INDIAN INSTITUTE OF TECHNOLOGY ROORKEE

NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)

Discrete Mathematics

Module-06
Relations
Lecture-02
Properties of relation

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Today we will continue our discussion on relations we will see that the relation on a set on a set from a set to itself that we have defined have different particular properties and we can classify relations based on these properties now before going into all the properties let us look at let us recall the definition of a relation.

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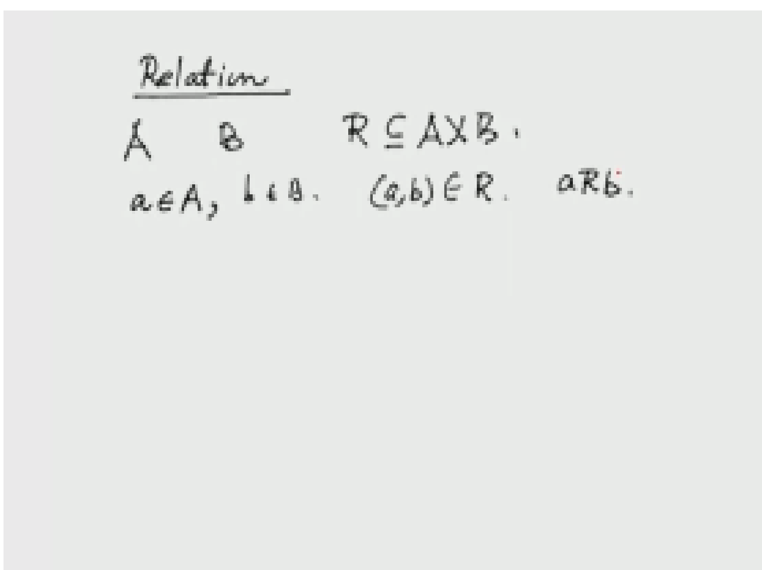

By relation from a set A to A set B we basically mean a subset of the Cartesian product of A subset of the Cartesian product of A by B so and any subset of A x B that is a Cartesian product of A and B is said to be a relation usually when we say that there is an element A ∈ a element a ∈ A and element b ∈ B they are related if by the relation R if the ordered pair a, b ∈ R and we write very often a R b.

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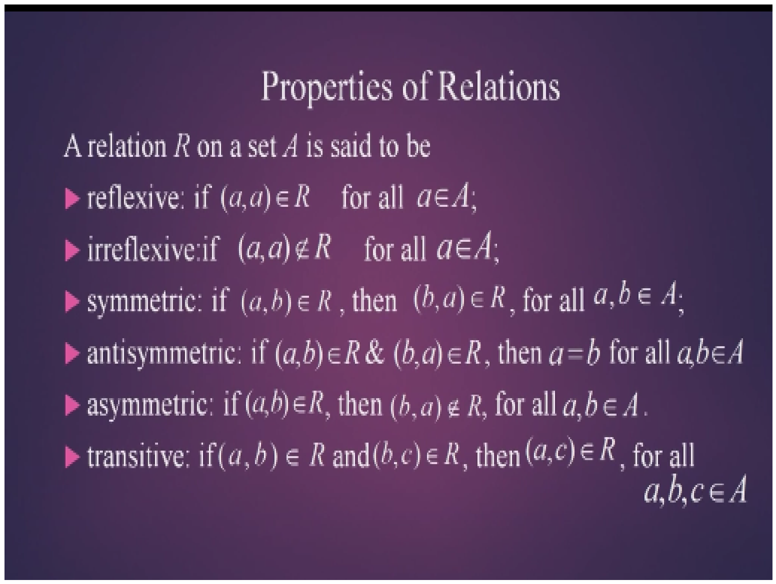

Now we go to particular properties of relations the first property in this context is called the reflexive property we say that a relation R on a set A again to recall that when we say that a relation R is on the set on A set a that means the relation R is from the set A to the set A that is R is a subset of the Cartesian product A x A now if R is a relation on A we call our a reflexive relation if a ,a ∈ R that is a is related to R for all a ∈ the set A let us try to check one example in this context.

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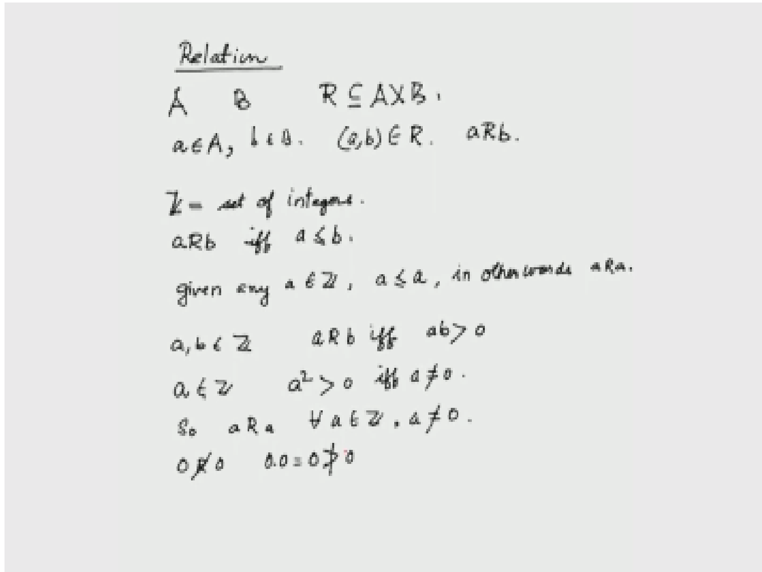

Let us see that let us consider the set Z the set of integers and consider the relation a R b if and only if a ≤ b so if a ≤ b where a b are elements of the set of integers we will say that a is related to b now we see that given any given any a ∈ Z a ≤ a in other words a is related to a and this happens for all a in the set of integers therefore ≤ relation is a reflexive relation now let us try to think about a relation which is not reflexive so let us let us take the same set Z and let us say that for a b ∈ Z a is related to b if and only if a into b is strictly > 0 please note that here we are specifying that a into b is strictly > 0.

