INDIAN INSTITUTE OF TECHNOLOGY ROORKEE

NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)

Discrete Mathematics

Module-06
Relations
Lecture-07
Warshall's algorithm

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Today we will discuss a method to construct transitive closure of a relation this is an algorithm which is called what warshall's algorithm and we will see that it makes computation much easy however before we go to what sells algorithm we will just recall some of the ideas that we have discussed in the previous lectures.

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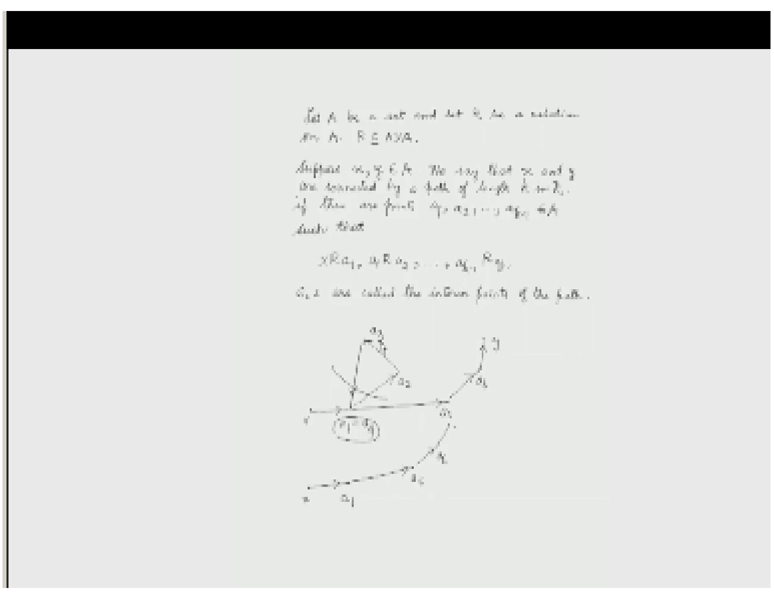

Now our starting point again is a set a and a relation are on a that is our is a subset of the Cartesian product of a with itself now we have already seen what we mean by a path connected connecting an element X to another element, element Y in a, we recall that that idea suppose x and y belongs to a we will say that x and y are connected by a path of length K in R we say that x and y are connected by a path of length k in r well if there are points a 1 a 2.

And so on up to a K-1 all belonging to a such that X related to a1 a 1 related to a 2 and proceeding soon up to a K -1 related to a, a k-1 related to Y why now what we have seen that when we are considering a path in general this path need not be through distinct elements that is to say the elements a eyes need not be distinct in fact.

These elements a eyes have a special name these are called interior elements of the path or interior points of the part ai are called the interior points of the path now these interior points as I have already said may not be distinct but what we realize is that we can always reduce any given path to a path quantanium only distinct interior points.

For example let us see this part that is X let us suppose it goes to a point a 1and then from Ava let us suppose we go to a 2 & 4 from a 3 let us suppose we go to a 4 and suppose this is not a 4 but let us call this a 3 right so from a - it will go to a three right then it goes to a four but suppose a 4 and a 1 are same so I have this and then suppose this goes to a point a 5 and then a 6 and then let us suppose we ultimately reach Y now here in the sequence of vertices a 1 and a 4 are same of course what we can do is that we can cut this loop out.

And we can have a path from X to Y as X going to a 1 then a 1 to a 5 then a 5 to a 6 and then to why now it is not difficult to see that proceeding in this way we can reduce any path to a path which contains only distinct interior points this fact has got some interesting consequences when the set a is a is a finite set now we have already seen that a transitive closure of a relation R.

