INDIAN INSTITUTE OF TECHNOLOGY ROORKEE

NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)

Discrete Mathematics

Module-07
Partially ordered Sets
Lecture-01
Partial Ordered relation

With
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In today's lecture we will be discussing partial orders and partially ordered sets let us consider a set A.

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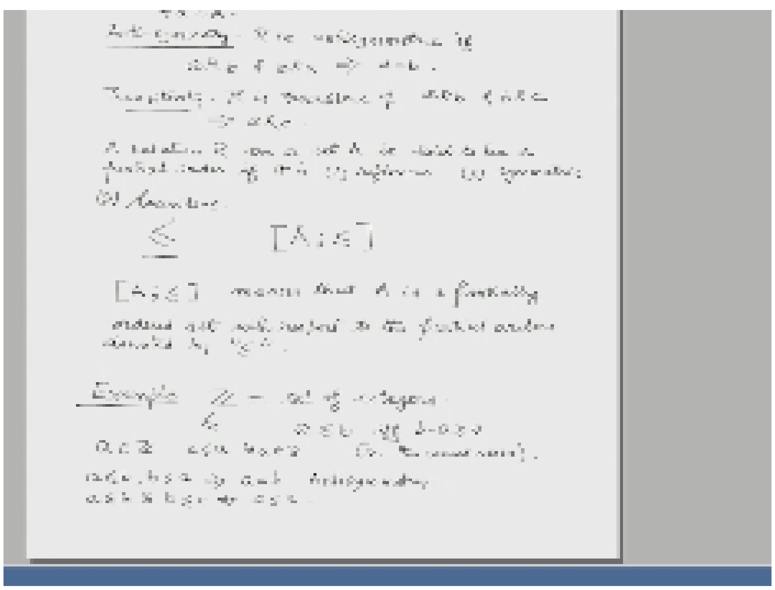

And relation R defined on the set we have already seen that if R is a relation on A then R is A subset of A cross A or in other words a relation on A is a subset of A cross A which we write in symbol as our subset of a cross a we have also discussed that there are special properties of relations and we will recall three properties that we have discussed before the first one is reflectivity's a relation R on A is said to be reflexive R is said to be reflexive if A related to A for all a belonging to A next property that we recall is anti symmetry.

A relation R on a is said to be anti symmetric if ARB that is A related to B and be related to a together implies a equal to B so R is R is anti-symmetric if AR B and B RA ⇒A = B finally the third one that we need in today's lecture is called transitivity R is said to be transitive if A related to B and B related to C ⇒ A related to C now a relation R is said to be a partial order if it is reflexive anti- symmetric and transitive a relation R on a set A is said to be a partial order if it is one reflexive to symmetric and three transitive.

This is a special class of relations that we study extensively and therefore we have a special symbol to denote partial orders and the symbol is well known less than or equal to so usually we will denote a partial order by this less than or equal to symbol but we have to keep in mind that it is something different than the usual less than or equal to that we use in the in the context of real numbers or integers and so on.

So if I have a set a on which a partial order denoted by this less than or equal to symbol is defined then we will write this whole system as a less than or equal to then all breasts by a third bracket and in between we put a semicolon so when I say this I can say that this is a partially ordered set so if I say that I am considering A ;< = symbol then this means that means, that A is A partially ordered set with respect to the partial order denoted by the less than or equal to symbol all right.

Now of course this less than or equal to although it is not really the less than or equal to that we are familiar with raises a question that what about our usual less than or equal to is it a partial order the answer is yes so the first example of first partial order that we will discuss in this lecture will be on the set of integers which we denote by this special symbol Z and the partial order will be denoted by less than or equal to and this is indeed the less than or equal to in the sense of the ordering.

In the set of integers that is a is less than or equal to B if and only if B - < =0 in the usual sense so I will write in the bracket in the usual sense now we can quickly check that this is indeed a partial order because if we take any element a inside the set of integers we do not have to explain much to tell that a is less than or equal to a for all a belonging to Z and then if we take two integers and suppose these two integers denoted by A and B such that A > = B as I have already written and B >=A then together they will imply A = B.

So we have anti symmetry over here finally again it is pretty much obvious that if we have A > = B and B >= C where all of them are integers then I can write A>= C which is transitivity therefore we see that the set of integers with usual less than or equal to relation is a partial order at this point I would like to mention that this Z said along with the less than or equal to relation is a partial order .

