INDIAN INSTITUTE OF TECHNOLOGY ROORKEE

NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)

Discrete Mathematics

Module-07
Partially ordered sets
Lecture-03
Lattices

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In today's lecture we will be moving on from lattices to a special class of lattices called Boolean function a Boolean algebra, now as we have already seen that a lattice.

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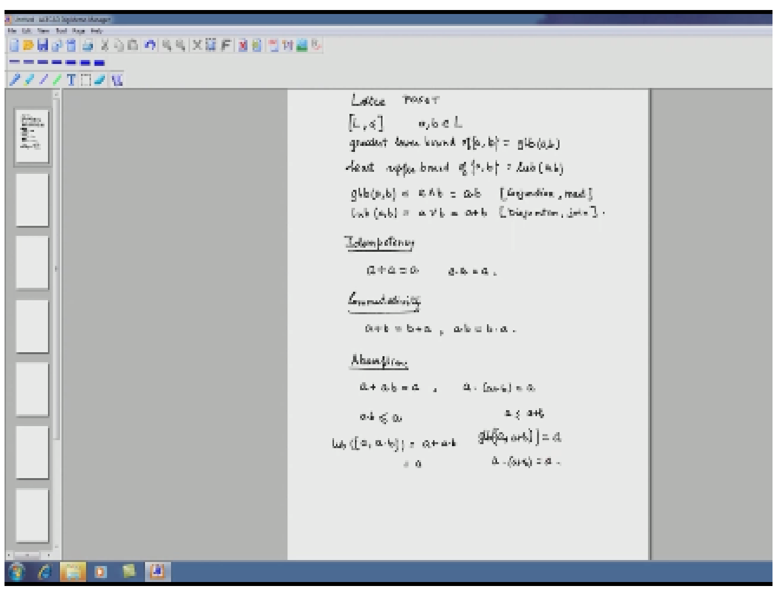

He is a partially ordered set written as posit with the property that any two elements in it has a greatest lower bound and a least upper bound we can rephrase whatever we have said now as suppose L is a lattice with a partial order denoted by the symbol less or equal then for any two elements a b & L we can define an element called the greatest lower bound of a b in short G L b of a b greatest lower bound of a b which we are writing as GL b of a b and least upper bound of a b.

We can as well write it as a set a b that is l u b of a b, now what the definition of a lattice tells us is that in a lattice any two elements will have a greatest lower bound and a least upper bound because of this we can in fact write this g l v and lub as binary operations over L, we will denote them by symbols like V and wedge so glb of a b is denoted by a and the wedge then b we will also sometimes denote it just by a .b and lub that is the least upper bound of a b is denoted by a v v sometimes will be denoting by a + b, now this a.b or this a wedge B is called conjunction, conjunction or the meet operation and a vb or a + b is called disjunction or the joint operation.

Now in this lecture we will be using plus and dot instead of the v and wedge, now there are certain properties of this meet and join that we will discuss, no one is Idempotent or idem potency property this states that a + a = a and a. a = a now this does not come as a surprise to us because, we know that when we take the plus operation it is essentially the least upper bound of the elements now if we have if my set is singleton so I have got a and a, so the least upper bound is a itself.

And similarly for greatest lower bound the next property that we discuss is commutatively if we have two elements a and b inside L then a + b = b+ a and a .b = b. a this is also somewhat easy to think of because a +b is nothing, but the least upper bound of the set a b and as we know the elements of a set are not ordered, so least upper bound of a b is same as the least upper bound of b a similarly a .b is the greatest lower bound of the set a b which is same as the greatest lower bound of the set b a.

So these two the first two properties of this glv and lub operations need no proof as such now we come to another property which is called absorption, now this absorption property needs a proof well we are starting with two elements a and b and we see that a. b is definitely something which is less or equal a the reason is that a .b is the greatest lower bound of a and b then of course a will dominate a .b and therefore, if we consider the least upper bound of the set a and a .b right so if we consider the least upper bound lub.

