INDIAN INSTITUTE OF TECHNOLOGY ROORKEE

NATIONAL PROGRAMME ON TECHNOLOGY ENHANCED LEARNING (NPTEL)

Discrete Mathematics

Module-05

Set theory

Lecture-06

Planar graphs

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In this lecture we will discuss planar graphs and Wyler's criterion.

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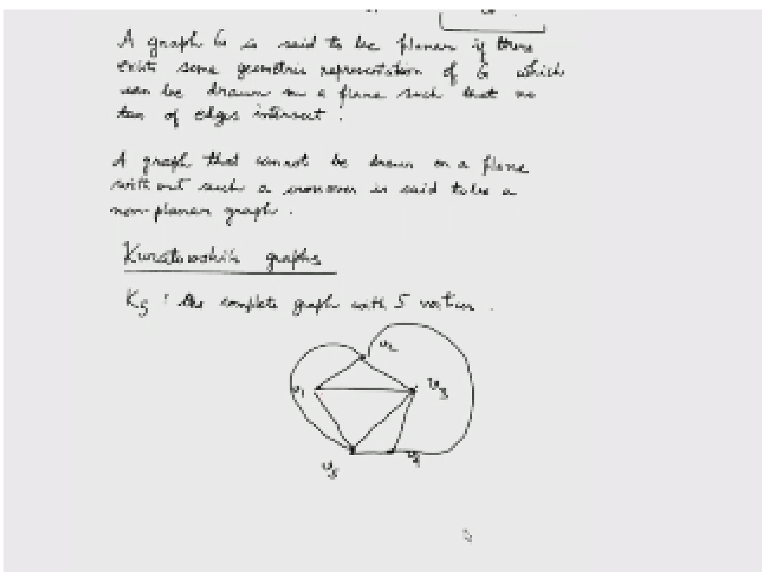

To decide whether a graph is planar or not now so far we have talked had talked about graphs and we have been drawing diagrams of graphs on a plane however we have not checked the issue that weather. I can draw a graph on a plane without its edges overlapping on each other for example I can think of a graph like this where these are the vertices and I have got two edges crossing each other on the other hand.

I could have drawn it on the plane in almost same way just by drawing an edge like this now we see that this graph G and the other one G ′ both are essentially same although if you look at this graph no edge is overlapping with another. I mean cutting another now our question is that when a graph can be drawn on a plane in this way and when it cannot be drawn in the in this way so we come to the formal definition of linearity of a graph we write a graph G is said to be planar if there exists some geometric representation of G.

Which can be drawn on a plane such that no two of its edges intersect a graph that cannot be drawn on a plane without such a crossover is said to be a non planar graph now the question is that can we create some examples of non planar graphs in this context there are two famous graphs called lot of cutoff skis two graphs. Which are shown to be non planar the first graph is the complete graph with five vertices or written as K5 and the second graph is written as k 3, 3 which is a bipartite graph a complete bipartite graph with parameters 3, 3.

So the complete graph with 5 vertices looks like this we have got 5 vertices v1 v2 v3 v4 v5 we have got a cycle here and then we have to connect v1 and v3 we connect that we have to connect v 3 and v 5 we connect that now we have to connect v 5 and v 2 we connect that and we have to connect v3 v2 and v4 we collect that and now we have to connect v4 and v1 and now we see that neither can we go in this direction or in this direction or nor in this direction without cutting one of the edges.

So I can write probably like this and like this and we know that there is an ∩ over here so we see that we are unable to draw a complete graph with five vertices without intersecting the edges so I will draw it again , so let us see the graph again we have got v1 then v 2 we draw like this is v2 then we draw like this is v 3 and then we come back to here this is with 4 then we have got v 5 and we join v 5 and v 1 and now v 1 and v 3 our joint then v 2 and v 5 are joint then v2 and v4 has to be joined.

We can join like this and then we join v3 and v5 and then we see that suppose I have to join is since we have to join before and v 1 we have no other way other than cutting or intersecting one of the edges so we may write like this or whichever we go we will intersect the edges so this is cut off skis first graph and let us look at the second graph which is as I said that a complete bipartite graph.