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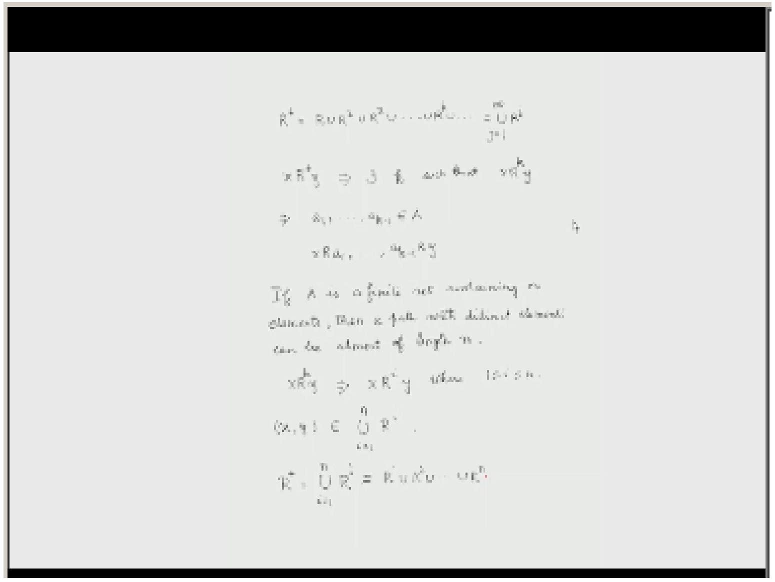

Which is denoted as R plus is essentially the union of R with R^2 then $R^3$and so on up to R^K but we do not end there we keep on going up to ∞ so in general it is an infinite union of our J where J starts from 1 and goes up to ∞ now this means that if we have two elements which are connected to each other by the relation R plus suppose now these two elements are XY then this means that there exists some.

Let us say K such that k such that X^{KY} now this in turn means that there are intermediate points a 1 up to a K -1 all belonging to the set a on which the relation is defined such that X are a 1 and so on up to XK - 1 are Y now this we have already seen now this means again that I can keep on reducing the number of interior points to the interior points which are distinct and then if the set is distant this is finite that is if, if a is a finite set containing n elements apart with distinct element can be at most of length n then a part with distinct elements can be at most of length n.

Now considering this fact we have already seen that X RR^{KY} will imply that X R^I of Y where 1 < I < to n and this will mean that this X Y this pair is inside the Union I = 1 to n of R^I this in turn means that our plus is a finite union of sets in case a has size n so R + I = 1 to n RI that is R 1 that is our Union R^2 Union and so on up to R^n now once we have understood this it is now clear to us that r+ will contents will contain pairs of elements of a which are connected to a path of maximum length n having distinct interior points.

Therefore if we can find all such elements which are connected then we have got essentially the transitive closure of our this idea is used in what shall L versus algorithm which we are going to discuss very soon but before that I will recall again the direct technique that we have seen that is to just get the matrix corresponding to our Plus which is essentially the matrix of our or matrix of R^2 according the special product that we have defined in previous lectures then MR^3 and.

So on and up to m r^n this sum of powers of course gives us the matrix corresponding to the transitive closure but the only problem is it is very difficult to compute these individual products and then take the, the or of all these products we will now move on to what sells algorithm but we will first see an example so let us now consider a particular set which is the set containing five elements.

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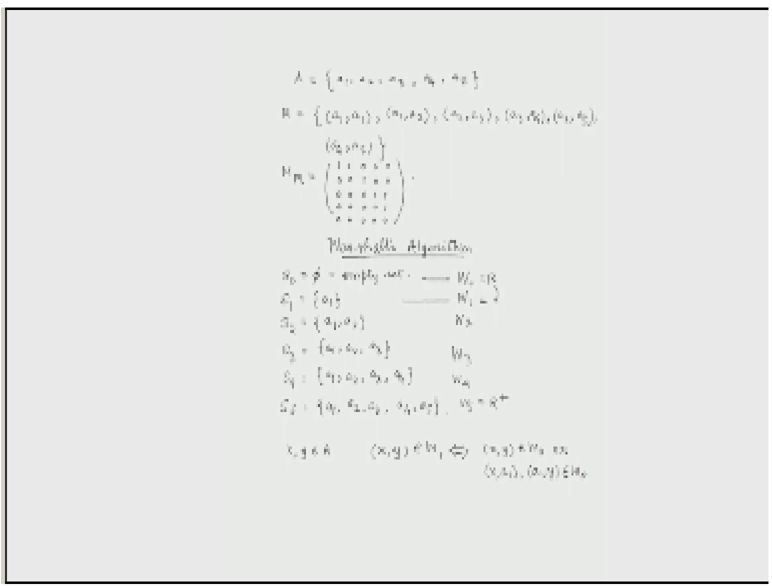

So this is a we name the elements let us say as (a1, a2) (a3, a4) and a5 right and we take a particular relation which is given by (a 1 , a 1)then (a 1 , a 2) then (a 2, a3)then (a 3 , a 4)(a 3 , a 5) and lastly we have another element (a 4 ,T 5) all right you check the matrix corresponding to our according to the ordering given in a then this matrix is of this type this is 1 1 0 0 0 then we have 0 0 then a 2 a 31 0 0 then we have 0 0 0 then 1 1 and lastly we will have 0 0 0 0 1 and the fifth row is all 0.

