INDIAN INSTITUTE OF TECHNOLOGY
ROORKEE

NATIONAL PROGRAMME ON TECHNOLOGY
ENHANCED LEARNING
(NPTEL)

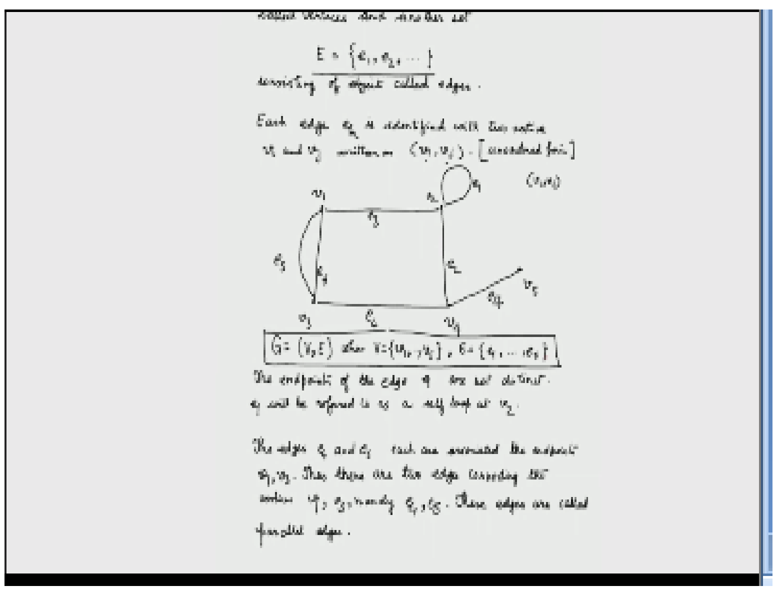
Discrete Mathematics

Module-05
Graph theory
Lecture-01
Basic definition

With
Dr. Sugata Gangopadhyay
Department of Mathematics
IIT Roorkee

In today is lecture we will start discussions on graph theory.

(Refer Slide Time: 00:48)



Now the question is what is a graph? A graph is an ordered pair of sets the first set is called the set of vertices and the second set is called the set of edges, however the set of vertices and set of edges are not independent. Every edge corresponds to 2 vertices which are called the end points or the end vertices of these two this of that edge. Let me write down the definition formally a graph consists of a set of objects V and the objects denoted by V V_2 and so on called vertices.

And another set denoted by e usually consisting of stages consisted consisting of objects which we denote by e_1, e_2 and so on consisting of objects called edges. Now what is the connection between an edge and vertices, each edge e_t is identified with two vertices V_I and V_J will be writing this fact by, now here we have to be careful and mention that this pair is unordered that means we do not consider the order as important here, so it is called unordered pair.

Later on we may introduce ordered pair and then instead of just edges we will have directed edges and instead of just graphs we will have directed graphs all digraphs but we are not considering those type of graphs right now, we are considering just graphs or undirected graphs and therefore an edge is associated to a pair of vertices, in fact we will let us see that it does not mean that the vertices has to be distinct.

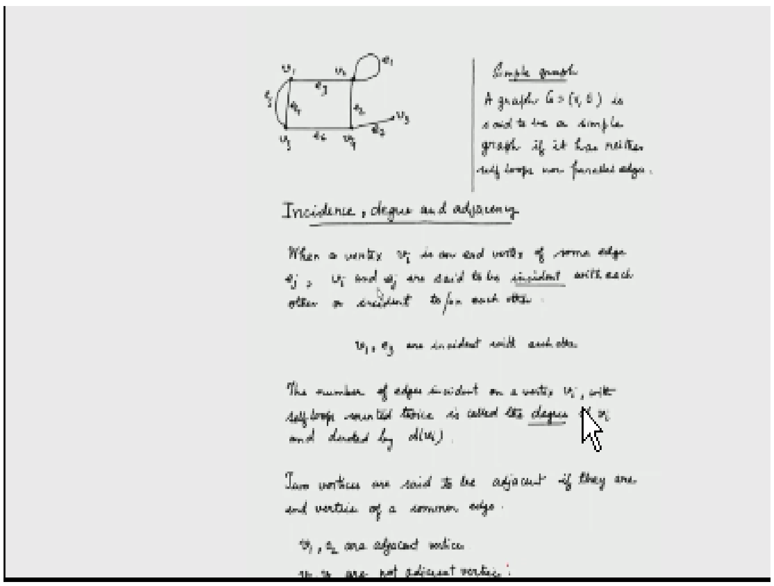
So what happens if the two vertices are same we will discuss that also, so it will associate to two vertices which need not be distinct. So let us look at an example of a graph, so we have got vertices which we write as points on a plane let us name these points call them v_1 v_2 v_3 v_4 and v_5. Now we start writing the edges the edges e_1 is in a very interesting because the vertices corresponding to e_1 are not distinct e_1 corresponds to the pair that are given, it is called a self-loop.