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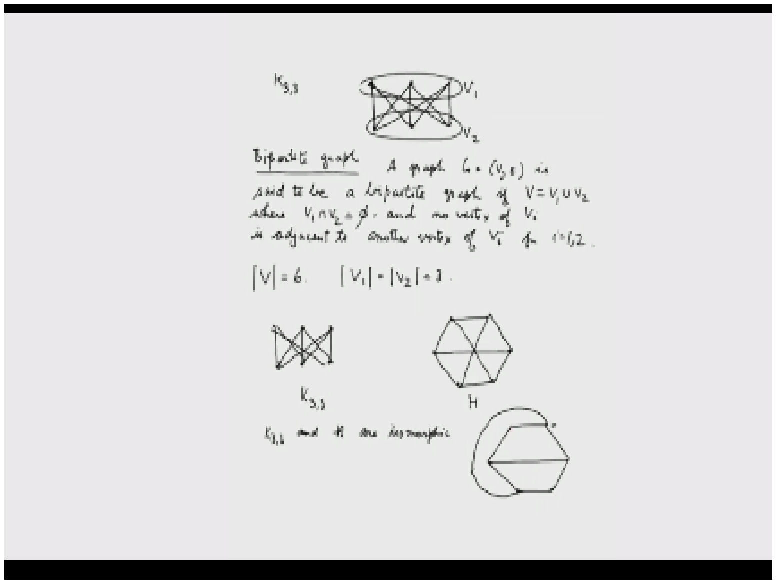

K 3 ,3 and K 3, 3 it has got the set of vertices aspartic what he says is partition into two subsets each of three vertices and then vertex from each vertex from one subset is connected to vertices of the other subset, so I will have like this then here like this and then like this in this context we define bipartite graph in general graph a graph G v e a graph G v, E is said to be a bipartite graph if v is partitioned into two subsets v 1 and V 2 where V 1 ∩ V 2 is empty and all the edges are connecting vertices of v1 to vertices of v2 and there is no edge connecting vertices of v so we write that where v1 v2 is the empty set.

And no vertex of vI is adjacent to another vertex of v I for I is 1 & 2 so the vertices of v1 are not adjacent to each other what is it so v2 are not adjusting to each other so now loop then we look at this graph what we denote as K 3, 3 we see that the three vertices over here are not adjacent to each other and the three vertices over here are not adjacent to each other whereas there are edges from one subset to the other now this graph k3t is something more here that cardinality of the set of vertices is 6 and cardinality of v 1 = cardinality of v 2 = 3 and if we note that each vertex of v 1 is connected or is a and adjust to each of the vertices of v 2.

And converse so we have K 3, 3 is a complete bipartite graph we draw it again now what we would like to prove here is that k33 is not planar now it is not difficult to check that k 3, 3 is isomorphic to a graph like this so this I leave for exercise that these two graphs are isomorphic so if I call this K 3, 3 and this simply H K 3, 3 and H are isomorphic. Now if I start drawing H all over again then here then we will start drawing the cycle over here and then. I can of course draw this one and then see that we can choose to draw this edge from the top.

But then if I want to draw an edge from here to here then I am forced to intersect over here this is the place where the ∩ will occur so this graph also looks like a graph which cannot be drawn on a plane but the proofs that, I have given right now are our intuitive proofs and we would move on to proof to approve which is more analytic for that we would like to have more definitions so first of all we would like to mention that whenever we have a simple planar graph.

We can embedded it on a plane by using only straight lines we do not have to have any crooked lines to embed a planar graph if it can be embedded it can be embedded by using a straight line using straight lines we do not give a proof of that but it is very intuitive and after that we define a very important concept called regions.