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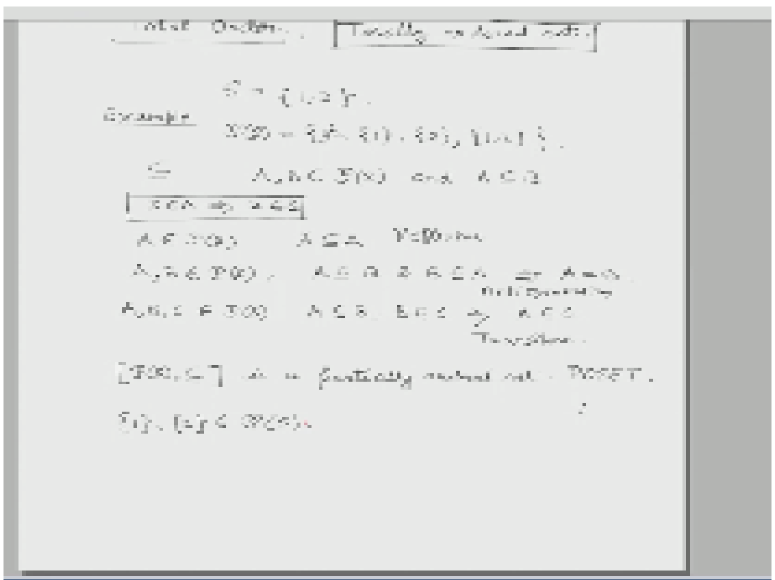

But it is much more than that because if we pick up any two elements let us say A and B belonging to Z they are not necessarily distinct then we know that either >= B or B > =A or of course both but in case both are true then we know that A= B but what is important here is that given any two elements in Z I can compare them so it gives a kind of ordering which is of course a partial order but it is a special case of partial order which is called a total order and the fact that given any two elements either.

The first one is less than or equal to the second or the second one is less than or equal to the first is told in language as they can be compared so that means that any two elements in the set Z can be compared and if that happens then we call that a total order and the set Z along with the partial order less than or equal to which is a total order will be called a totally ordered set now we move on to a partial order which is not a total order this is important because otherwise we do not have to study much because then we will say that okay ,all partial orders.

Are total orders but it is not so but to establish that we need an example now we start with an element sorry we start with a set containing two elements let us call that set S then we construct all the subsets of this set S and we denote it by P(s) which is first the empty set then the set {1} and {2}and {1,2} now the relation that we choose is the subset equal relation that is conveniently denoted by the well-known symbol of subset equality so if I am saying that A B belongs to PS and A is subset of B.

Then that means that all the elements of A are in the elements of B in other words X belonging to A ⇒ X belonging to B all right but this is essentially the well-known definition of subsets of another set and here of course I do not exclude that possibility that A= B now this relation is also a partial order the reason is that if we take any subset a belonging to the set of all subsets of S then a is A subset of a then if we consider two elements of the set P S and suppose.

A is a subset of B and B is a subset of a then we will have A= B so the relation is anti-symmetric and if we have three elements a b c inside PS and A is subset of b and B is a subset of C this will imply a is a subset of C therefore we have transitivity so I write here this is reflexive this is anti symmetry and the last one is transitive therefore.

We can say that the power set of s with respect to the relation subset equal is a partially ordered set in short we write a partially ordered set as a posed so we will be sometimes writing as posits now we would like to investigate whether this partially ordered set is a totally ordered set or not for that we consider two elements the set one and the set to both belonging to the power set of s now we see that one is not a subset of the set containing only two and the set containing only two is not a subset of the set containing only one therefore.

I have been able to produce from this very small set PS which contains only four elements I have been able to produce two elements namely the set containing only one and the set containing only two which are not comparable but I have checked that the properties of the partially ordered set that is reflexivity anti symmetry and transitivity all hold for this set with respect to the partial order therefore it is a partially ordered set which is not totally ordered now we will move on to more examples on partially ordered set.

We will we will take up a slightly bigger example then what we have done just now with the power set of 1 2.

