INDIAN INSTITUTE OF TECHNOLOGY
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NATIONAL PROGRAMME ON TECHNOLOGY
ENHANCED LEARNING
(NPTEL)

Discrete Mathematics

Module-07
Partially orders sets
Lecture-02
Partially orders sets

With
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In this lecture we will continue the discussion that we have we started in the previous lecture that is on partial order relations and partially ordered sets.

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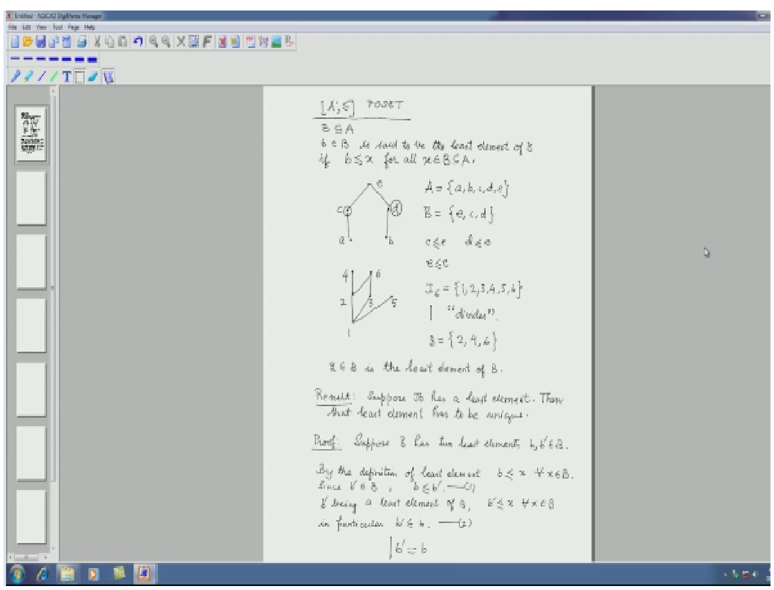

So a set A along with a relation which is usually denoted by a ≤ is said to be a partially ordered set in short written as a POSET if the relation on a is reflexive anti symmetric and transitive we have also seen that if we have a finite partially ordered set then we can represent it by a diagram called has a diagram which is extremely convenient to visualize a partially ordered set in this lecture we continue in the same direction and introduce some special elements in a partially ordered set.

So we have the partially ordered set A ≤ and suppose we have a subset B of A an element b ∈ the B is said to be the least element of B if b is related to x for all x ∈ the subset B of the partially ordered set A now we concentrate on the fact that whether a least element will exist always so let us consider a has a diagram of this type now suppose this a, b, c, d, e is the set A and well suppose the partial order is given by this hasse diagram.

Now let us consider the set B which is only e, c and d now we shall see that there is no element in e, c, d the set containing e, c, d which can be called the least element if we consider the element c here of course c is related to e which we sometimes just say c ≤ e but she has no connection with d similarly d is related to e this is d over here but d has no connection to c and of course e is only related to e and nothing else therefore this is an example of a subset which does not have a least element.

Now let us take up an example where there are least elements so suppose we have a situation like this right so we have seen this example in the previous lecture this is essentially the set I6 which consists of elements from 1 to 6 the integers from 1 to 6 and the partial order is divides which is denoted by a vertical line now here we if we consider the set B as {2, 4, 6} then we note that the element 2 is the least element of this set 2 ∈ B is the least element of N now we note another fact that is if B has a least element then definitely this least element is unique we can give a short proof of that.

Suppose B has a least element then that least element has to be unique we give a proof of this fact now let us suppose if possible B has 2 least elements suppose B has to least elements b and b ` both ∈ B now by the definition of least element b must be related to all the elements of B by the definition of least element b is related to x for all x ∈ B now b` is an element of B therefore since b`∈ B b is related to b` exactly using the same argument we can say that b` being a least element of B b` is related to all the elements of b in particular b` is related to b.

Now we can number these 2 relations that we have obtained one is b related to b` the second one is b` related to b therefore combining them and remembering that we are after all working on a partially ordered set where the relation is a partial order that means it is anti-symmetric which means that b` = b.