So it cannot be 0 now let us consider the whether a is later to itself so take any a ∈ Z and consider the product of a with itself that is a^2 now a^2 > 0 if and only if ≠ 0 so a is related to a for all a ∈ Z except a ≠ 0 but when = 0 then a0 is not related to 0 since 0 into 0 = 0 is not strictly > 0 therefore we see that this relation is not reflexive because there is just one element in the set which is not related to itself.

Therefore we have to be very careful when we are checking for reflexivity of a relation we have to you have to check for each and every relation that each and every element that that L that element should be related to itself now let us check the next property now we say that what happens if a relation is not reflexive so if a relation is not reflexive then well it is it is something that it is possible that a is not related to a for at least some a inside the set on which the relation is defined but if it. So happens that a is not related to itself for all a ∈ a then we say that the relation is irreflexive that is a second property that we are considering here.

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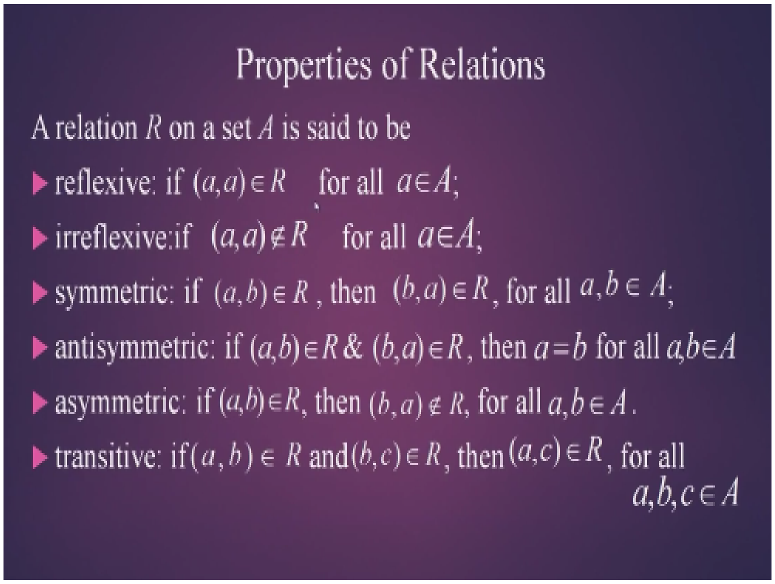

That is irreflexive if a, a not in R that is a is not in R for all a ∈ a now let us try to find out one example we can modify the relation that we were looking at again we check S we say that we define a relation let us say R.

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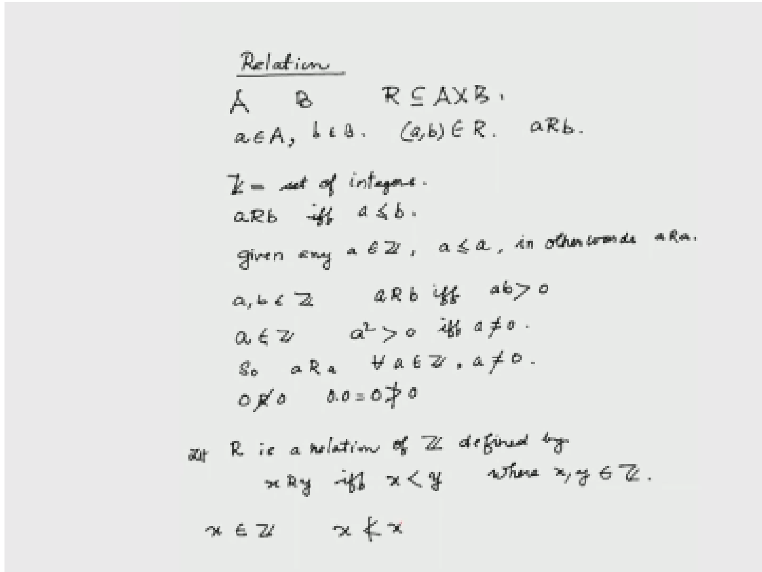

Okay and R is a relation on Z the set of integer defined by X related to y if and only if x is strictly less than y where x, y ∈ Z now take any x ∈ Z of course x is not strictly < x therefore this relation is irreflexive the next in line is the relation the properties symmetric we say that a relation is symmetric if a, b ∈ R ⇒ b, a ∈ R for all a, b ∈ a that is we consider the elements of a and if a for elements a b ∈ a and if it so happens that a is related to b ⇒ always b is related to a then we say that the relation is symmetric this is because this means that this when we are said there is a symmetry about the relation.

So suppose a related to b if we switch b and a that the statement will still be true if the relation is symmetric so let us look at example of symmetric relations.

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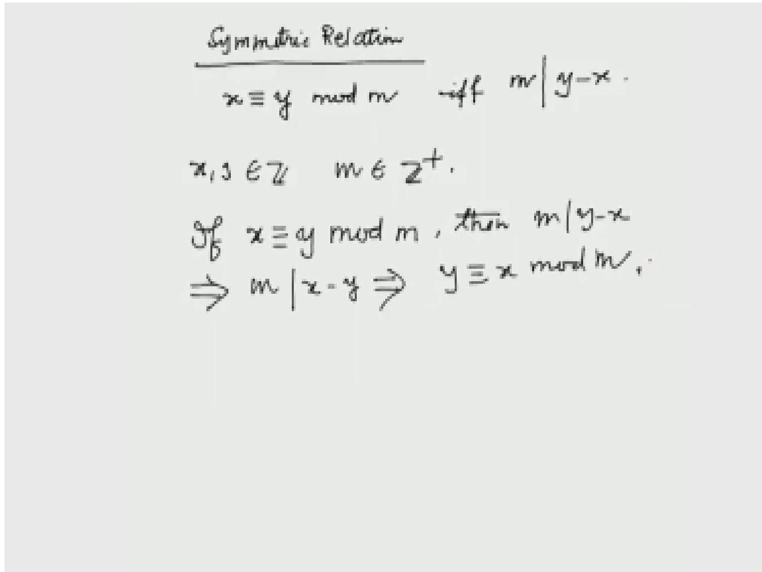

Now we have studied congruence modulo M relation so x is congruent to y mod M if and only if m | y - x this is the congruence modular relation where x, y ∈ Z and m ∈ Z + now we see that if x is congruent to y mod m then m |y - x which ⇒ m | x - y which in turn ⇒ y is congruent to x mod m therefore congruence modulo m relation is a symmetric relation on the set of integers.