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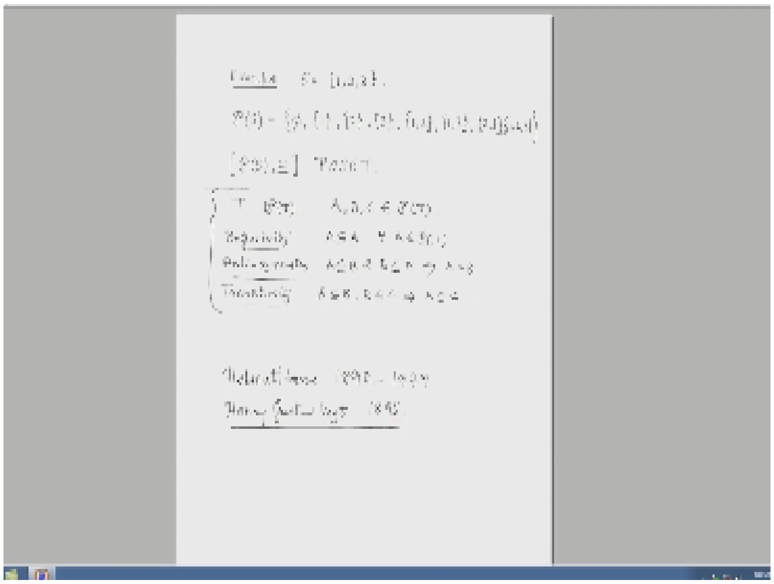

Our example we will be the power set of the set {1,2,3} now we can write the power set explicitly as this it is Л then set {1}.{2},{3} then we have set {1,2} {1,3} the set {2,3} and finally the set {1,2,43} this is the power set and just like before we can consider the subset equal relation and we can see that BS with the subsets equal partial relation is a partially ordered set or insert a posit so of course we can again ask why it is so we can finish off this whole chain of examples by a single proof by taking any set and the corresponding PS and show that PS with subsets equal is a partial partially ordered set I will give quickly an outline of that proof which is pretty straightforward.

So I have now a set essentially let us call it T so it is any set I am considering the set of all subsets of P which is which is P (t) now we only have to consider the subset equal relation and suppose ABC are subsets of T and there are four elements of PT from very basic set theory we know that reflexivity holds that is a is a subset of a for all a belonging to PT including the empty set then anti symmetry that is if a is a subset of B and B is a subset of a this will imply A= B and finally transitivity that is A subset of B and B subset of C B will force.

Here to be A subset of C these are very basic properties of set inclusion and equality therefore we do not have to really worry much about these things so we have proved essentially once for all that BS with respect to subsets equal is a partial order now we will talk about representation of a partially ordered set we go back to about hundred years to around the a towards the end of the 19th century so we will so there was one mathematician Helmut hassle from 1898 to 1979 who used certain diagrams.

To represent partial orders these diagrams are referred to as hazy diagram following the name of Helmut hassle but historically there is another person who used Hassid diagram before helmet has a and that is Henry Gustav vote in about 1895 it is recorded that vote was using diagrams which were essentially the same as hassle diagram but since has I have been using it extensively the name of the diagram is Jose diagram now we will take up the partial order that we have already seen.

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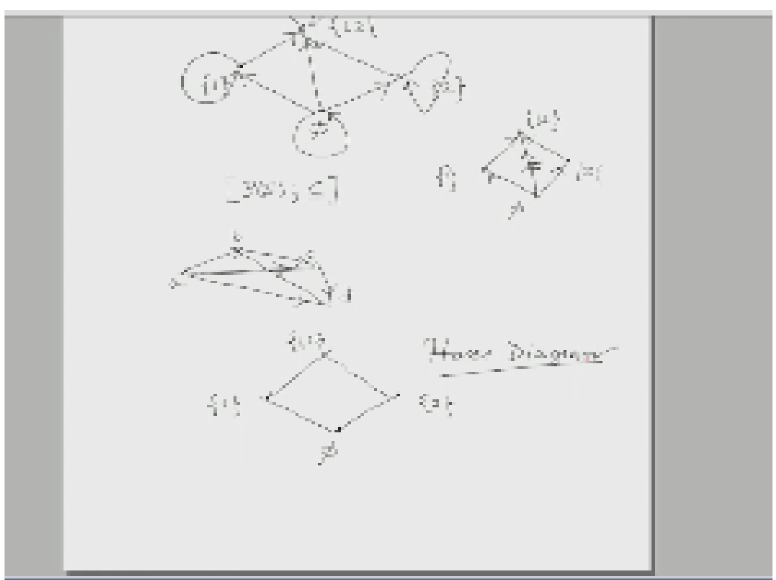

Let us see that one that is the set {12} and the power set of the set {12} which is Л 1 2 & 1 2 now suppose we try to draw the digraph corresponding to this partial order then we will write four points on the plane so we write Л which is the MC set then one which is the singleton containing one two this is a singleton containing two and the set one two now we will start joining these points or nodes by directed edges so first of all if I start from Л I know that Л is rated to itself.