Because a 5 is not related to anything so we have all 0 5 throw in order to start warshall's algorithm we consider a sequence of subsets of a so the first one in the sequence is called S 0 which is the empty set the second one is s1 which consists of only one element a1 the second the third set is even a to the fourth that is S 3 is (a1 a2 a3) S 4 is (a1 a2 a3 a4) and lastly s5 is (a1 a2 a3 a4 and a5) now corresponding to these subsets we will construct relations.

So corresponding to a zero we have the relation w 0 which is same as R then corresponding to S 1 we will construct a relation which we will call w 1 which is well something that you will derive from R and in a specific pattern iteratively we will keep on defining new relations w3 and w 4 corresponding to s 3 and s 4and then ultimately we will get w 5corresponding to s 5 which we will claim to be R Plus that is the transitive closure of R now let us see how we get from W 0 to W 1.

Now we define in this way that consider two elements X Y belonging to a then this pair X, Y belongs to W 1 if and only if X , Y belongs to W 0 or (X , a 1)and (a 1, Y)both belong to W 0 this is very important so we must have a close look at it what I am saying here is that we are defining a new relationww1 from W 0 and what is a new relation so given a pair of elements x and y I should be able to say whether despair the ordered pair to be more precise.

Whether this order pair ordered pair belongs to the relation W 1 or not I will I will say that X ,Y the ordered pair belongs to W 1 if and only if X , Y belongs to the previous relation W 0 or it there is a path connecting X to Y through the set S 1 so this means that I have X over here and Y over here and there is a relation from X to a 1 that is X are a 1 and then xa1 ry and we know that a R and W zero are same.

So we have a path from X to Y if this happens or if x and y are directly connected through the relation R or W zero whatever we say then we say that it is in W 1 the question at this point is that how do we construct the matrix corresponding to W 1 based on the matrix corresponding to W 0 we see that the matrix corresponding to the 2 W 0 is same as the matrix corresponding to R that is M R so I can safely right over here that mw 0 is 1 1 0 0 0 0 0 1 0 0 0 0 01 1 0 0 0 0 1 and 0 0 0 0 0.

And then when I am constructing the matrix corresponding to W 1well I will put one at all the places where there is one in mw zero so I will put one in these two places and these are undecided these are undecided but of course this is one and this is like this and then I get these are undecided points but of course these are once and then here we will get one over here and in the last one I will write like this.

Now I have to decide whether to put 1 or 0and let us say this position that is first row and third column for that the corresponding elements are a 1 and a 3so I have to check whether I have a connection from (a 1 to a 1) and (a 1 to a3) so that means in the matrix M W 0 I have to see what is their entry of the first row first column so let us now write it as a symbol these entries as s IJ this is a five by five matrix.

And let us suppose by symbol I write this as TIJ five by five so I am to decide whether t 1 3 is 0 or not for that I have to check whether a 1 is related to a 1 and a 1 is related to a 3 now that means a 1 is related to a van is given by the entry S 1 1 if a 1 is related to a 1 then s 1 1 must be 1 and this a 1 is related to a 3 then S 1 3 ought to be 1alsobut we see that s 1 3 is not 1 but it is 0 so this is not correct is 1 C 0 therefore T 1 3 is 0.

So we see that we have another simple rule we do not have to really think much we just put the entries 1 wherever it is1 in the previous matrix and for the rest of the entries we just write like in this case the 1 3 is product S 1 1 s1 3 and in this case it is 0 therefore I will put 0 over here and if I go on like this I will see that I will get 0 in the other places as well so if we consider t1 4 according to my rule I have to only check S 1 1 and S 1 S 1 1 is 1but S 1 4 is 0 so it is 0.

So I will put 0 over here and then T 1 5 right so please see that T 1 5 is S 1 1 and S 1 s1 5 is this entry which is 0 therefore this is 1 times 0 so it is 0 so I have resolved this now if we go on in this way we will find the other entries to be zeros so I will put it put all of them to be zeros you can check, check that on your own my intention of doing this is that I would like to see what happens in mw2 so suppose now we have got mw1and I would like to construct mw2.