This is nothing but a + a. b this of course will give me a therefore I get the first absorption rule and then in the next in the next case, we note that a +b is definitely dominating a the reason is that if you consider a + b it is the least upper bound of a and b therefore a will be < a +b but then if we compare a and a + b then the greatest lower bound of the set a and a + b is going to be a in other words a . a + b is going to be a, so these are the first to absorption laws over here and next we move on to associativity.

Which states that a + b + c = a + b + c and a .b. c = a. b . c now in a lattice even these rules need a proof I will give the proof of the first associativity law and then leave the other part for exercise, now let us look at the rule again we want to show that.

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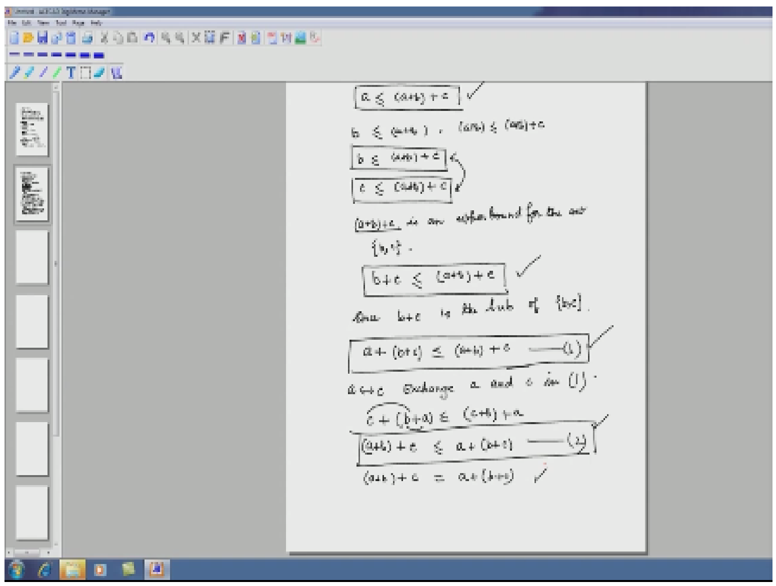

a + b + c = c in order to do prove that we first start by checking a, a is definitely less or equal to a +b and a + b is definitely < to a +b + c well the reason is that a + b is the least upper bound of a and b therefore it must dominate a and a + b + c that I write here is the least upper bound of a +b and c therefore it must dominate a + b, so by using the transitive law, now because we know that after all we are looking at partial order relations using transitivity of the partial order, we have a less or equal a + b + c.

Now let us look at b and c, now let us put this first in a box and then start off with b now b is definitely less or equal a + b and a + b < = a + b + c therefore we can write again by using transitivity of the partial order involved be less or equal a + b + c we again put it in a box the next candidate is c start with c and of course c is less or equal a + b + c we put it in a box, now we would like to combine these two results, if you look at this we'll see that in the both two boxes the right-hand side is same that is a + b within bracket + c and b < a + b + c is also less or equal a + b + c now we know that we are working in a lattice therefore b and c will have a least upper bound definitely a +b + c upper bound for the set bc we write it down a + b + c is an upper bound for the set bc.

Therefore any least upper bound will be dominated by this upper bound so we know that the existence of the least upper bound of b and c is guaranteed so which we denote by b + c therefore b + c will be less or equal a + b + c why the reason is that since b + c is the least upper bound of the set containing bc, now we put a bracket over here as well now we see this and this again we see that a + b+ c dominates a that is a is related to a + b + c it is less or equal a + b + c whatever us a remembering.

That it is a lattice and b + c is also less or equal a + b + c, now of course we can we are we can take the least upper bound of a and b + c which is a + b + c and this will be less than a general upper bound any general upper bound and one of them is a + b + c thus we see that we have been able to prove that a + b + c is related to a + b + c as we choose to say in a lattice now what about the other way round, now if we look at this last relation we will see that in the bracket in the left hand side.

Where bc and in the right hand side we have ab and outside we have a and c that is a remaining symbol, so this gives an idea what about if we exchange a and c that is exchange a and c where let us call it equation one or it's not an equation it is call it the relation one in one if we do that we will get c + b + a < = c + b + a rearranging the terms using commutatively and yeah only we need only commutative we will get a + b + c, so I am using commutative here and commutative here is less or equal again using commutative a + b +c.