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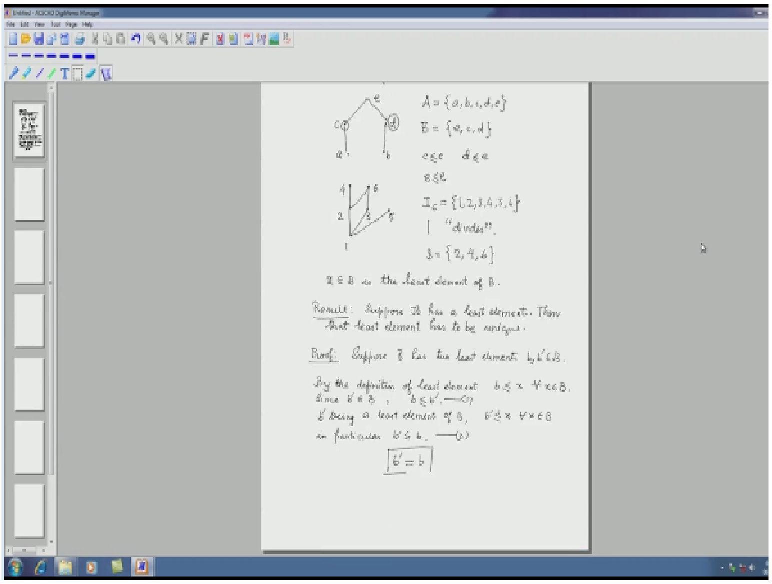

This proves that we cannot have more than one distinct least elements now this same argument is going to work for greatest elements that we are going to define shortly so we I would not be doing this that proof again but the argument is going to be the same so we have talked about least elements now just like great a least element we have the greatest element.

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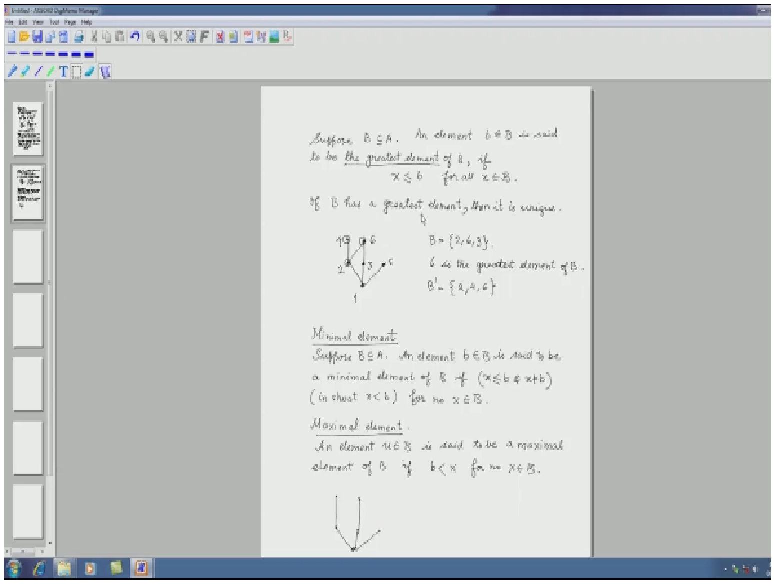

So suppose B is a subset of A suppose B is a subset of A an element b ∈ B is said to be the greatest element of B if x is related to b for all x ∈ B again by using the same argument as before we can we can infer that if greatest element exists then it has to be unique so we do not prove it again we just write if B has a greatest element then it is unique now just as before we can have situations where B does not have greatest element or it may have greatest elements so let us go back again to our example of I6.

So it is like this 1, 2, 4, 6, 3 and 5 now let us try to find out a subset which has a greatest element well if we consider the subset {2, 6, 3} 6 happens to be the greatest element of b right but we can also construct in the same partially ordered set sub sets which do not have greatest elements one such subset is well let us write B ` which = {2, 4 and 6} so here we are considering this node {2, 4 and 6} and we see that we do not have any greatest element.

Now we move on to the definitions of some more special elements and some special elements which are very typical of a partially ordered set we define minimal element of a set now in a partially ordered set A a subset B has a minimal element if there is an element b in that subset which does not dominate any other element in that subset now let me write down the definition so suppose B is a subset of A and element b ∈ B is said to be a minimal element of B if x ≤b and x ≠ b together.