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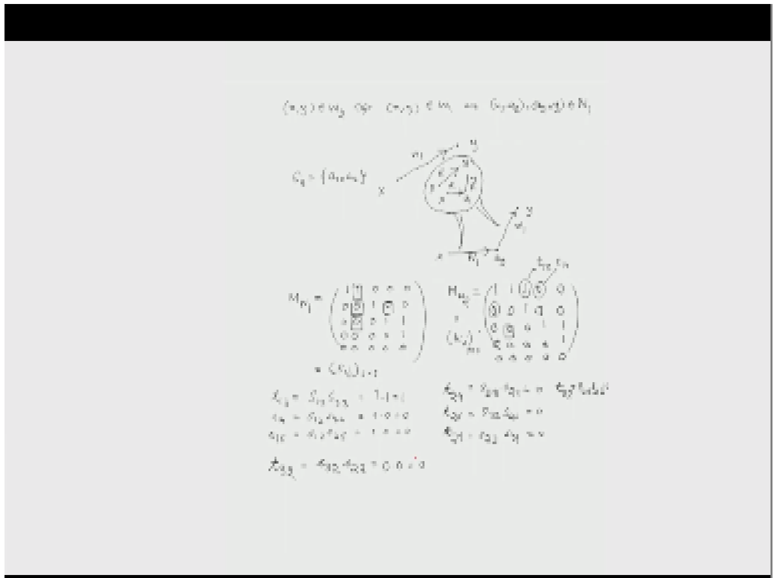

But first of all what is w 2 X, Y right belongs to W to if and only if X, Y belongs to W 1 or X, Y belongs sorry or now it is a question of intermediate point oh it is X , a 2 and a 2 , Y both belongs to W 1 so if we now look at the digraph corresponding to this so suppose I have got x and y so then in W 2 if there is a direct connection through W 1 so I do not have to worry about I do not have to worry about what happens in between.

But possibly either if we if we, we reduce it even further then there are two cases either you have a direct connection from X to Y or we have a connection from X to a 1 and a 1 to Y these are through the relation R so combining these two relations I will say that X is directly connected to W 1 and then the next possibility is that X is connected to a 2 through W 1 and then a 2 is connected to Y through W 1.

Now each of this each of these segments can be blown up to something like this right so you can have intermediate points but this basically means that X is connected to Y in such a fashion that the intermediate points of the paths are, are lying inside the set S 2 which is a1 and a 2 it is kind of straightforward but this is something that goes on over and over again and eventually when we come to WN then if two elements are in WN that means that.

They are connected through a path consisting off of the elements in a as interior points but well that is all about getting the transitive closure if we if we if we have a relation which connects two elements through that relation in whatever possible paths containing the elements of a then that relation is of course transitive closure because transitive closure is union of our eyes where I runs from 1 to M now.

Now let us see what happens when we when we want to construct the matrix corresponding to M matrix corresponding to w 2 that is mw2 but we have already got mw 1 well and let us write this matrix all right how we have this matrix and we would like to construct the matrix corresponding to mw - I am sorry the matrix corresponding to w2 which is MW -according to our rule we are quite safe if we put all these ones wherever they are right and in place of zeros.

Let us put small dashes these are the positions that we have to fill in and lastly we must not forget the last row that is all zero now we start from here this is right now t13 please note that it is also quite reasonable to change the S and T symbol now mw1entries will be referred to as is J's and mw2 entries will be referred to as TI J's all right so now we are interested in find out finding out t13well it is undecided.

Because in mw 1 it is 0 so let us patiently try to find out all these elements right so T 1 3 is now S 1 to S 2 3 why it is as 1 2 because I am now interested in path through the point a 2 so S 1 2 is 1 S 2 3 is also 1so it is 1 I will put 1 here the next entry is T 1 4 let us compute T 1 4 this is S 1 2 which is 1 s 2 4 S 1 2 is this very special element now which is 1 and which is going to appear in many places and s 2 4 is s 24 is the element here 24 which is 0 so 1 into 0 it gives me 0.

So I will put 0 over here and then in the next entryx-15 this is s 1 2 and s 2 S 1 2 is of course 1+ S 2 5 is zero so I get zero so I will write zero over here now we come over here this entry is S sorry this is this entry is T 2 for T 2 for is s 2 2into s 2 4 S 2 2 is 0 therefore 22 is 0 therefore I have zero now I write zero here and then I come to s- 5 which is s 2 2 and s 2 5 which is also 0 because s 2 2 is 0.