Let us call it 2 so if we have got now 2 and we have got 1 we remember that after all we are working in a lattice, so therefore it is anti-symmetric so these two relations together imply that a +b + c = a + b + c thus we have proved just now that in a lattice associativity holds for addition and what I leave as exercise is the meet operation or product whatever you say to prove that a .b .c = a. b .c what I can tell you is that it is almost the same the underlying ideas are more or less same, now we make a statement that we do not prove explicitly.

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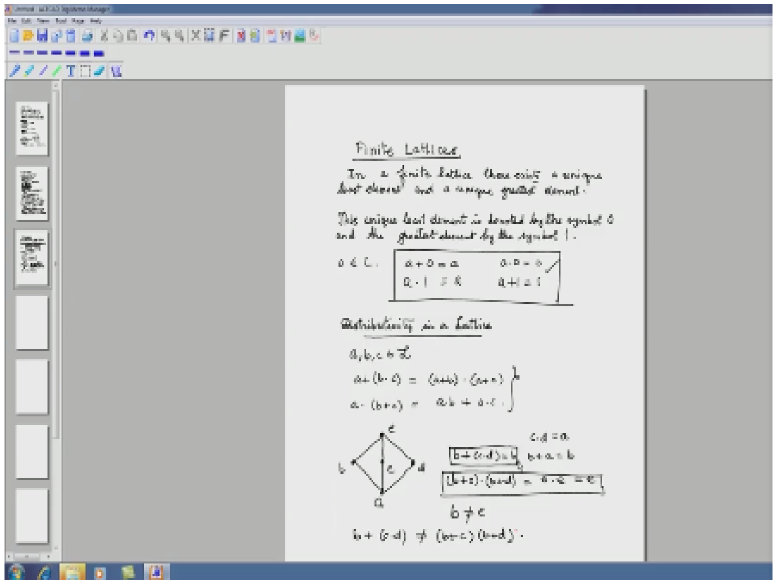

But it is more or less clear we are considering finite lattices we state that in a finite lattice in a finite lattice there exists a unique list element and a unique greatest element the question is why the reason is that given any two elements I can find the least element and if I just pick up all the elements are mutually tech least element and the least element of the least element ultimately I will arrive at the least so let me reward it given any two elements I get a least upper grade lower bound.

So if I if I keep on doing that for all the all the elements and then construct all the greatest lower bounds and then do the same thing for all the greatest lower bound, so ultimately I will arrive at one element which is the least element in the whole lattice the reason is that we don't have infinitely many elements, so whichever chain by using whichever chain we go downward we will stop at one element and which is going to work as the least element of the whole lattice and similarly we have the idea of least upper bound so we go on like that and get a get greatest element this unique least element is denoted usually by the symbol zero.

This unique least element is denoted by the symbol 0andthe greatest element by the symbol one and given any element a in the lattice L we have the properties a + 0 = a. a.1 is a. a0 is 0 and a + 1is 1, so these are the things which are again more or less straight forward because after all 0 is the greatest element, so if I take the least upper bound of a and 0 its sorry 0 is the least element so if I and so whatever element I take a it is going to dominate 0 therefore if I take the least upper bound it is going to be a and 1 is the greatest element.

Therefore if I take the lower bound of 1 and a for any a it is going to be a similarly the other two cases, now we come to the question of distributivity in a lattice, now a lattice is called a distributive lattice if for any a b c in that lattice and I must mention here that a b c need not be distinct we must have a. b. c = a + b. a + c and a .b + c = a .b + a. c, now the next natural question that comes here is that is any lattice distributive and the answer is no there are lattices which are not distributive.

I will give examples of two lattices which are not distributive so first one is like this, so suppose this is a this is b this is c this is well this is c, so this is c this is d and this is e and I connect them and this gives me a hassle diagram, now let us consider the product be well I am using the other notation, so I changed my notation let us consider b + c alright now let us evaluate c. d you see and d this is a and b + a gives me b, so I can write that b + cd is b all right if we consider b + c C and b + d this will mean b + c.

Is e and b + ds e therefore e meet e gives meet we put this in a box comparing these two we see that of course b is not e therefore in this case b + cd is not equal to b + c meet b + d, so this lattice.