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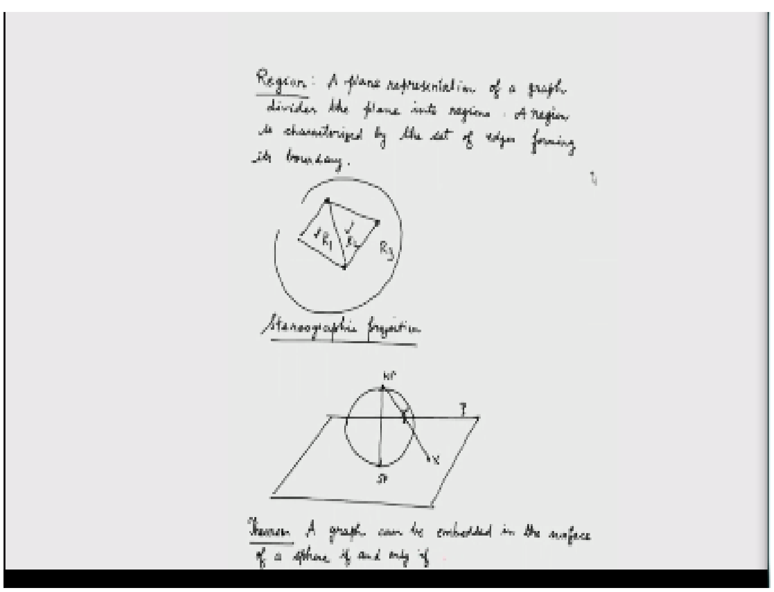

In the context of planar graphs, so a plane representation of a graph a plane representation of a graph divides the plane in two regions a region is characterized by the set of edges forming it is boundary. No let us look at some examples let us look at a planar graph like this which is reasonably straightforward kind of graphs now this is a planar graph and we see that its edges are forming regions and there is another region which is outside this graph so I can have regions which are both finite and infinite so if the region is enclosed by the edges.

Then it is finite but of course. I will have one graph one region which is infinite which is in intuitively outside the graph so here we have got three regions let us call them r1 r2 and r3 now we can remove this distinction between finite and infinite region by embedding a graph on a sphere on the surface of a sphere by using stereographic projection now I will quickly give an idea of the stereographic projection of a plane onto a surface of a sphere. So suppose I have got a sphere like this where this is the South Pole this is the North Pole.

And I put the sphere on a plane let us call this P and what I do is that I take a. X on a on the plane and then connect X to the North Pole and it is bound to cut the surface of the sphere on another point let us say X 'now well it is not difficult to see that whatever line or whatever set of points are there on the plane in this way. I can map it on the sphere and this map is one-to-one and on to accept that I will have the all the infinite points getting mapped to the North Pole rest of the all the points are mapped to a single point only there where whichever direction I go it will be mapped the North Pole.

Now it is again intuitively very clear that if we have a graph on a plane. I can use the stereographic projection to map it on this on a sphere spherical surface and vice versa and this gives us that theorem like this which I will state without a concrete proof but which is intuitively clear from what I have already told this is theorem a graph a graph can be embedded in the surface of his fear if and only if it can be embedded in a plane well and the next theorem is again intuitive that will prove.

We will not prove but state that is a planar graph maybe embedded in a plane such that any specified region can be made the infinite region made infinite can be made the infinite region now we are in a position to start looking at a surprisingly elegant theorem by Euler which connects the regions number of vertices and edges in a planar graph so this is called Euler's formula.

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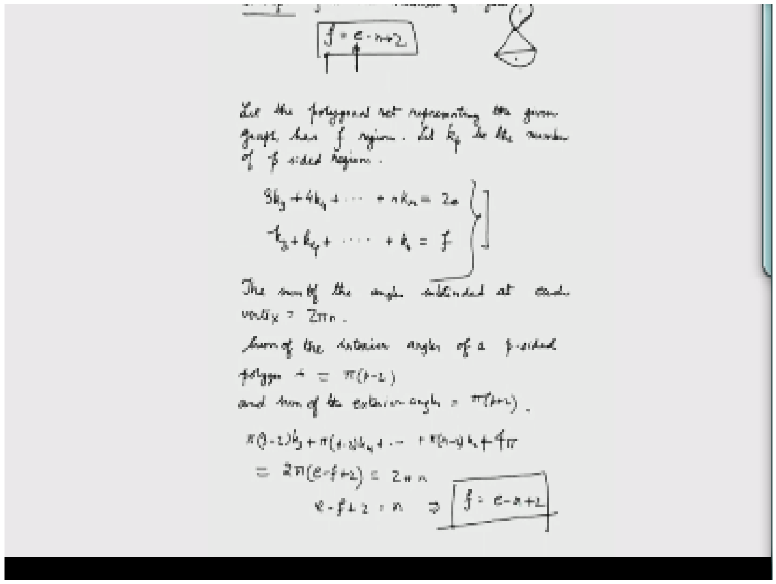

It states that a connected planar graph with n vertices a connected planar graph with n vertices and E edges has e-n+2 region. Now indeed this is a very surprising result and who would like to give a proof of this result so first of first of all we will observe that it is enough to prove this result for simple graphs the reason is that suppose F is the number of regions, then if you if you see that if I start increasing the number of edges over a simple graph by introducing more parallel edges or self loops.