Okay we have to remember that x of course x is a b ≤b so if you do not take that case so we take the cases where x ≤ b and x ≠ b and we say that this should not happen so if x ≤ b that is x related to b and x ≠ b now here this right in short x strict < b for no x ∈ B so this means that there is no x other than b inside the sub set B which is related to the elements b we can define maximal element of a sub set in the same way so I write down maximal element well just to avoid any confusion let me change the name of the element.

So I write u an element u ∈ B is said to be a maximal element of be if b strict < x for no x ∈ B that is B is not dominated by any other element of the set B well what we must understand here is that there is a difference between minimal element maximal element least element and greatest element now what is that for example let us look at again the lattice that we considered sorry again the Hasse diagram that we considered just in the just before so I draw the Hasse diagram again. So here we have 4, 2 going to 1 and then 6, 3 going to 1 we have 5.

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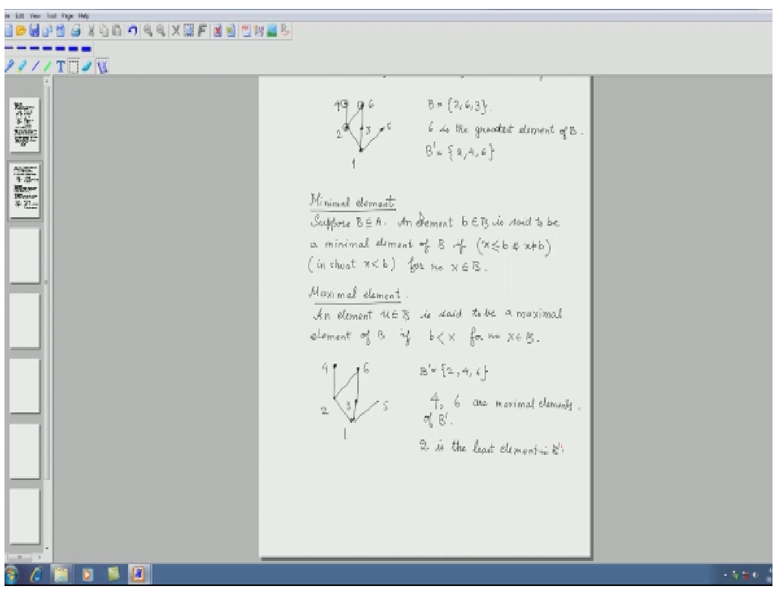

So we number them this is exactly what I drew above all right, so now let us consider the set B` that we again considered before B ` which is 2, 4, 6 we saw that this set does not have a greatest element of course because there is no element which such that all the elements in B is related to that element but that 2 elements 4 and 6 are maximal elements of the set B` that is because there is no element in the set B ` such that 4 or 6 is related to that element so it is a maximal element,.

So we can write that 4, 6 are maximal elements of B` now we see that B` has a has a least element it is a it is of course a unique element and that is 2 is the least element in B ` all right and 2 is of course a minimal element because there is no element other than 2 which is related to inside B inside B` thus we have seen the ideas of least element greatest element minimal element and maximal element for subset in a partially ordered set now we shall move on to another idea which is the idea of bounds now our starting point is again a lattice I am sorry.

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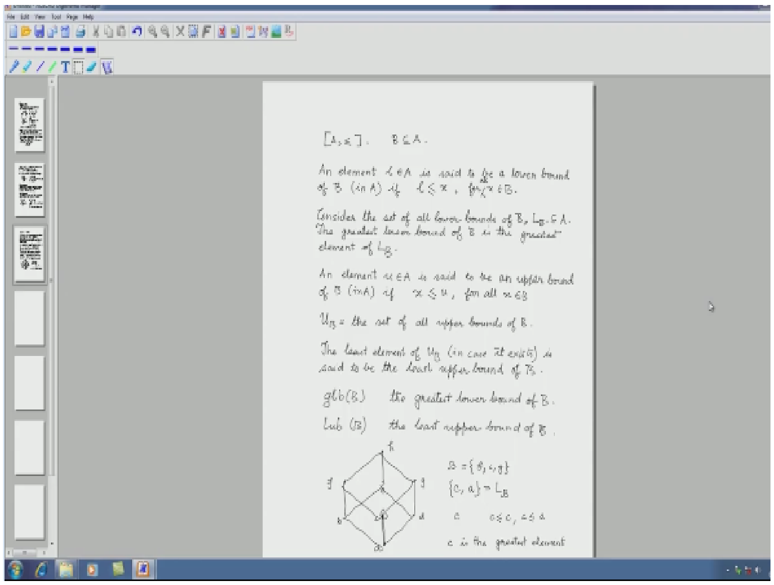