So the endpoints of the edge e_1 are not distinct, e_1 will be referred to as a self-loop at v_2, now where is e_2 e_2 is a is an edge like this then e_3 is this e_4 is this now we come to another phenomena if e_5 now we see that we have drawn two edges in between V_1 and V_2 V_3 it is also possible because when I am talking about the set of edges and edge e_k is associated to end vertices or end points let us say V_I V_J and there may be another some other edge associated to the same unordered pair of vertices.

These two edges will be called parallel edges, so we write this as the edges e_4 and e_5 each are associated to the endpoints V_1 V_3 thus there are two edges connecting the end points or vertices v_1 v_3. Thus there are two edges connecting the vertices v_1 v_3 namely V_4 V_5 these edges are called parallel edges. Now let us look at let us draw some more e_6 and then e_7 thus we have got here an example of a graph the type of objects that we are going to consider in this lecture and in some subsequent lectures.

Now we can take this the whole thing together and denote this graph by an ordered pair of sets let us call it G V e where we consists of the vertices V $_1$ up to V $_5$ and E consisted consists of the edges even up to e $_7$ this is our graph. Now we will consider the same graph and discuss some more terminologies which are related to graphs according to our definition this is a graph we can restrict our definition a little more and talk about simple graphs.

(Refer Slide Time: 14:03)



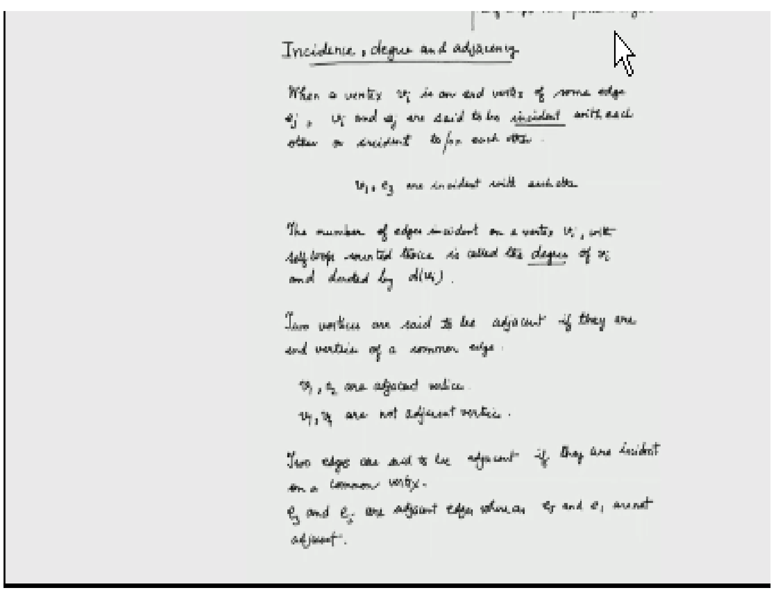
A graph G v_e is said to be a simple graph, if it has neither self loops nor parallel edges, sometimes we will prove certain results related to simple graphs but for the time being we consider the graphs which may have parallel edges or self loops. Now we talk about two important concepts related to vertices and edges, namely incidence and adjacent Z and degree.

