 then we can enumerate very easily all the points of BN that is B2 those are 0 0 0 1 1 0 and 1 1, we can list them in a table and since we are interested in a function from B 2 - B we can write a rightmost column in the table some values from B.

So it can be just 1 0 1 0, so suppose this column I am designating by the function f which is essentially the function that we are defining from B 2 to B we can say that this function takes the value 1 when the input is 0 0 it takes the value 0 when the input is 0 1 it takes the value 1 when the input is 1 0 and it takes the value 0 when the input is 1 1. At this point we define something else we define what we call a Boolean variable, so a symbol X is called a Boolean variable if it can take the values 0 or 1.

So we are essentially defining variables that are the set B. A symbol X is said to be a Boolean variable if X can take values from B which is essentially the set containing 0 and 1. Now we look again at the table that we have already discussed in the context of the function f the each coordinate point of B 2 can be assigned to a Boolean variable. So we rewrite the same table like this, we assign the Boolean variable x1 to the leftmost coordinate and x2 to the rightmost coordinate.

So we get the values like this 0 0 0 1 1 0 & 1 1, and now the function f can be thought of as a function on X 1 and X 2 and its values are 0 1 1 0 a table like this for a Boolean function is said to be its truth table, it is of course obvious that if we have a function from B^n to B for any positive integer value of n then we will be able to write the truth table of that function, at this point it is worthwhile to to check the ordering in which we write the values of the vectors of BN.

So we look at B3 suppose X_1 X_2 X_3 are the Boolean variable corresponding to the coordinates of the vectors in B3 then we will write the values in this order first we will write the all 0 vector then we will write 0 0 1 then we will write 0 1 0 and lastly we will write 0 1 1 and then we will write 1 0 0 then 1 0 1 and then 1 1 0 1 1 1.

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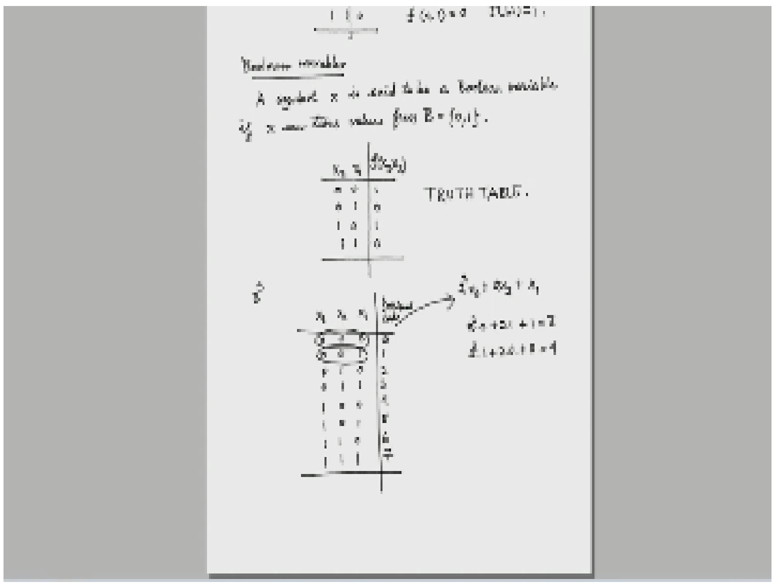

In what follows we will always write the table in this order this has a very useful connection to decimal numbers what we can do is that we can say that each of these vectors or strings or elements can be related to a number which is given by X_1 + 2 times X_2 + 2^2 times X_3. Now let us evaluate each and every point when it is 0 then of course the result is 0 when it is only one that is alt is 1 but when it is 0 1 0 then it is 2 when it is 0 1 1.

Now let us check when X_3 is 0 so that is 2^2 x 0 then X_2 is 1 that is 2 x 1 + X_1 is 1 that is 1, so this gives me 3, so this is 3 and the next one is 2^2 x 1 + 2 x 0 + 0 = 4 continuing in this way we will see that the rest of the elements correspond to the decimal numbers 5 6 & 7 so this is something that we will be calling decimal code. Now of course we can extend this and go to the situation when there are n Boolean variables.