I am our starting point is again a partially ordered set with a partial order and we also consider a subset B in A now the question is that what do we mean when we say that an element l is a lower bound of B in A that is precisely this an element l ∈ A is said to be a lower bound of B you can you can add in A if l is related to all x ∈ B now we consider the set of all lower bounds of the set B now these lower bounds may or may not ∈ that subset B but of course they have to ∈ A so consider this set of all lower bounds of B.

Now how do we name this set let us call this set L subscript B which is of course a subset of A now we will consider the case when L sub B has a greatest element in case L sub B has a greatest element the greatest element of L sub B is said to be the greatest lower bound of B in A of course the greatest lower bound of B is the greatest element of L sub B now exactly in the same way we can define upper bounds and least upper bounds we do that and element u ∈ A is said to be and upper bound of B of course in A if x is related to you for all x ∈ B here also we should write for all x.

Now suppose U_B is the set of all upper bounds of B then in case this set of upper bounds has a least element we call that element the least upper bound so we write the least element of U_B in case it exists is said to be the least upper bound of B now there are some short forms we will be writing the greatest lower bound of B simply as glb(B) the greatest lower bound of B and the least upper bound of B will be written as lub (B) the least upper bound of B.

Now we will consider we can consider some lattices particularly we can look at some hasse diagrams of lattices and check least upper bounds and greatest lower bounds of certain subsets now for example if we consider the a lattice of this type let us take this example all right and let us name the points possibly let us name this as a then b, c, d, e then f, g and h alright now if we consider the subset f, c, g let us call this b then we will see that we have got 2 lower bounds of this set namely c and a because c is over here c is of course ≤c c ≤ if c ≤ g and a ≤ to c a because it is connected to f through apart if is a ≤f a ≤ g therefore both are lower bounds.

So this is the set L_B the set of lower bounds of B and among this set we see that we have the element c such that c ≤c and c ≤ a so c is the greatest element of L_B c is the greatest element of L_B

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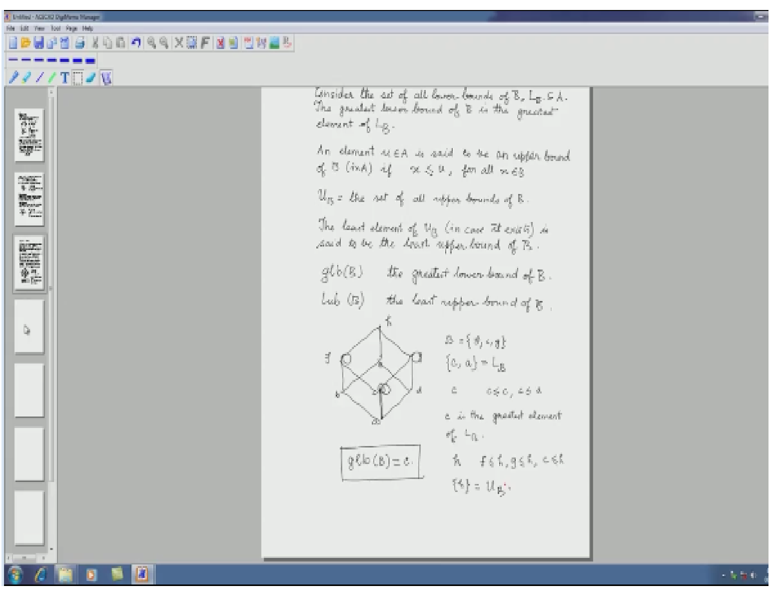

And so we can write that the greatest lower bound of B = C now the question is that does this set have a least upper bound for that we have to first try to find out the set of upper bounds my set contains the nodes c, f, g now we see that the element h is such that if ≤h g ≤h and so is c therefore this set h is U_B the set of upper bounds of B and since it is a singleton set this itself is the least upper bound of B.