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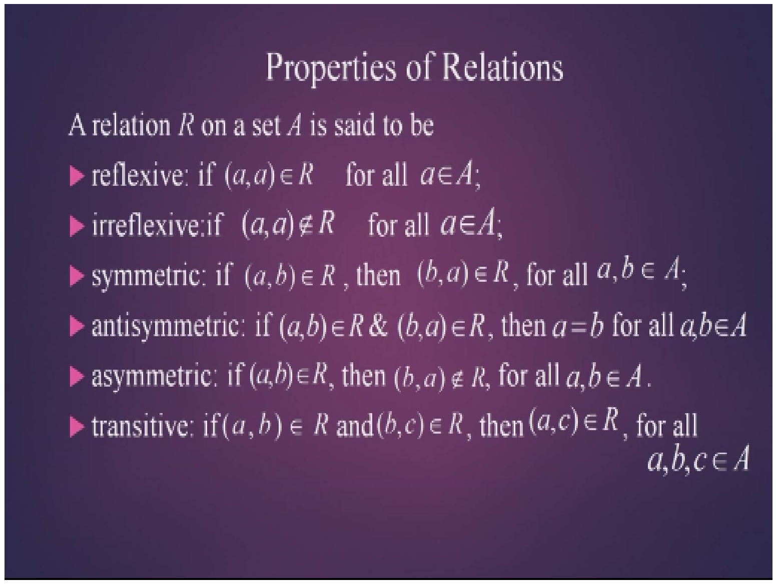

We move on to another property which is particularly important that is called anti symmetric property now a relation is said to be anti symmetric if a related to b and be related to a at the same time will mean a = b now let us look at an example of anti symmetric relation we again consider the set Z and define a relation that that is ≤ so the relation is defined as a ≤ b we know that these fields at a ≤ b if b - a > 0.

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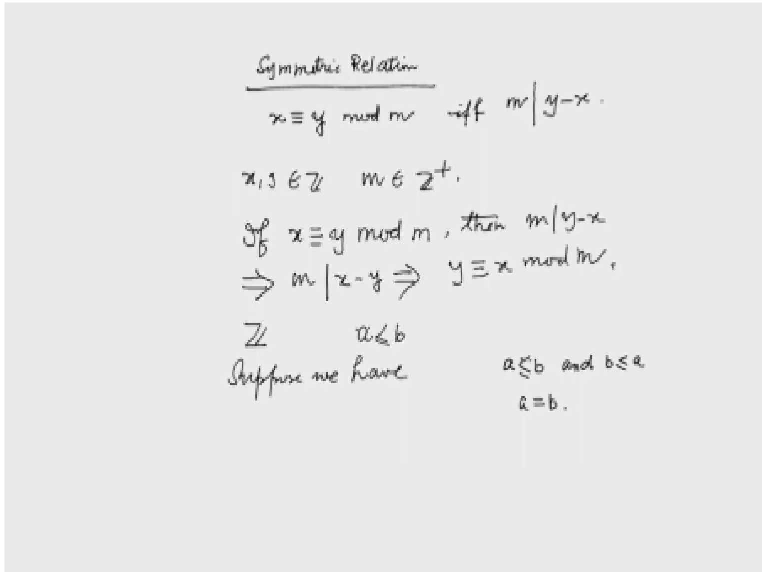

Now suppose we have a ≤ b and b ≤ a then of course a = b thus this is a anti symmetric relation the last property that we will study for the time being is called transitive relation.

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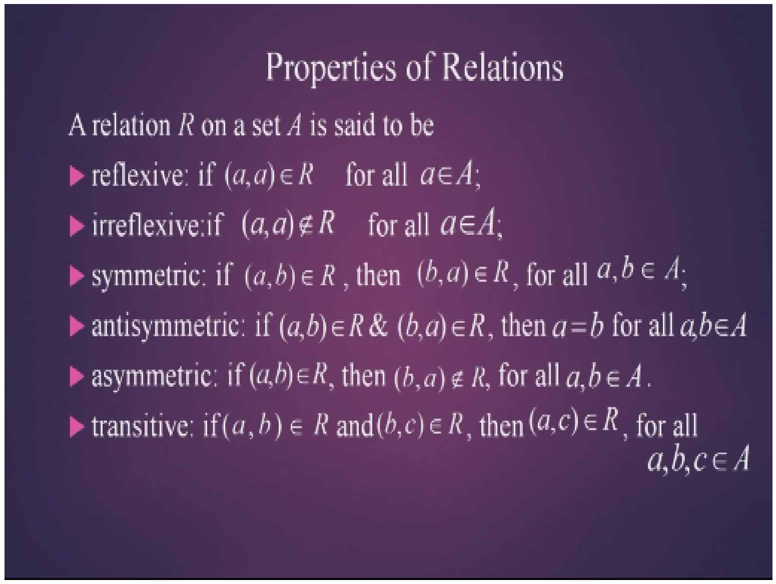

So it is like this that if a is related to b and b is related to c then a is related to c now let us look at examples of transitive relation.