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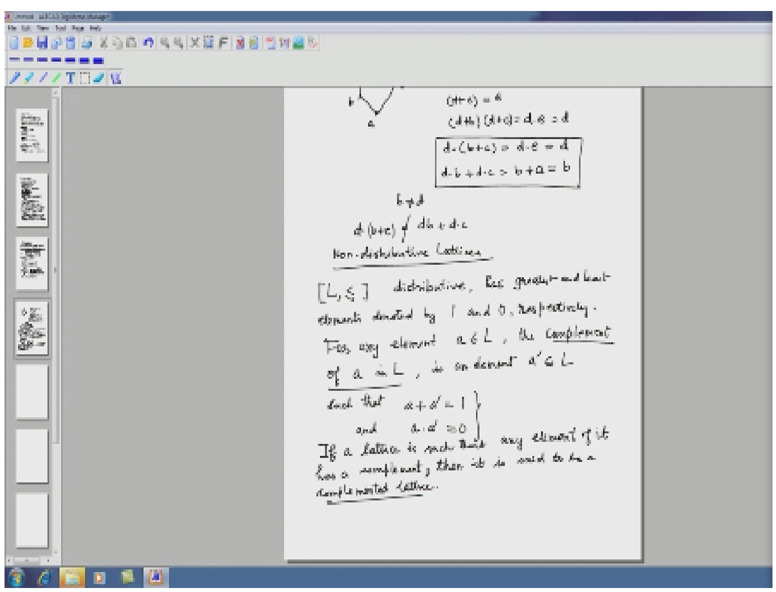

Is not distributive next we check another lattice which is not distributive which is given by a hassle diagram like this we have, now five elements we have got a over here then b then d and at the top well let me write r here because I will put c over here, now let us see whether we can find out some combinations where which will not be a which will not be distributive, so let us start from here d and then let us consider d + b e, now this is d + this is not easy size c so this is cd + bc, now bc is a so d +a.

So this gives me d on the other hand d + b gives me d and d+ see gives me e and if I take a product d + b product d + c well this I see that this does not work out I get again d, so it does not work out, so we need some other combination let us let us try some something else let us let us try the other way, so let us consider d. b + c which is d meet b + c e this gives me d and d meet b + d meet cd meet b will definitely give me b and d meet b gives me a, so this gives me b all right.

So here we see that for this particular case this does not work out but if we if we change the plus with dot and dot with plus when we see that we have a situation here where of course b = d therefore we have db + c = db + dc they are not equal thus we see that these lattice as well as the one we discussed previous to this both are non distributive, so these are examples of non distributive lattices however for our purpose we will be considering only distributive lattices and not only that distributive lattices which has complements.

So we define another operation on a lattice which has got a greatest element and a least element so I am considering a lattice L with a partial order which is distributive has greatest and least elements denoted by 1 and 0 respectively, we are considering this and we define unary operation on it which is called complement for any element a belonging to L the complement of a in L is an element a prime we will be writing this as a prime of the original element such that a + a' = 1 and a . a' = 0.

If a lattice has a if a lattice is such that any element of it has a complement then it is called a complimented lattice, we have to note here is that this complementation is not restricted to the distributive lattices we can have complemented lattices which are not distributive in fact a closer analysis will show us that two non distributive lattices that we constructed just before are complemented lattices, but now we are not considering that we are considering distributive lattices having greatest.

And least elements and which are complemented which is a complemented lattice as well and these lattices are called Boolean algebras, so I am now in a position to define a Boolean algebra.