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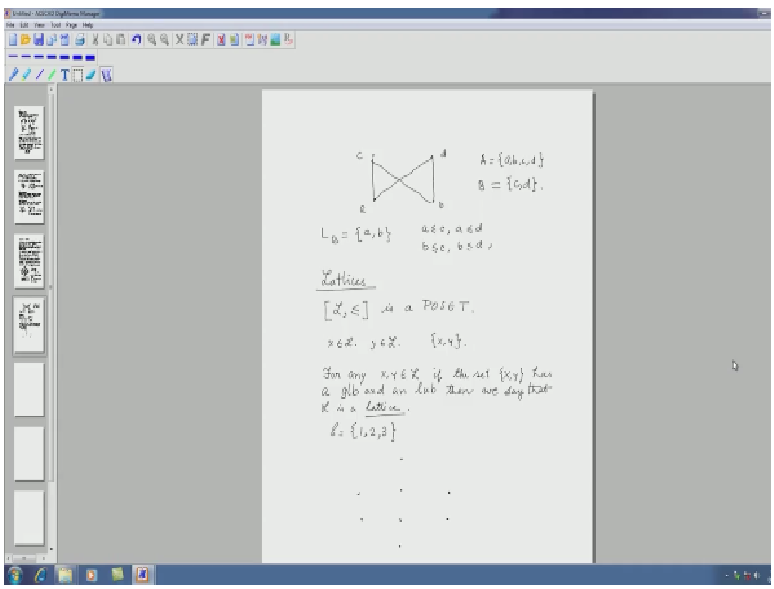

Now we come to a question that is it possible to have a partially ordered set and a subset inside that partially ordered set which has lower bounds but no greatest lower bound and we can invert the question and replace lower by upper and say that is it possible to have subsets in a partially ordered set which has upper bounds but no least upper bound now the answer is yes let us look at a hasse diagram of a possible partially ordered set suppose we have 4 points we name the points as a b c and d.

And suppose we write like this so I have drawn a hasse diagram we see that the totality is a, b, c, d and let us consider the subset B which = c, d now if we try to construct L sub B we will see that L sub B consists of 2 points a and b because a ≤c a ≤d b is also ≤c b ≤ d okay therefore both of them are lower bounds of c, d but which one is the greatest lower bound the answer is there is no greatest lower bound the reason is that the elements a and b are not comparable now this leads us to if a particular question.

We would like to construct partially ordered sets such that some specific subsets of that partially ordered set will have some properties related to greatest lower bound and least upper bounds now one definition has been has proved to be very useful that is the definition of a particular class of partially ordered set called lattices suppose L with a partial order ≤ is a POSET now this L will be called a lattice if given any 2 elements of course not necessarily distinct we will be able to find a least upper bound and they greatest lower bound of that subset.

So suppose the elements are same suppose I consider an element x ∈ L and of course well I consider another element y ∈ L and then construct the set x, y what I say is that this set x, y will always have a least upper bound and a greatest lower bound all right so this makes a lattice now the question is that do we have examples of lattices the answer is yes so first of first example that comes to my mind is the example that we construct by taking the power set of the set 1, 2, 3 the hasse diagram corresponding to the POSET defined by using the set subset equal relation is this.

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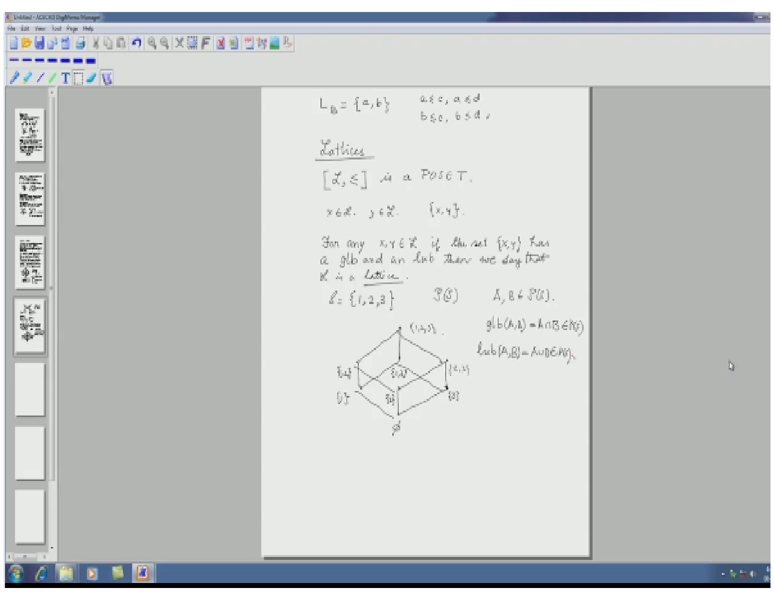