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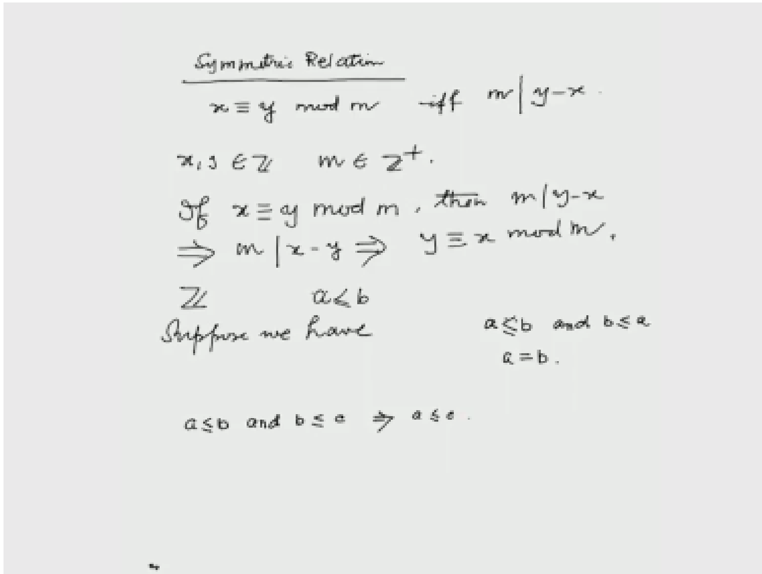

We can use the relation that we have already studied that is a ≤ b so we see that if a ≤ b and b is ≤ c this ⇒ a ≤ c thus it is a transitive relation we will see other examples of transitive relation soon now let us take some more some examples.

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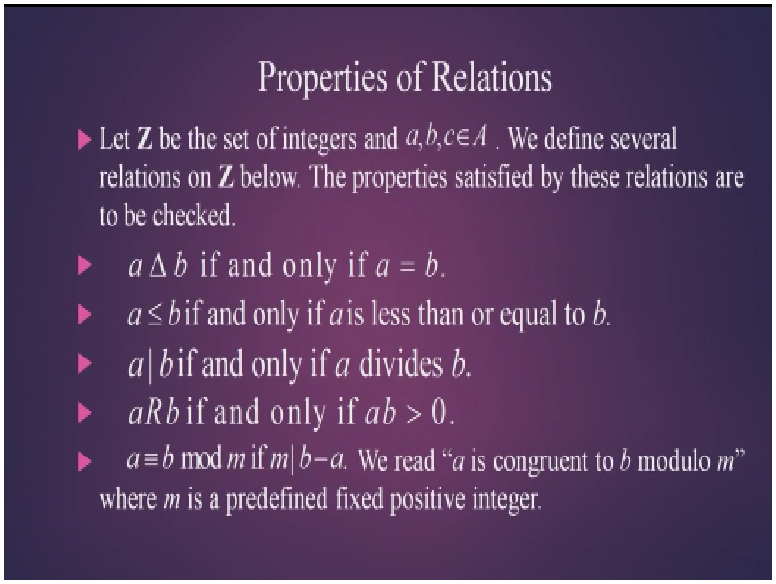

Now we are considering Z to be the set of integers and a, b, c are elements of Z there is a standard notation to define the equivalence relation that is this the inverted Δ so a inverted Δ this is a Δ right so this Δ a Δ b if and only if a = b so this is the Equality relation we can easily see that the Equality relation is reflexive because a is always equal to a for all a ∈ Z and then a it is symmetric because if a = b then of course b = a it is transitive because if a = b and b = c then a is = c and in fact it is also anti symmetric because if a = b and b =a of course a = a.

If a = b and b = a of course this means that a = b but of course we know that it is a = b the next relation ≤ relation that we have already seen a ≤ b if and only if a ≤ b and then we have the division a vertical a vertical line b if and only if a divides b we will be reading this symbol as a divides b and then another relation that we have already seen that is a is related to b if and only if a b is strictly > 0 and also lastly we have also seen the congruent modulo M relation.

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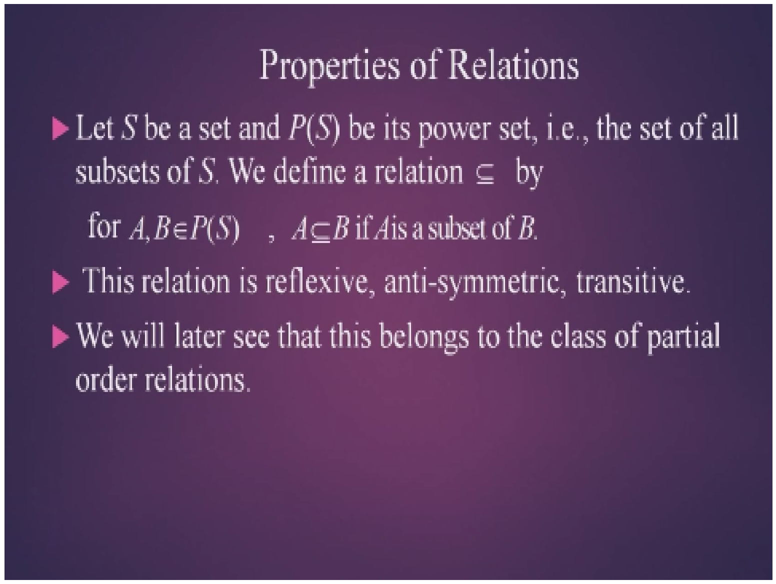

Now we come to the set containment relation suppose s is a set and P(S) is the power set of S and we will say we will define the relation which we denote by this sign if for a b ∈ the power set a contains b if and only if a is a subset of b this is a well known relation this relation is reflexive ant symmetric and transitive we will very soon see that this relation makes up an important class of relation called partial order relation.

Now we take up some particular type of relations the first one that is considered is called equivalence relation a relation R on a set a is said to be an equivalence relation if and only if.