And then I come to tee off wait a moment here s2 for ya it is 0 sos2 4 this is s2 4 this is 0 and 2 5 this is also 0 but we forgot s2 1 which is s2 2 and s 2 1 which is of course 0 and we have t2 2 which is again s 2 2 into s 2 2 which is zero so I have got this as zero again if we use the same way if we compute the other, other entries we will see those are all zeros I am not doing that here but I live it has exercise please use this same way.

For example let us consider this element what is this element this is third row second column now that is this is t3 2what is the value of t3 to so I have to simply write s3 2 and s 2 2 now what is the value of S 3 2 if we now check the matrix we will see that S 3 2 is S 3 2this is this entry which is 0 S 2 2 is of course 0 so again 0 into 0.

So we get 0 next we take this matrix that is mw2 and move on to consider considering n mw3we see when we compare mw 1 and mw 2 we see that there is only one change that is the first row third entry has become1 and rest are all same so now I will write down mw 2 again and try to find out mw 3.

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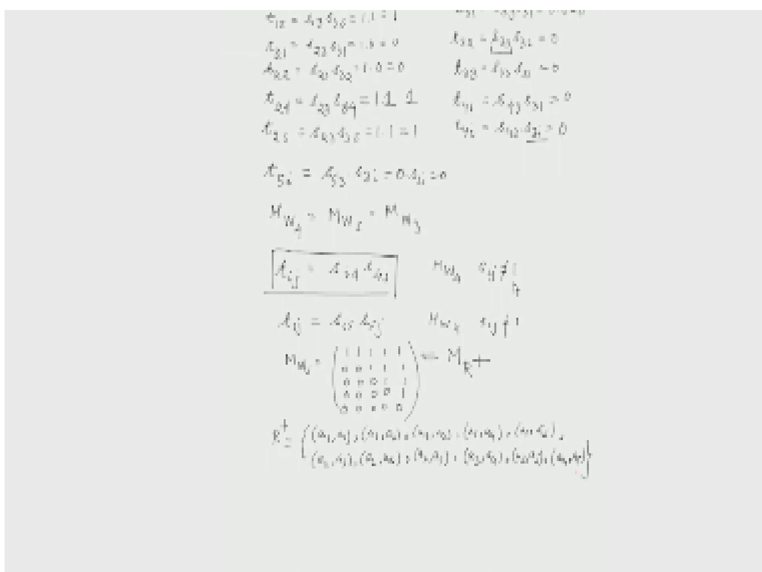

So mw 2 is 1 1 1 0 0 0 0 1 0 00 0 0 1 1 0 0 0 0 1 and the last row is all 0 now when we are considering mw 3well before that now change S and T is now these are - I J's of course 5 / 5matrices and we consider mw 3 well whose entries will be denoted by T I J's and I hope that we will see some action here because up to up to this point we are saying that very few entries are getting changed right.

So we, we hope that we should get something nice over here that a lot of lot of lot of zeros should become once so let us see whether it happens or not but first to start with we will put once he wherever there are ones so I will get like this and let us agree to put small dashes wherever we are undecided so we get something like this and again put thread ashes and then I give one over here and a 1 at the end.

And then we have got one two three four, four dashes and a one and the last row consists of all dashes now we will see what to do well we will start with this entry which corresponds to t1 for so let us let us start writing t1 for is s1 now what we have exhausted 1 2 and now it is 3 so s13 + s1 4sorry I am wrong here it is not as one for we have to consider parts which go through three so it is S three four.

So let me change this portion right so we get here S three four now what are these elements S 1 3well this is s 1 3 and s 3 4 this is s 34 both are once therefore 1 into 1 is 1therefore we put a 1 over here and then we have T 1 5 which is s 1 3 and s 3 5 now s 1 3 well we have we can lock onto this element for the time being is S 1 3we know that it is 1 and S 3 5 well it is a third row and we go up to this.

So it is 3 5 this is 1 again so 1 so 1 so it put1 over here I get a 1 here so the first row is complete we go one step below to the second row where we have T 2 1 which is s 2 3 and s 3 1 according to our agreement and now we search for s 2 3 in the matrix well this is the second row and we go up we locate the point here so this is s 2 3 we can safely target this one for the time being so s 2 3 is 1 and s 3 1 well s 3 1 is this one which is 0.