So we have the null set Ø at the very bottom then one we connect it then this we will write to here then this is 3 we will write 3 here then this is 1, 2 this is 1, 3 and this is 2, 3 these are the subsets of S and eventually we have 1, 2, 3 we join this as well now pick up any 2 any 2

elements from this lattice which is essentially the power set of S if we pick any 2 elements they are basically subsets of S and the suppose we write them as A, B what we can find is that greatest lower bound of A, B is simply the subset A ∩ B which is of course an element of PS.
And least upper bound of A, B is A U B which is an element of PS thus this is a lattice we can take up another example of a lattice let us look at that.

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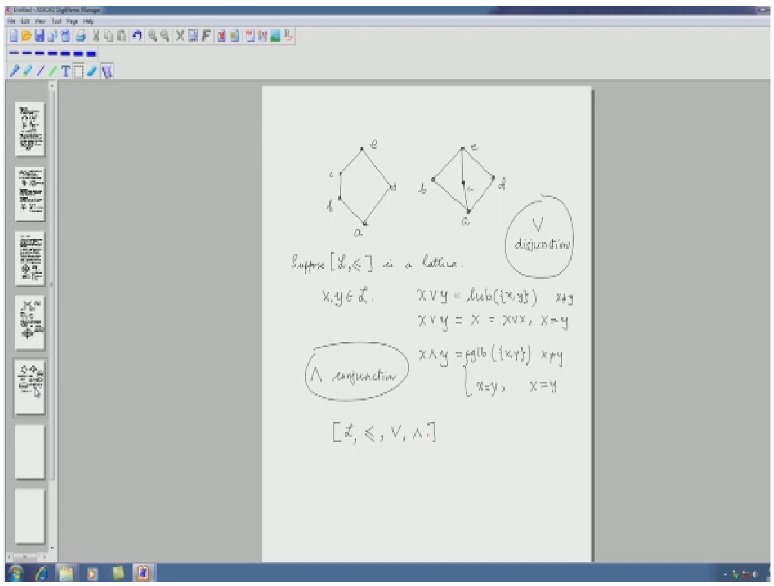

So let us consider something like this or like this these are also lattices let us name them a, b, c, d, e and a, b, c, d and e what you can do is that you can pick up any 2 elements and you will see that you will be able to construct a greatest lower bound and a least upper bound of that set there are some differences between the lattices that we the first lattice that we saw and these ones these differences will be discussed in the next lecture but I will stop this lecture by introducing another concept related to lattices.

What we find is that this idea of greatest lower bound and least upper bound and the fact that any subset containing 2 elements in a lat in a lattice will always have a greatest lower bound and a least upper bound leads us to define some kind of binary operation of on lattices for example suppose we start with L with this and suppose that it is a lattice suppose L is a lattice then take 2 elements in L in this case these 2 elements do not have to be distinct what I will do is that if they are distinct then I will define x this y this symbol that I draw will be called disjunction.

All right x disjunction y will be nothing but least upper bound of the set x , y in case $x \neq y$ in case $x = y$ I will simply define x disjunction y as x which is basically x disjunction x so this is the case $x = y$ I could have I could have written y also similarly I define x conjunction y as greatest lower bound of the set x y when $x \neq y$ and well = x or you could write y as well in case $x = y$ and this symbol is called conjunction.

Now we can consider a lattice along with the partial order defined on it a some kind of algebraic system having these 2 operations conjunction and disjunction so I can write the whole set up as L then the partial order then the disjunction and the conjunction this is for today's lecture in the next lecture we will build algebraic systems on this setup, so for the time being I stopped thank you.

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