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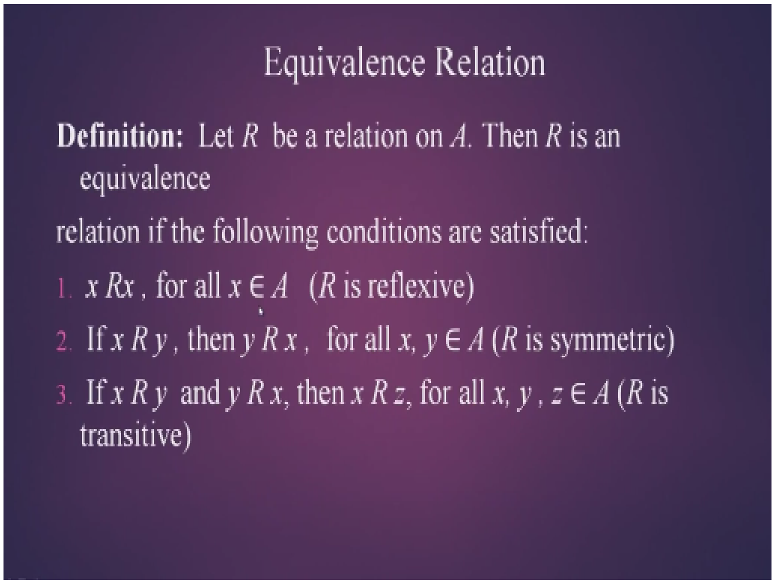

The following properties are satisfied x related to x for all x ∈ a that is r is reflexive x is related to y then y is related to x for all x y ∈ a that is R is symmetric if x is leaded to y and y is related to x then x is related to z for all x y z ∈ a that is R is transitive in other words if we have a relation which is reflexive symmetric and transitive then we call it an equivalence relation possibly the most famous equivalence relation is the congruence modulo m relation that we have studied in the previous lecture.

And we have considered the example just a while back so we are considering the set of integers and we say that x is congruent to y mod m.

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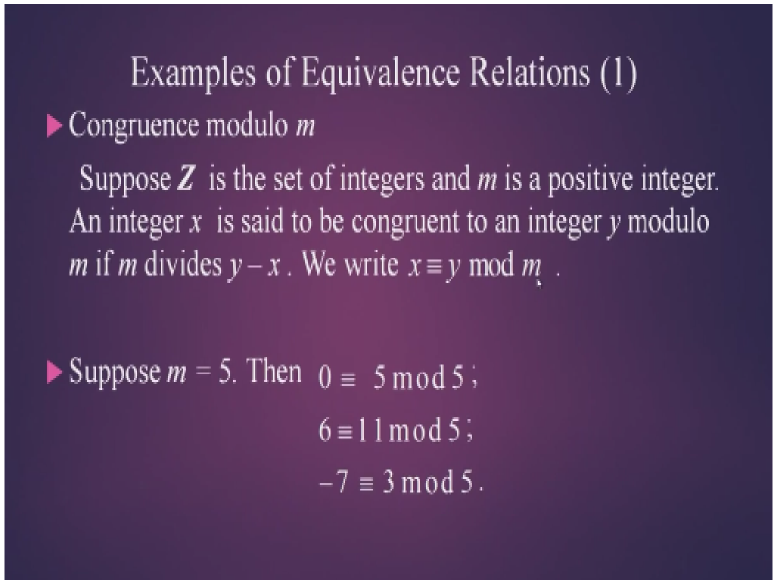

If y - x is divisible by m for example if we take m = 5 then we see that 0 is congruent to 5 mod 5 6 is congruent to 11 mod 5 - 7 is congruent to 3 mod 5.

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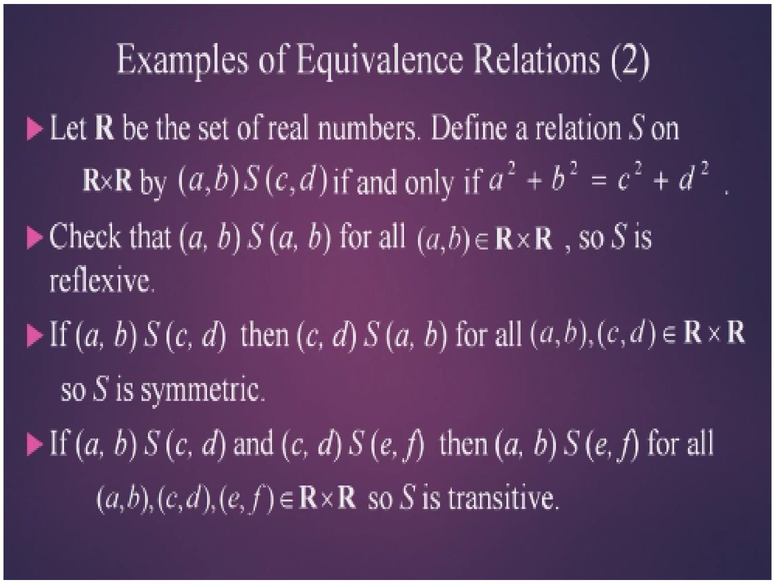

Now let us move on to another relation this is we consider the set of real numbers and we say we consider the Cartesian product of the set of real numbers so we are now in f in the real plane so our points are ordered pairs we say that an ordered pair a, b in R x R is related by the relation S to an ordered pair c, d in R x R if and only if $a^2 + b^2 = c^2 + d^2$.

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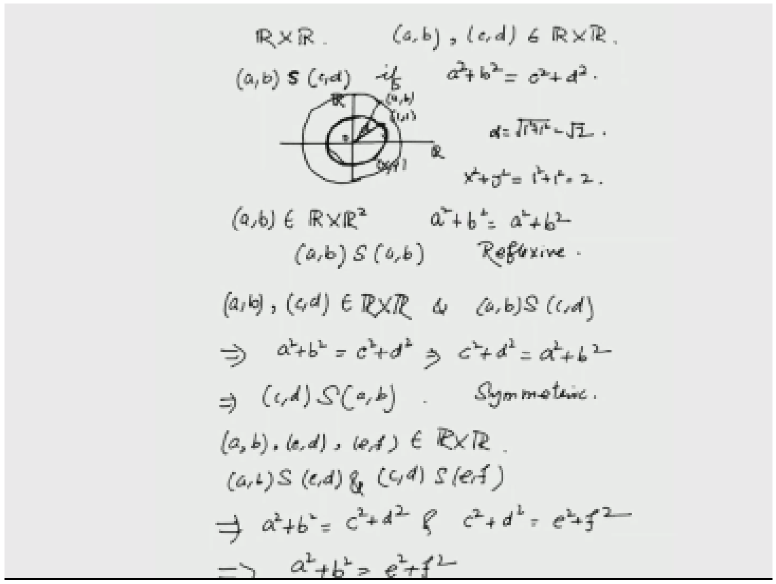

Now let us start checking what happens see we are considering R x R and for two elements a, b and c, d ∈ R x R we are defining them to be related by a relation let me yeah so they are defining by the relation S a, b S c, d if $a^2 + b^2 = c^2 + b^2$ what does it mean if we consider the plain R x R then we will see that suppose let us consider the point 1, 1 here then if we draw a circle around the origin o containing 1, 1 then all the elements that are in the circle that are on the circle has the same distance as that of 1, 1 the distance d here is $1^2 + 1^2$ it is √ 2.