So I draw a self loop and I also know that Л is related to one so I draw an arrow like this then I know that Л is related to two therefore I draw arrow like this and finally I draw an arrow from Л to 1 to put a Arrow head in the proper direction then I start from one I know that one is related to itself so I draw like this and I know that one is related to one two so I draw an arrow like this I come to 2 now 2 is related to itself.

So I draw a self loop over here and 2 is related to 1 2 so I draw arrow head with arrow and finally 1 2 is related to itself so I draw like this what I have here is a digraph corresponding to P yes with respect to the subset equality partial order so I can write like this it is PS ; the partial order written inside a bracket what we note here that there are many edges that we could have removed first of all when we are dealing with partial orders then we know that each node will have a self loop so there is no point drawing a self loop at each node.

If we if we are certain that we are we are discussing partial orders therefore we can do have with the self nodes so we can remove the self nodes and so if we do that let us in rough see what will happen here so it you just get something like this okay so it put four nodes over here arrows are still there and then I will put 5 over here 1 over here - over here and 1 2 over here once I do that and know here I will have something more I will have this edge once I have this I know it that this edge is also redundant.

That question is why because we know that we are dealing with partial orders and since we are dealing with partial orders there therefore it is transitive therefore if for example there is one node let us say A which is related to another node B which is again related to another node C then of course you will have an edge from A to C for example if then C is related to another node D then will have an edge from B to D and from A to D all right now what I say is that we really do not have to consider these edges that we derive later.

We can say that it is transitive so there is an edge between two nodes if there is a path from the first node to the other so we can remove those edges therefore ultimately we will have a very simple diagram removing this but wait a moment we can do something more ,we can take the elements in the partially ordered set in some kind of hierarchy and draw them in a kind of layer by layer going bottom up so that the arrowhead is also understood in this case for example if you consider.

You will see that it is kind of the lowest node or lowest point in the partial order set because no point gets connected to it from below therefore I will start from here and I will now write two points 1 and 2 on a layer above noting that there is no point in between Л and 1 because we do not have any element such that Л is related to that element and that element is related to 1.

So therefore I write 1 and 2, 2 is an element like that and of course 1 2 is not an element like that therefore I write 1 and 2 and then connect them I deliberately do not draw arrowheads it is understood that the arrow is moving upward and then after that I have one two I will draw in this way and finish up with the diagram, so I have a diagram like this which I call a hassle diagram which is much more simplified than the digraph corresponding to a relation of course we can do it only in case of a partial or partially ordered set.

That is only in case set is reflexive the order relation is reflexive anti-symmetric and transitive now we will take up the problem of drawing hassle diagrams so we start with the hazard diagram of S= 1 2 3 and the power set corresponding to that set so let us now look at this example.

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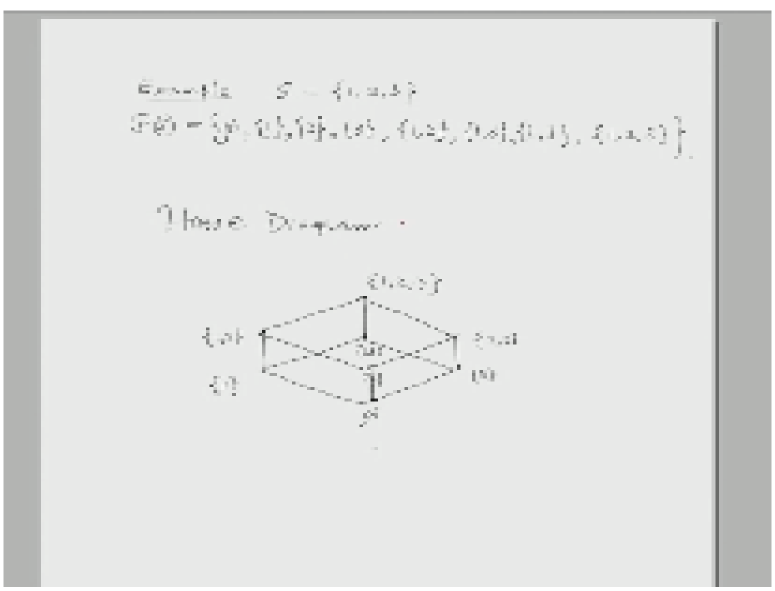

We have S 1 2 3 and we are considering the power set of the set 1 2 3 denote by PS and let us write explicitly all the elements so it S Л then 1 2 then we have 3 then we have fun - then we have 1 3 then we have 2 3 and lastly we have 1 2 3 just same as before we see that the element

Л is let me reward my sentence no element in PS is related to Л therefore we start with Л we draw this is five now we will try to pick up elements such that 5 is related to those elements but there is no intermediate element in between and the singleton sets will serve my purpose so I will write here the set one here.