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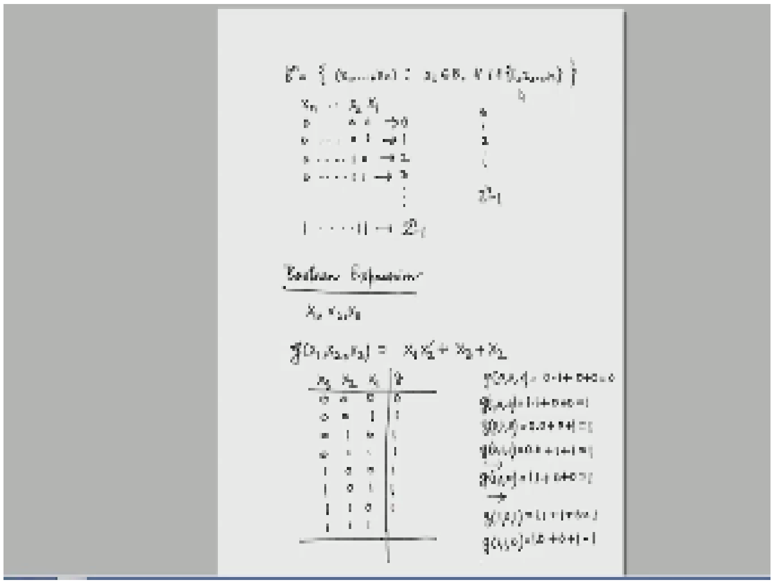

So B_n is typically the set consisting of X_1 and so on up to X_n where X_I belongs to B for all I belonging to 1, 2 so on up to n, now we will be writing the elements of BN in the order which is exactly the extension of what we have seen before, so the first one will be all 0 so the decimal coding will be 0 then it will be all 0 and 1. So the decimal coding will be 1 then it will be all 0 and 1 0 so the decimal code will be 2 and then all 0 and 1 1, so the decimal code will be 3 it will go on in this way and eventually we will have the pattern which consists of only all 1 and this is in this ml code it will be 2^n-1.

And if we see carefully this will give us all the positive integers from 0 1 1 to 2 $2^n – 1$, we will be using this ordering always in this in these lectures and otherwise also in the theory of Boolean functions and Boolean algebra this ordering is predominantly used. Now what we realize coming to this point is that we can talk about something called Boolean expressions, a Boolean expression is any formula that we can build up by using Boolean variables plus dot and complement that is Boolean variables or and compliment or conjunction, disjunction and complementation.

So for example we can suppose we fix the number of Boolean variables to be three so we are considering here X_1 X_2 and X_3 only we can build up any expression let us call one expression like this as G X_1 X_2 X_3 which is equal to possibly X_1 X_2 compliment plus $X_3 + X_2$, now what we can do is that we can put as inputs the values of X_1 X_2 X_3 from the Boolean algebra B_3 and see what happens let us try that now.

So I write X_1 then X_2 and X_3 I put 0 0 0 then the next entry is 0 0 1 the next is 0 1 0 the fourth one is 0 1 1 v 1 is 1 0 0 & 1 0 1 then 1 1 0 and then 1 1 1 so I have got all the 8 entries over here and I will put the inputs in G according to that the first one is 0 0 0 0 0 0, so here X_1 is 0 X_2 is 0 complement so 1 + X_3 is 0 and X_2 is 0 so it is 0 so I will write it over here. So I am designating the right-hand column has the function G then let us consider G of now X_1 is 1 rest are zeros now X_1 is 1 and see here X_2 is 0 so 0 complement is 1 therefore I have 1 over here + X_3 is 0 and X_2 is 0 now this gives me 1.