So any element for any element on the circle let us say xy $x^2 + y^2$ has to be equal to $1^2 + 1^2$ that is equal to 2 so if you consider any other point let us say a b the all the elements related to a b will lie on the circle around the origin o.

Now let us try and start checking the properties we claim that given any a b ∈ R x R $a^2 + b^2$ of course trivially is equal to $a^2 + b^2$ which means that ab is S of ab therefore the relation S is reflexive next suppose a , b and c , d both belong to R x R and a , b S c, d comedy and they are related which ⇒ $a^2 + b^2 = c^2 + d^2$ this in turn ⇒ $c^2 + d^2 = a^2 + b^2$ which means that c, d is related by S to a , b.

So the relation S is symmetric now let us consider 3 elements a, b c, d and let us say e, f all elements of R x R and a, b related to c, d and c, d related to e, f and this together will mean that $a^2 + b^2 = c^2 + d^2$ and $c^2 + d^2 = e^2 + f^2$ combining these two equations we get $a^2 + b^2 = e^2 + f^2$.

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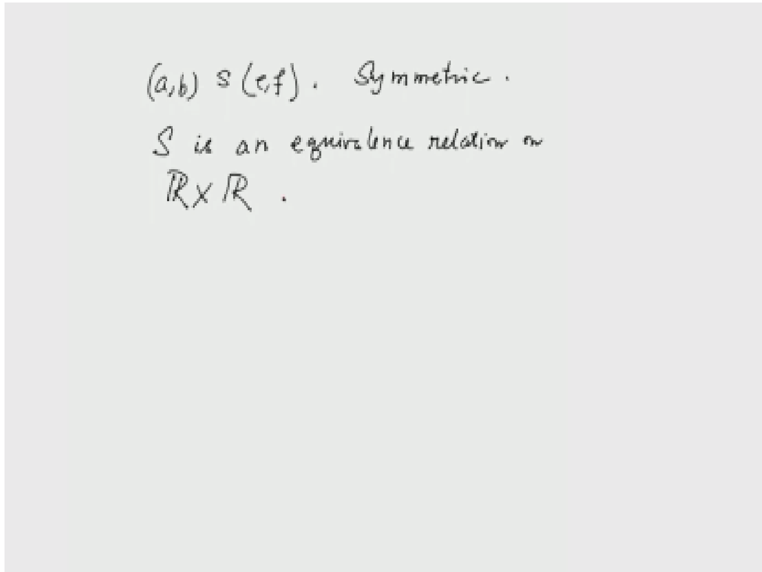

This will mean that a , b is related to e, f just like us go back we had here $a^2 + b^2 + d^2 + f^2$ that means you now have a, b is related to e, f this means that the relation S is symmetric thus S is an equivalence relation on R x R.

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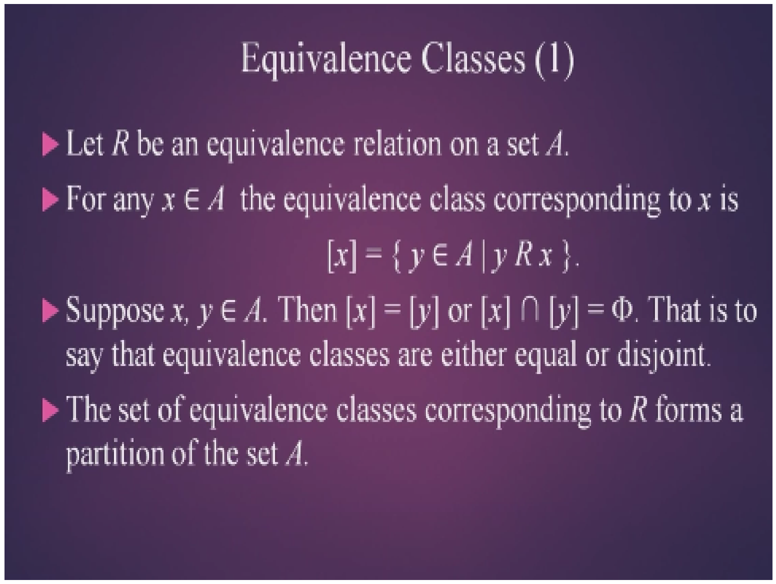

There is another concept which is intrinsically connected to the idea of equivalence relation is the concept of equivalence classes now if R is an equivalence relation on a set A we choose any x ∈ A and then we consider a set which consists of elements of A which are related to x so let us let us try to visualize that we have a relation on a set A and we are picking up one element from that set A set A let us call it x and we are considering all the all the elements in A which are related to x.

And we are building up a set which we will call the equivalence class corresponding to x now in this way we can keep on building equivalence classes of each and every element of a then we start questioning that how what is the relationship between different equivalence classes and then we come to a to an observation that if you pick up two elements from a set a then the corresponding equivalence classes are either equal or they are disjoint now this needs a proof so we will check how to prove this fact.