Incidents degree and adjacent c when a vertex V_I is an end vertex of some edge e_j V_I and e_J are said to be incident with each other or incident to/or on each other. Now let us look at the graph that we have here and see some examples of vertices and edges incident with each other. If we consider v_1 the vertex v_1 and the edge e_3 they are incident with each other or we can say that e_3 is incident to V $_1$ or incident on V $_1$ or we can say that V $_1$ is incident to or incident on e_3.

Similarly we can say that V $_1$ is incident with e $_5$ we can say that V $_1$ e $_4$ our incident with each other and so on. Now it is not difficult to see that if we are given a graph and we can pick up then in that case we can pick up each vertex and check how many edges are incident on it. Now that number is called the degree of that vertex, the number of edges incident on a vertex V_I width self loops counted twice is called the degree of V_I and denoted by D V_I.

So we have talked about incidence we have talked about degree now the question is what do we mean by adjacent vertex vertices or adjacent edges? Two vertices are said to be adjacent if they are end vertices of a common edge. Now again if we look at the graph above we will see that $V_{1,}$ v_2 are in are adjacent to each other because v_3 is common to them, v_1 v_2 are adjacent vertices but if you consider v_1 and v_4 they do not have a common edge therefore they are not adjacent V_1 V4 are not adjacent vertices, next we talk about adjacency of edges.

(Refer Slide Time: 22:50)

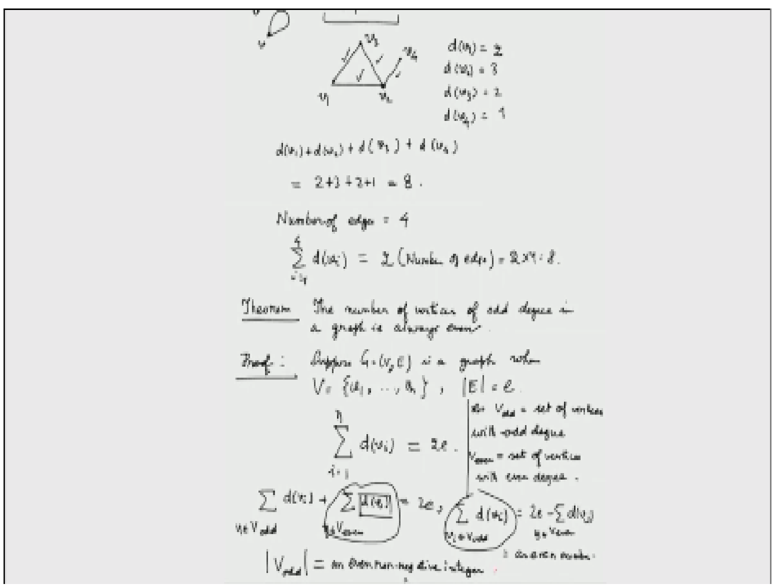


Two edges are said to be adjacent, if they are incident on a common vertex, so if we again look at our diagram we will see that the vertex each, the edge e_3 and e_5 are inside are adjacent because they have a common endpoint or end vertex V_1 e_3 and E_5 are adjacent edges, whereas e_5 and e_1

are not adjacent. At this point we have to remember another thing that when we talk about degree of a vertex we have to consider the self loops are contributing degree two to each verte.

The idea essentially is this that supposes we are at V $_2$ over here and we have got a self loop e_1. So it is coming out of V $_2$ in two places or coming in two places that mean, the edge e_1 is incident on V $_2$ twice and therefore because both the ends are incident on V $_2$. Therefore when we are counting the degree of V $_2$ even will contribute to and as we see from the diagram e_3 and e_2 will contribute one each, so the degree of V $_2$ is going to be 4 and not, next now we would like to see what happens if we sum all the degrees of all the vertices of a graph.

(Refer Slide Time: 26:54)



Let us consider a graph with e edges and n vertices, let us denote the vertices by v_1 v_2 and so on up to V_N. Now if we sum the degrees of all the vertices we get a sum like this d v_i-i = 1 to n. Now the ^{0} of a vertex is the number of edges incident on it, therefore each edge for each edge we will get ^{0} 1 the contribution of 1 to the ^{0} of the vertex and for self loops we will get 2 and the edge that is incident on a vertex.