So I write 1 over here and the third one is 0 1 0 here X_1 is 0 then X_2 compliment is also 0 then X_3 is 0 and X_2 in this case X_2 is 1 therefore again I will get 1. So I write 1 over here, now we come to the fourth one that is 0 1 1 remember that we are starting from this way onward so X_1 is 0 X_2 is 1 complement so it is 0 then X_3 is 1 then X_2 is 1 so I again get 1 I write 1 over here then I have 1 0 0 so X 1 remembering that I am $X_{1 2}$ X_3 is in this direction.

So X_1 is 1 X_2 is 0 complement so it is 1 then X_3 and X_2 both are 0 so I get 1 I right over here then CG 1 0 1 so it is X_1 is 1 X_2 complement is 1 then X_3 is 1 X_2 is 0 this also is 1.

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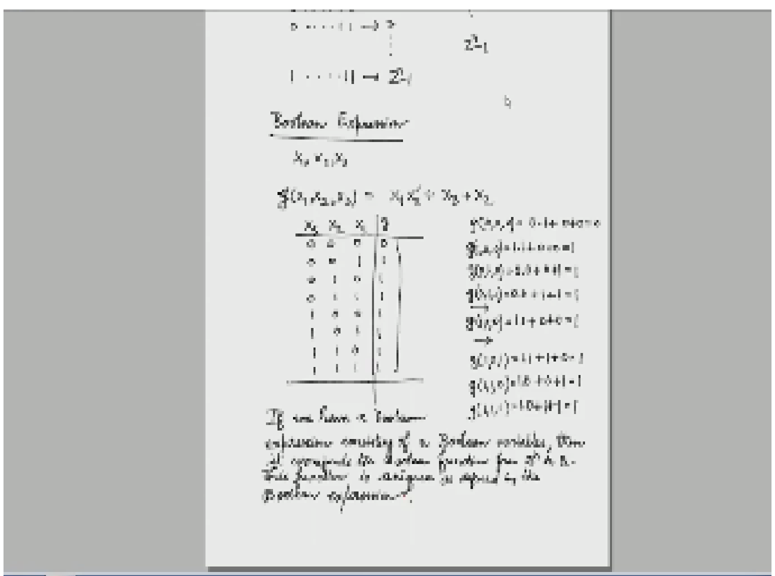

And then G 1 1 0 is X_1 is 1 X_2 complement is 0 then X_3 is 0 then X_2 is 1 this is again 1 and G 1 1 1 we get X_1 is 1 X_2 complement is 0 X_3 is 1 and X_2 is 1, so again I have got 1 so I get a pattern like the one I have got in the right hand side but this is a truth table of a Boolean function therefore what we have seen is that if we have a Boolean expression then we have a Boolean function. If we have a Boolean expression consisting of N Boolean variables then it corresponds to Boolean function from BN to B and of course this function is unique, so we see that a Boolean expression corresponds to a Boolean function.

Now the question is the other way around if we have a Boolean function can we construct a Boolean expression the answer is yes and we will see an example how it works.

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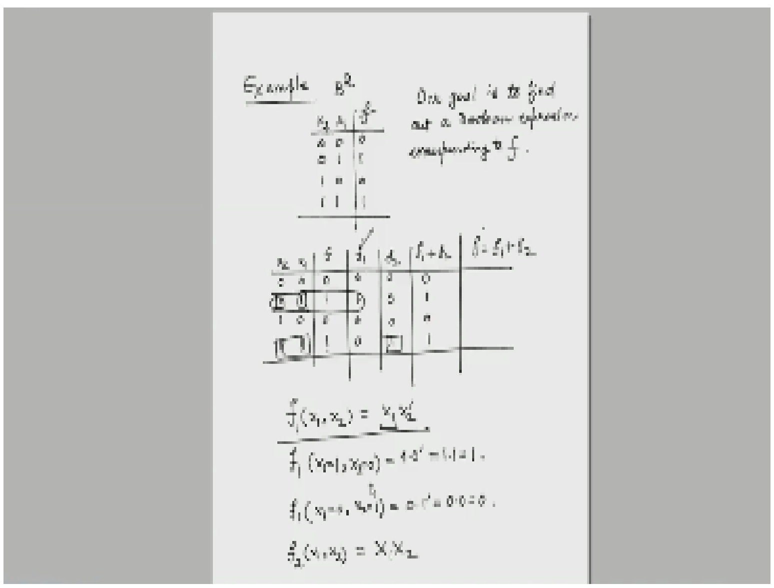