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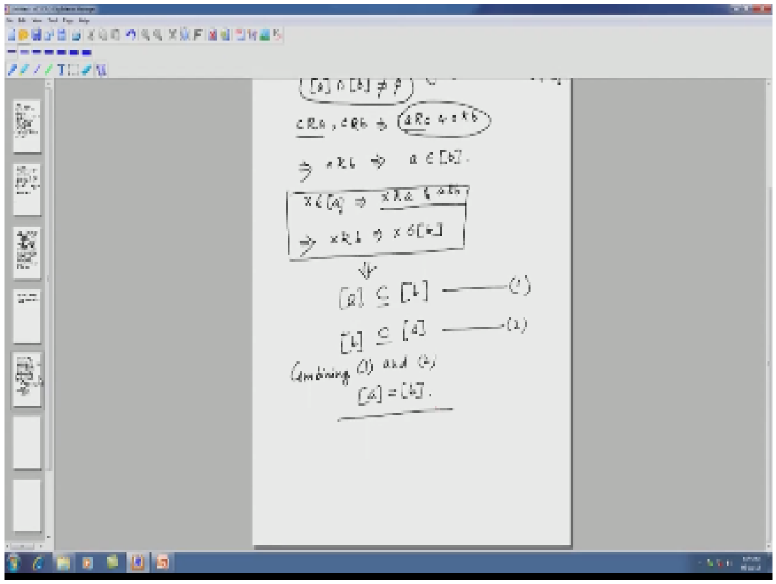

So let us let us recall x is the equivalence class corresponding to x so it consists of all y ∈ x on which x is defined such that y related to x now suppose we consider two elements let us say small a and b ∈ a and build their equivalence classes which is [a] and [b] now let us suppose that they have a non-empty intersection that is to say that the intersection of the equivalence class corresponding to a and the equivalence class corresponding to b is not empty therefore there exists a c ∈ A such that c ∈ A ∩ B.

Now this in turn means that c is related to a and c is related to C is related to b, now this implies that a is related to c and c is related to b, since R being an equivalence relation is symmetry therefore I can switch a and c they are symmetric series and related to a means he related to C and now we will use the transitivity of R because R after all is an equivalence relation and write that this fact implies a related to B.

But if a is related to b this will imply that a belongs to the equivalence class of b, so we see that if the equivalence class corresponding to a and the class corresponding to b have a non empty intersection then a is an element of the equivalence class of b, but then let us consider any element in the equivalence class of a suppose x is an element in the equivalence class of a which implies that x is related to a, and since we already know that a is related to b and we also know that R being an equivalence relation.

Is transitive we use the transitive property over here when we write x is related to b and x is related to b means x is inside the congruence the equivalence class of b and therefore this whole thing together implies that the class corresponding to a is contained in the class corresponding to b let us call it, one now we could just change the symbols a and b and then everything is will hold true and in exactly the similar way we can prove that is class corresponding to b will be contained in class for responding to a and combining one and two.

We have a equal to b because we see that if a and b the class corresponding to a and b has non-empty / then they are equal, so we come to our conclusion that these equivalence classes are either equal or disjoint fact that this proof is that the set of equivalence classes corresponding to an equivalence relation.

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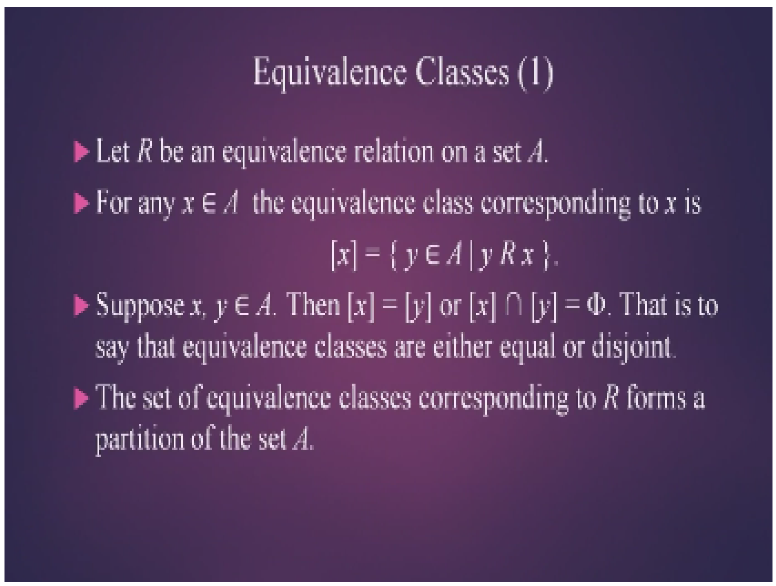

Are on A set a partitions are set for that we have to first recall what we mean by partition a partition on a set is A set of subsets of that set which are mutually disjoint and which covers the whole set that is for example, if we consider if we consider a let us look at the so let us consider he said A.

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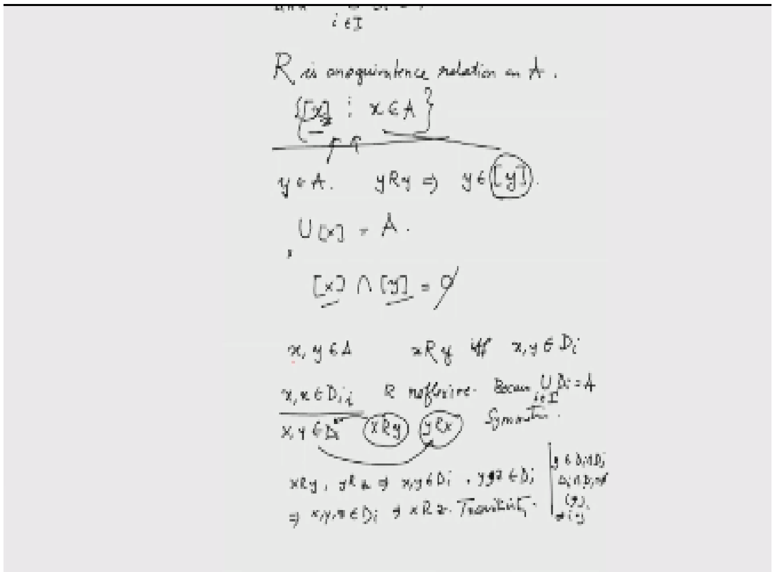

And P consisting of let us say some subsets of A where this I varies over some indexing set I will say we say that B is a partition of A if and only if Di / Dj is empty for ij belonging to I I not equal to J and U of Di when I varies over I gives us the whole set A, so suppose R is an equivalence relation on a consider the equivalence classes, so consider all that these joint equivalence classes so suppose you consider this set and we are only considering since it is a set only the no element repeat.