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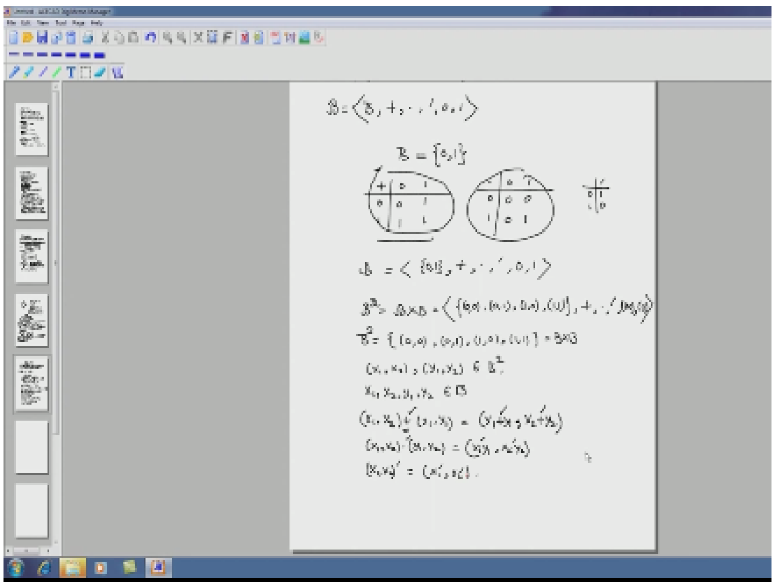

Boolean algebra is complemented distributive lattice with the operations +. and complementation defined as about, so here in all these context we must remember that we are not only looking at a set we have a set, but on that set there are operations there are elements and all that and together it is forming and algebra, so suppose I have a set b on this set b we define a partial order suppose with respect to that partial order it becomes a lattice and because it becomes a lattice we have two operations defined on it that is + and. which is a least upper bound and greatest lower bound.

And then suppose the greatest element and least element exist and we can define a complement operation which is a unary operation, and suppose it is a complimented lattice with respect to the this complementation and that complemented lattice is a distributive lattice we have discussed the distributive law, so we put all these operations together with the symbols for the greatest element and least element and what we get is called a Boolean algebra, and we can denote it by a script B.

Now the question is what is the simplest possible Boolean algebra and we indeed know probably the simplest possible algebraic structure that can ever be and that is a Boolean algebra that contains only two elements here please note that if we have just a set with one element we can have nothing, so we have a set with two elements let us called call them 0 & 1 call this set B and define + as this 0 + 0 is 0 0 + 1 is 1 1 + 0 is 1 and 1 + 1is 1 and let us define dot in this way 01 0 1 0 0 0 1 is 0 1 0 is 0 & 1 1 is 1and let us define compliment the unary operation in this way that 0 is changed to 1.

And 1 is changed to 0 and what we have is Boolean algebra and this is denoted by you know we can denote it by 0 1 which we are then +. this complement 0 1, so this all together give me this may be now if we want to generate more Boolean algebras the best way is to take Cartesian product of this B, so for example we can take B^2 which is the Cartesian product of B and B then we get this set well 0 0 0 1 1 0 1 1 + . complement and the least element is 0 0 greatest element is 1 1, now one can ask a question over here that what do I mean by this + and what do I mean by this. and the complement operation here.

We note that if we consider just a set let us say B which is essentially 0 0 0 1 1 1 1 0 & 1 1 any two general elements B can be written as x_1 x_2 and y_1 y_2 both belonging to B then x_1,x_2 y_1,y_2 belongs to I am sorry I change it this is B^2 this is B x B this is B x B, so here it is B^2 but this x1 x_2 y1 y_2 are in B so x_1 $x_2$will behave as elements in B therefore we can define the app the + and . by using the operations on B that is x1 x_2 + y_1 y_2 will be x_1 + x_2 y_1 + y_2 no this is this is not correct, so I will change it this will be x_1 + y1 x_2 + y_2 now we do not have to worry about this + is already defined over here and this is the + that we are defining on V x B by using the addition that we have already defined similarly we can define x_1, x to meet y1 y_2 as x_1 y_1 do their x_2 . y_2.

And we again remember that this meet or dot or product whatever we say is same as this one and we can check that the operations defined in this way follow the rules of Boolean algebra and for the complement again if we have x_1 x_2 complement we will just define it as x_1 complement x_2 complement as we see that given our very small and simple Boolean algebra we can take on take Cartesian products of this and generate new Boolean algebra and we will we will see later on that any finite Boolean algebra is essentially Cartesian products of B with respect to the operations that I have just now described for B^2 so for today we shall stop in the next lecture we will continue our discussions on Boolean algebras and functions on Boolean algebras thank you.

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