So we now consider our small Boolean algebra B_2 so not BN but just B_2 and we write down an arbitrary truth tables involving B_2 suppose X_1 is again the leftmost variable and X_2 is the next variable so the input values as 0 0 0 1 1 0 and 1 1 and let us consider the function 0 1 0 1, let us call it F, our goal is to find out a Boolean expression corresponding to this function. So I write here our goal is to find out a Boolean expression corresponding to earth what we do here is a very standard trick we extend that table a little bit here again I have got X_1 and X2 I write the input values and this is my function F so this is 0101 I write some so-called sub functions maybe yeah, so these are F_1 and F_2 so what I have done over here is that for each one in the truth table of the function f I have constructed a function.

So for this one I have a function F_1 for the next one I have a function F_2 and if we take $F_1 + F_2$ then we will see that we will arrive at the function f because $F_1 + F_2$ the first row is 0 the second one is 1 then 0 and then 1, it is of course clear to us that F is equal to $F_1 + F_2$. Now the question is that can we find out an expression for A_1 and F_2 if we can do that then we have a an expression for F to do that we concentrate on the function F_1 we see that if 1 is a function which is 1 at only this input point and otherwise it is 0 the question is that how can we do that and a simple observation shows that F_1 is nothing but x_1 and x_2 compliment.

What I am doing over here is that I am checking the point at which if one is one at that point I am checking the corresponding Boolean variables we see that corresponding to x_1 the component value is 1 therefore X_1 is kept as it is I see that corresponding to X_0 the component value is 0 therefore I take X to complement. Now if I give the input 0 1 2 this function say F_1 I put X_1 equal to 0 I am sorry I put here X_1 equal to 1.

So I have got Xx_1 equal to 1 and X_2 equal to 0 this gives me 1.0 complement that is 1.1 = 1, so at least I know that this expression that I have written this expression evaluates to 1 at the point $X_1 = 1$ and $X_2 = 0$. Now if we scan all the other points we will see that this pattern will never match, therefore in other places either X 1 will be 0 in that case is F_1 will be 0 or X_2 will be 1 in that case X_2 complement will be 0 for example if I evaluate the same function f 1 at let us say X_1 equal to 0 and X_2 equal to 1 this will evaluate to 0.1 complement which is 0.0 which gives me 0 we can check that this is going to happen in all the other input points.

Now so what is the rule of getting these functions these functions which we may call component functions of F or at this point or whatever these functions are 1 only in one input points and 0 in all the other input points so check that pattern, and wherever the Boolean variable corresponding Boolean variable values are 1 keep the variables intact corresponding to the other entries just take the complement of those variables take the product of them and get a essentially a product terms of some Boolean variables and their complements.

Now in the same way we can get the expression for F_2 which is $F_2 X_1 X_2$ now please see that F_2 is 1 only at the input point 1 1 and so all the Boolean variables are 1 over here, so I will simply write this is $X_1 X_2$ and as we have seen if we take or R + of these two columns we get back our function F.