Let us see like this suppose I have a vertex V and the edge E is incident on it now this edge will have another end point let us call that V/ and so this edge will contribute another ^{0} to V` in case this edge is a self-loop then this will come back to V itself and therefore it will contribute ^{0} to the ^{0} of V because of this we can say that each edge is going to contribute to this sum and therefore the sum is equal to twice the number of edges.

We can check this by examples, let us look at one small graph like this so let us call it v_1 v_2 v_3 and before, now ^{0} of V one = two ^{0} of V 2 is = 3 degree of V_3 = 2 and ^{0} of V_4 =1. Now if I add all of them degree of V_1 + ^{0} of V_2 + ^{0} of V_3 + ^{0} of V_4 = 2 + 3 + two + 1which gives me 8 and now if I count the number of edges this is 1, 2, 3, and 4, so the number of edges =4.

So the sum of ^{0} I = 1 to 4 D V_I = 2 times the number of edges, that is 2 times 4 = 8, this rather simple observation has another interesting consequence and that is the first theorem on graph theory that we are going to prove. So we have this theorem this theorem states that the number of vertices of odd degree in a graph is always even. Now the question is why and that is what we will do when we check the proof.

Now suppose that I have a set of edges and number of edges n, suppose G = V ,E is a graph where V = V_1 up to V_N and the number of elements in E is small e, now we already know that if I sum up all the degrees of the vertices σ $_I$= 1 to n dV_I is going to be 2 times of e. Now what we observe that in capital V that is a set of vertices there are two types of vertices, the vertices which has got odd degree and the vertices which has got even degree.

Now let us write over here, let V_{odd} = the set of vertices with or ^{0} and V_{even} is the set of vertices with even degree. Now if we have these then the sum that we are considering can be split up into two partial sums, one sum dV_I where this is over V_{odd} and the other some is again dv_j where this is over V even. We can write all V_I belonging to V_{odd} and all V_J belonging to V_{even} and this is equal to 2 times e.

Now at this point we realize that, at this point we realized that this term is an even number the reason is that each of dv_j by definition is even because these degrees are even and if we sum up even numbers no matter how many of them we are going to get an even number and therefore we will have σV_I belongs to V_{odd} $D(V_I)$ = 2 e - σ V_{even} D of V_J V_J belongs to the e_l this is equal to an even number.

Now since it is an even number then the sum d V_I where V_I varies over V_{odd} is going to be even an even number but we know that individual dv_i are odd. Now if I add some odd numbers odd number many times then I am going to get an odd number, so the sum the number of terms in the sum here has to be even and that means that the cardinality of V_{odd} is going to be and even positive.

I should write an even integer non-negative integer, because of course it can be zero, this completes the proof of the theorem, so what we have proved that no matter whichever graph I take and if I compute the degrees of the vertices then we will see that number of vertices with odd degree will always be even, with this we come to the end of today is lecture thank you.

Educational Technology Cell
Indian Institute of Technology Roorkee

Production For NPTEL
Ministry of Human Resource Development
Government of India

For Further Details **Contact**

Coordinate, Educational Technology Cell
Indian Institute of Technology Roorkee
Hoorkee-24/667
Email:etcell@iitr.ernet.in,etcell.iitrke@gmail.
Website: www.nptel.iim.ac.in

Acknowledgement
Prof pradipta Banerji
Director,IIT Roorke

Subject Expert & Script
Dr.Sugata Gangopadhyay
Dept of Mathematics
IIT Roorkee

Production Team
Neetesh Kumar
Jitender Kumar
Pankaj Saini
Meenakshi Chauhan

Camera
Sarath Koovery
Younus Salim

Online Editing
Jithin.k

Graphics
Binoy.V.P

NPTEL Coordinator
Prof.Bikash Mohanty

An Educational Technology Cell
IIT Roorkee Production