So we are this a set of subsets which are distinct and we know that distinct equivalence classes are disjoint we claim that this gives us a partition, because if we have any if we take any Y belonging to A then Y is related to Y which implies that Y is in the equivalence class of Y itself therefore for some 4 therefore this set Y must appear in the set, so this shows that if I consider the union of all the equivalence classes where X varies over all the disjoint equivalence classes then I will get the whole set a further.

If we have already proved that if we take any two equivalence classes either they are same in which case this X equivalence class corresponding to X an equivalence class corresponding to Y will give us only one entry over here otherwise they are disjoint, so they will they will they will contribute to different entries and therefore if we take all the disjoint equivalence classes we will get a partition on the other hand, if we are given a partition on a set a then it naturally gives us an equivalence relation.

Let us look at the partition that we have already discussed we can define equal equivalence relation in this way that suppose we consider x, y belonging to A we may say that x related to y if and only if x, y belonging to B_I, now we see that x, x belongs to Di because of course x is a single element, so this is trivially true so R is reflexive we again see that if x belongs to x & y belongs to Di then both things this will imply both things at x or y and y or x therefore x related to y means this which in turn of course means this.

Therefore we know that it is symmetric and further at the end if x related to y and y related to z this → x and y belongs to Di for sum I and y & z belongs to Dj for some j, but that means that let us see aside that means that y belongs to Di / Dj we know that Di / Dj is π, if I not equal to j which → that i = j so we can say that this implies that x, y, z belongs to Di which means that x is related to z therefore we have transitivity over here at this point I will again emphasize that given any x we will have x belonging to some bi.

Because Union I varying over capital I Di is the whole set A therefore this happens this is the property of partition that we are using thus we have given a proof of the fact that if we have a set and an equivalence relation on a set, it will generate a partition on the set and if we have a partition then that it will generate a an equivalence relation.

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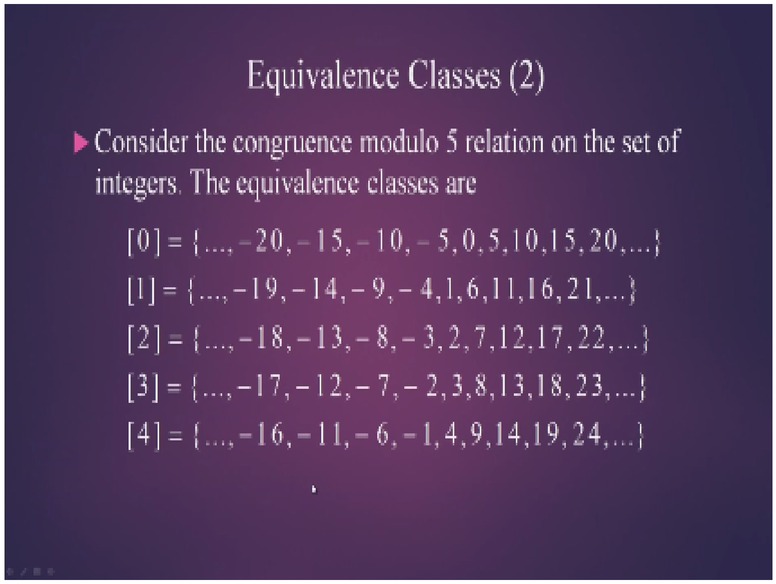

Now let us look at some examples of equivalence classes, now we go back to the example that we saw some time back which is here if you take m equal to 5 and we have congruence modulo 5 relation and we come over here now corresponding to congruent modulo 5 if we consider 0 let us see the equivalence class generated by 0, so we have got 0 5 - 5 10 - 10 15 - 15 20 - 20 and so on now if you want to find out the congruence class equivalence class related to 1 then we see that it is 1 then so instead of 0 it is 1 instead of 5.

It is 6 instead of 10 it is 11 and so on so this whole equivalence class that we have got corresponding to 0 we have to shift by 1 and we get this one shift by 2 we get this one shift by 3 we get this 1 and 4 and now we know that if we shift this by another one, so if we add one to all the elements will go back to the congruence class corresponding to congruence or equivalence class corresponding to 0, so these are the equivalence classes corresponding to the congruence modulo five relation these are also sometimes called congruence classes.

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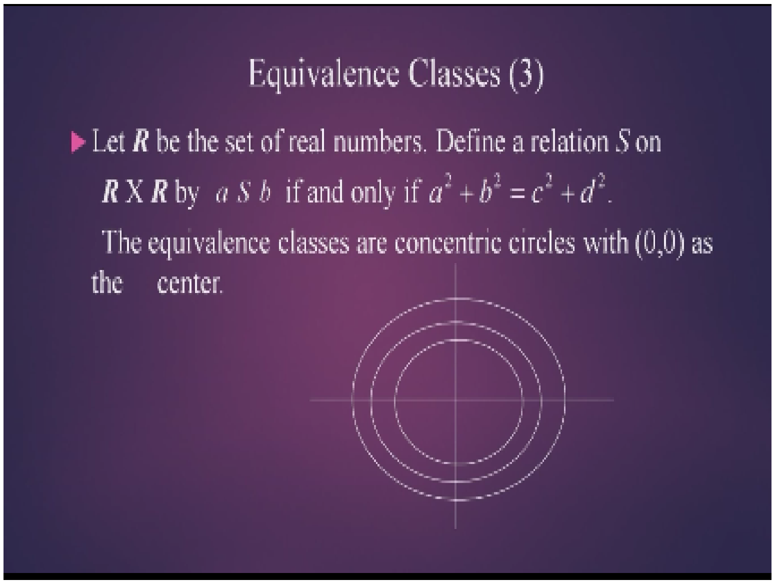

Now we look at the other relation that you considered so the equivalence classes here are concentric circles around the origin, so who will have infinite number of concentric circles around the origin which are the equivalence classes corresponding to the equivalence relation S which is to recall that ab, cd are related by S if a $b^2+b^2 = c^2 + d^2$, so this is the end of this lecture so thank you.

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