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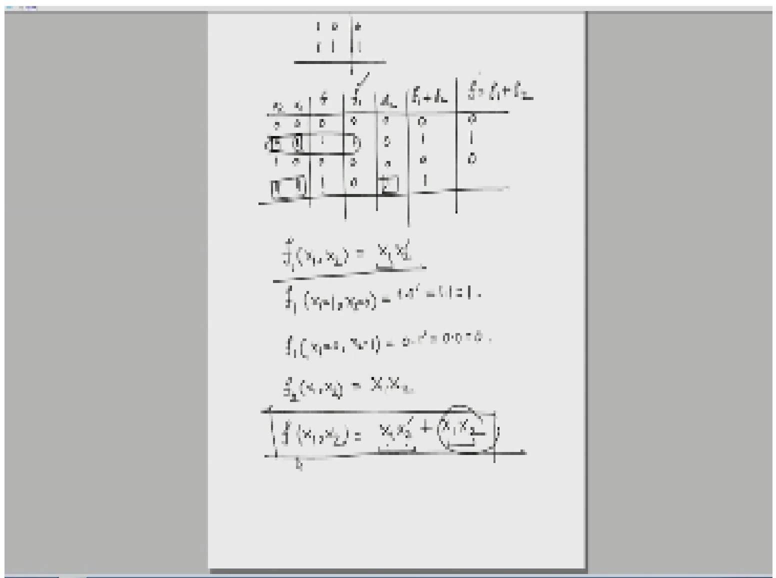

So we can write F $_1$ X_1 X_2 = X_1 X_2 complement X $_1$ +X $_1$ X $_2$ we do not write the 1 here because this is the whole function F. So we see that F is X_1 X_2 complement + X_1 X_2 we can evaluate this function here and check that it is indeed. So let us take the first point when both are zeros then X_1 equal to 0 X $_2$ equal to 1 and X_1 0 so I get 0 then when X $_1$ = 1 and X $_2$ = 0 then this is 1 and this is 1, so I get 1 over here when X $_2$ = 1 and X $_1$ = 0 then this is evaluated to 0 and this is also evaluated to 0 so I get 0 over here and when both are 1 they and this is one therefore I will get one over here.

So we see that the expression that we have got matches with the Boolean function, this example more or less convinces us that we can get a Boolean expression for any Boolean function on any number of variables. The basic rule is same just decompose the function into functions having one at only one place and correspondingly write the product terms and add them up at the end. Here what we have seen is that we are getting the Boolean expression in the form which can be expressed as sum of products.

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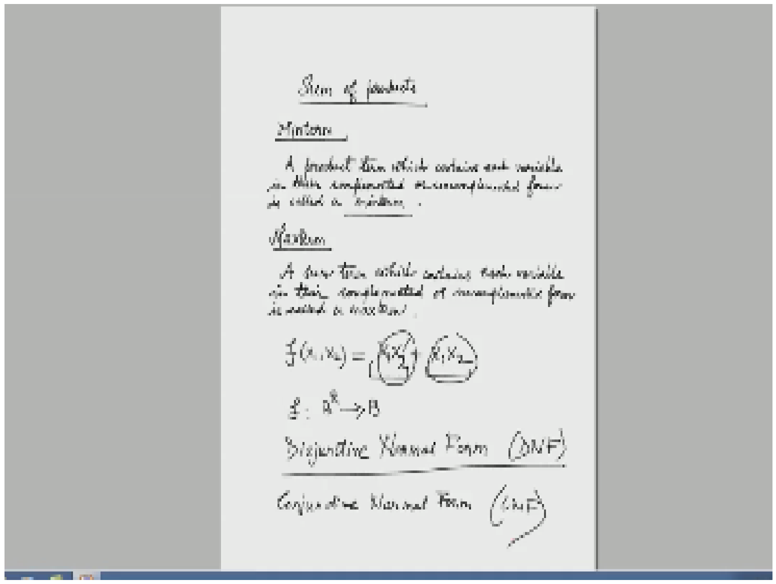

Indeed we sometimes call these expressions as sum of products; we also see a salient point over here that is each product term contains all the variables. So we come to a definition which is called min term, a product term which contains each variable in there complimented or uncomplimentary for is called a mean turn. Similarly we define something called max terms which we will be discussing shortly.

So I will define them in the same goal a max term is a sum term which contains each variable in there complimented or uncomplimentary form is called a max turn, what do you have seen just now is that the function if $X_1 X_2$ is $X_1 X_2$ compliment plus $X_1 X_2$ here the salient point of this expression is that this is 2 salient point of this expression is that it is some of min terms this function f is from B_2 to B so only two variables are involved and among these two variables both the variables are involved in all the product terms whose sums we are taking.