I will write the set 2 and here I will write the set 3 now we see that I have a direct relation from Л 2, 2 so this is a straight line this is a straight line, and this is a straight line, now what I do is that I draw next layer of elements this is one two so one is of course related to one two so I will join one and one two and two is of course related to one two so I will join two and one two then I will draw one three one is related to 13 ,3 is related to 13 and here I will write 2,3 and join them.

So I have got I have got other the next layer of elements all right and ultimately I have got a single element 1,2,3 and I can complete my diagram in this way so this is the Hassle a diagram corresponding to the set PS with subsidy correlation. Where = 1, 2, 3 what is up we will end today is talk by another has a diagram now let us look at that one first.

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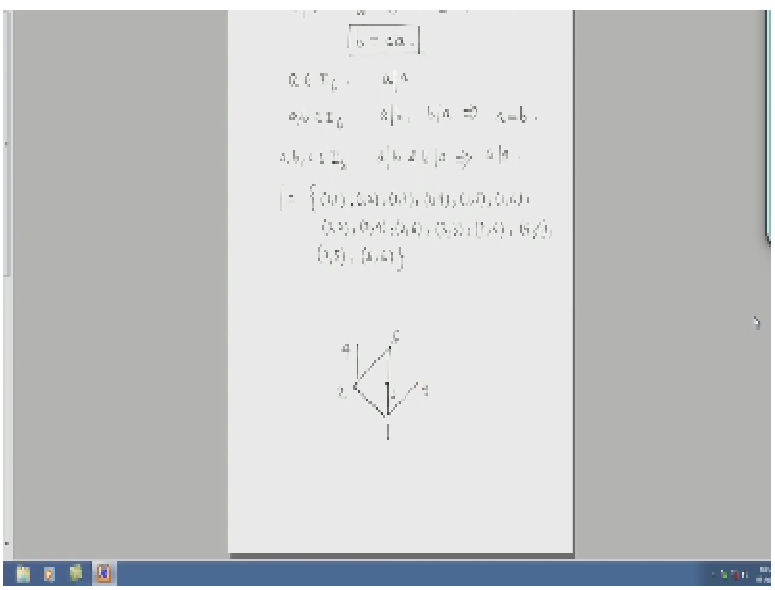

Let us consider a partially ordered set I call it I 6 which contains integers between 1 to 6 and the partial order that we define is divides and we denote this partial order by a vertical line so we will write A / B if and only if A /B in the usual sense strictly speaking I can say that A/ B if and only if there exists belonging to Z such that B = C times a so this is the standard definition of division that we are using now on the set of all integers between 1 to 6 let us consider this relation it is not difficult to see that it is also a partial order.

The question is why because if I take up any element a inside I 6 then of course a divides a and if I take up any two elements inside I 6 such that a divides B & B divides a then from very basic property of division I can infer that A= B and finally for A,B, C three elements in I 6 A / B & B / C automatically means A/ C so these are properties of division now I would like to write down explicitly the partial order corresponding to this divides so see that I will have 1, 1 and then.

I will have (1 2) then I will have (1 3) then I will have (1 4) then I will have (1 5) and (1 6) then starting from I have 2 , 2 and then 2 4 + 2 6 starting from 3 I have only(3 ,3) and(3 ,6) starting from 4 I have only (4, 4) and then I have (5, 5) and I have (6 ,6) this is the relation now if you want if I want to draw a if I want to draw a has a diagram corresponding to this relation I will keep one at the bottom because nothing divides1 and then check out all the numbers.

Particularly the prime numbers so 1, 1 ,1 I need not consider so I get 1 - right so I will draw like this and then let us, consider 3 also so I have got 3 the one - I have got 1/3 and then 1/4 but I am not going to draw one fold because 2 is related to 4 so it will start from here and go upward to 4 this is 4 and I am not going to draw 1/6 because 3 is related to 6.

So I will start from 3 and go up to 6 this is 6 and C 2 is related to 6 also so I have got this then I have got 1 2 3 4 6 there is only one element that remains outside this set that is 5 I draw 5 over here so this is a hazard diagram this is different from the ones we have seen before like this there are other hazard diagrams that one can draw and that one has to try as exercise so we stop our discussions on partially ordered set and representation of partially ordered sets by using has a diagram here thank you.

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