So it is oh so it is zero so we have got a zero over here and then s - t so this is s 2 3 and s 3 2 which is again 1 into s3 - see this is s 3 2 which is 0therefore we put a 0 over here so we get0 now we come to the to the other entries that is to the right of the 1 in the second, second row and let us see what happens over here we have got T 2 3which quickly gets translated so 2 s 2 3and s 3 3 now we search it is s 2 3 we have already locked it which is this one and.

This is 1 therefore you put a 1 and s 33 well s 33 this is 0 so I get 0 over here another 0 and by the way there is a mistake over here it is not 3 because 3 is already taken care of from the previous matrix so I will cut this off it will be in fact a changeover here I will cut this off right so we have got T to 4 this is a fourth entry so we have got T - 4 = s to 4 and S again is not S to 4 it is it is s 23 right.

So we will have s 2 3 and S 3 4 but that is something different because S 2 3 is 1 and S 3 4so we have to search here and we come to this point this is S 3 4 this is also 1so we get 1 so instead of 0 here we will get a 1 so we will get a 1 over here right so we go now to the element t25this element is s 2 3 and s 3 5 now s 23 as we have seen that it is 1 we have to search what is s 3 5 s 3 5 is this guy and this is also 1 therefore you have got 1 over here.

So we have got 1 over here and again we now look at the remaining elements remaining are the remaining dashes small dashes so what is this one this one is T 3 1 T 3 1 is s T3 1 is s 33 and s 3 1 now s 3 3 is 0 because that is this entry so it is 0 into s 3 1 S 3 1 by the way is also 0 so it is 0 so I put a 0 over here I know that it is 0 then 332 that is s33 and s 3 2 please see that I really do not have to worry about what is s t2 because once I have seen that s 3 3is 0 it is 0 so I put over here 0 and P3 3 which is again s 3 3 and then s 3 3of course this is 0.

So this is 0 now next we come to tea for one which is P for three know which is s for three and into S for 3 into s 3 1 so this is s4 3into s3 1 now what is s 4 3 is 4 3 is 0so I do not have to worry it's 0 and so it is 0 and whenever I am starting with T 4 and whatever it may bet 4 let us say T 4 I this is going to best 4 3 and s 3 I whatever is the value of s 3 I it is 0 because s 4 3 is 0 because s 4 S 4 3 right.

So I come here yeah s 43 is 0 so I do not have to worry it is 0 so that is why all these are zeros andcs5 3 is always 0 so we can in this context generalize a little bit so for example now we are interested in finding out the fifth row and so in general it will be p5i and that gets decomposed into s53 into s3 I but s5 three when we search over here this is the element s53 which is 0 therefore it is 0 into s3 I which is equal to 0.

So we really do not have to come compare each and every element in the last row we can put all 0over here now we can do this thing another two times and if you do that you will find that mw4 is same as mw3 our way of obtaining is same and same for mw5 all are equal to mw3 this is something that I keep as an exercise please see by using the same rule just remember that when you are trying to find out mw4 from mw3 put all once.

And wherever there are zeros for those entries the rule will be d ij = sI for S 4j in case of mw for when s i j is not 1 so if sij is not 1 then the corresponding in the next matrix you get t IJ do this and for mw for t IJ corresponding tij is tij is the entry of mw v now and sij of mw for use this idea right so this is for mw for when sIJ = 1 so we do like this so ultimately we will see that the matrix mw v is indeed 1 1 1 1 1 0 0 1 1 1 0 0 01 1 0 0 0 0 1 and then 0 0 0 all 0 and we claim that this is equal to m r plus.

And therefore this relation m r plus can be written as C this is r plus we can write from the beginning (a 1, a 1)(a 1, a 2)(a 1, a 3) (a 1, a 4)(a 1, a 5) then(a 2, a 3)(a 2, a 4)(a 2, a 5) then (a 3, a 4)(a 3, a 5) and lastly 1 element (a 4, a 5) and that is all this is the transitive closure of the relation R thus in this lecture we have seen by an example how to compute transitive closure of a relation by using what sells algorithm although we have seen a particular example you will see that it is fairly easy to extend it to a general case where a is a finite set a 1 up to a n because the F.