So these are mean terms and we are summing the mean terms to get the expression of the function and if such is the case then this expression is called a conjunctive normal form I am sorry a disjunctive normal form. So this is called disjunctive normal form or in terms DNF I repeat again if a Boolean expression is written as sum of mean terms then it is said to be a disjunctive normal form of the corresponding Boolean function.

Now we will soon see that we can write a function as product of some terms that is product of max terms in particular and that will be called conjunctive normal form or C n F to do that we will take up again the same function that we have discussed just before and try to find its conjunctive normal form or in other words try to write a Boolean expression which is product of max terms, so we look at the truth table again.

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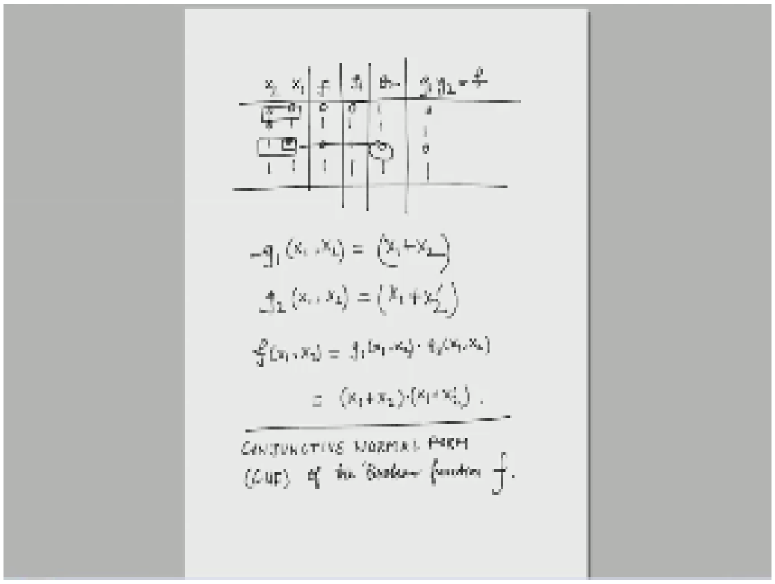

And we have 0 1 0 1 0 1 0 1, now what we do is that we now try to locate the zeros, so we have now a function G_1 which is 0 in the first place and 1 in all the other places and a function G_2 which is 0 in the third place and 1 in all the other places we will show that G_1 and G_2 can be written as sum of the Boolean variables which are involved in this Boolean algebra and if we take the product of G_1 and G_2 that is end of G_1 and G_2 then we will get 0 1 0 & 1 which is nothing but our original function f now the question is how to get that so our rule is as follows we have G_1 we write X_1 X_2 we want it to be 0 when the input is 0 0 in order to do that we just write $X_1 + X_2$.

Because this expression is 0 when both X_1 X_2 are 0 but if anything else happens and it is not 0 and it is in fact 1. Similarly we concentrate on this input vector which is gives me the value this so it is G_2 X_1 X_2 and this G_2 X_1 X_2 is going to be zero if I put well as X_1 intact because anyway X ones value is zero over here and instead of X_2 I could X_2 complement please check that these two are the functions that we need and the rule is very clear that we have only in one input value that we have to consider check each component if the component value is zero just write the corresponding Boolean variable in the sum if it is one right its complement if you do that you will get exactly the function you need and then F X_1 X_2 is G_1 X_1 X_2 . G_2 X_1 X_2 = X_1

+X 2 .X 1 + X 2 complement and this is what is called the conjunctive normal form of the function F.

Conjunctive insert in short CNF of the Boolean function thus we have seen two very important normal forms of Boolean functions. Now we will move on to another useful form which is of course very difficult to write when the function is large but when the function is not on a very large space we can write that and this relates to the decimal codes that we have discussed before we will check this way of representation by an example.

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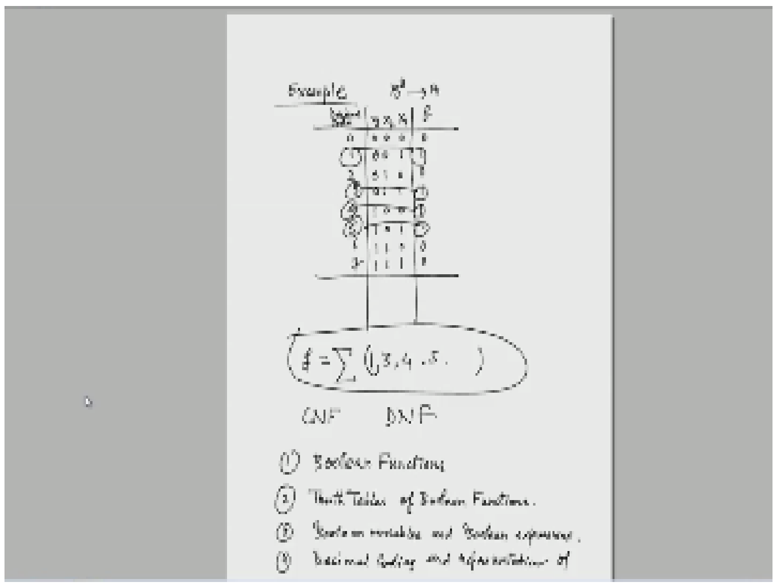

So we have an example on a function from B_3 to B and let me write down all the all the points of B_3 along with the decimal coding now we write the variables as X_1 X_2 and X_3 as before, so I write 0 0 0 0 0 1 0 1 0 0 1 1 1 0 0 1 0 1 1 1 0 and 1 1 1 and now I write the decimal code which is 0 1 2 3 4 5 6 & 7 and suppose my Boolean function if is something like this 0 1 0 1 1 1 0 0 what I can do is that I can simply write this function with a summation notation and put within bracket all the points at which the function is 1.

So it is 1 and 1 so I write 1 over here and then it is 1 at 3 so I write 3 over here one at 4 I write 4 over here 1 at 5 I write 5 over here of course this representation may not be convenient if the Boolean function is on a large number of variables but for a small number of variables it's going to be convenient and a very common problem is to write the CNF or the DNF of a function which is given in this way and this is what I would like to propose as an exercise to you.

So in this lecture we have discussed one Boolean functions two truth table then we have discussed Boolean variables and Boolean expressions we have discussed decimal coding and representation of Boolean functions.

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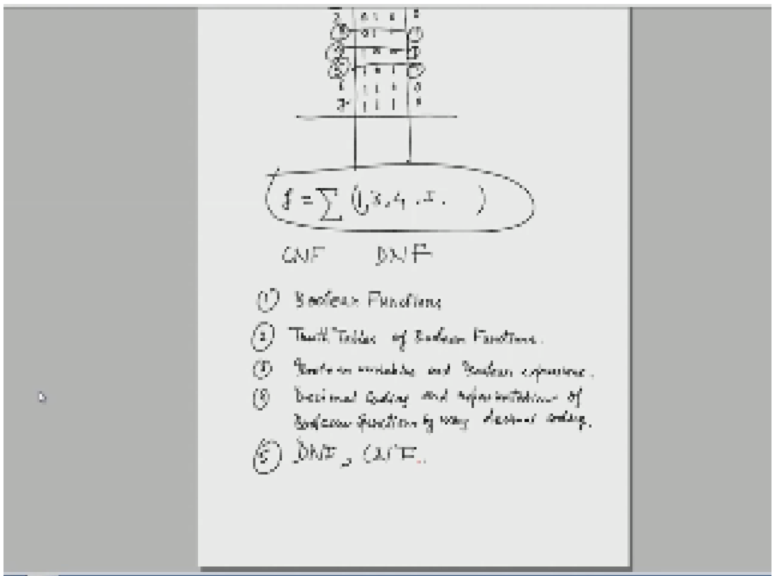

And finally the most important fact of Boolean functions is DN F and C n F that is disjunctive normal forms and conjunctive normal form and their interrelations this is for today, thank you.

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