We will see that each edge will generate an extra face and therefore if I have something like that this that F = e - n + 2 for a simple graph suppose I have got a simple graph and for this is true if I introduce one self-loop then I am introducing one edge so II will increase by 1 and one region will also increase so this will increase by 1 again I introduce 1 II edge then again even increase by one and F also will be increased by one so if this formula is true for simple graph then it is true for any graph so therefore we will only deal with simple graphs now let the polygonal net representing the given graph has F regions which we have already told.

Let KP be the number of P sided regions so what we are saying here that we can write a planar graph if at all it can be drawn on a plane by the vertices and the joining edges straight lines therefore I will always be able to represent a polygonal .I represent a graph by a polygonal net so the faces will be polygons and let us say that KP be the number of P sided regions that is P sauted polygons as regions.

Now if this happens then we see that 3 into K 3 that is there are K 3 many three-sided regions and so the number of edges associated will be 3x K 3 then similarly 4 x K 4 and similarly if we go on R x K R and we know that each edge is going to be present in two regions therefore I will have 2 times E and if I sum up all these face sides so they will give me ultimately number of faces because K 3 is the number of faces or regions with 3 edges and 4 and so on up to some R.

So therefore if I add all of them I am going to get the number of which is sometimes called phases so, I have got two expressions over here and I also know that the sum of the angle subtended at each vertex is 2Лn sum of the angles subtended at each vertex =2 Л n next some of the interior angles of AP sided polygon =Л-2 and some of the external exterior angles = Л+2 so in that network that polygonal network of the graph when drawn on a plane we will have F -1 bounded regions or finite regions and one infinite region and therefore.

If I sum up all the interior and exterior angles and I am going to get a sum like this Лx3 - 2 k3 + Л x 4 - 2 K 4 +and so on up to Л x R - 2 kr ++ 4Л and this is going =2 Л e - F +2 this is by using these two expressions and this is=2 Л N and from this we get e- f +2 is =n which ⇒ that F = e - n +2 which is Euler's formula once we have done this we will check one corollary to this formula which says that in any simple graph.

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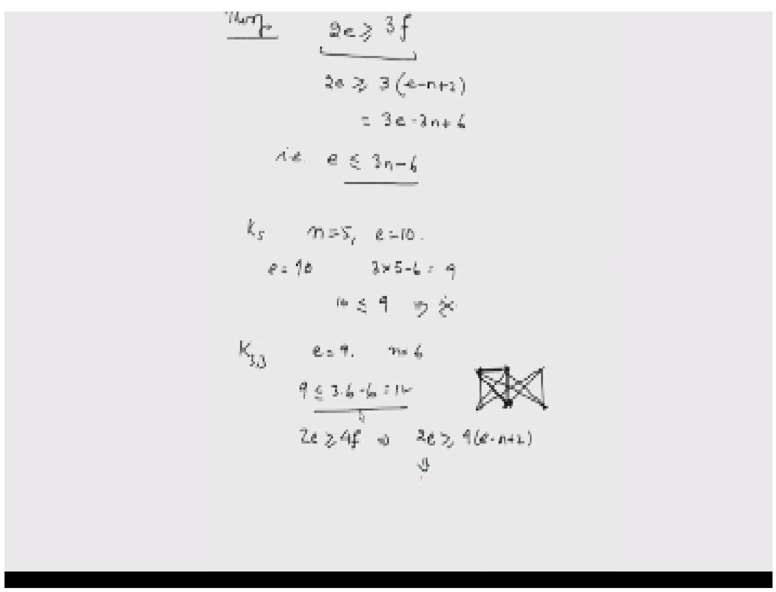

With F regions in vertices and e edges where e >2 e >= 3 / 2 F and E <3 n - 6 now let us try to give a proof of this result now what we observe here is that suppose. I have got F many regions each region will have at least 3 Edge's and so the total number of edges is 3f and we know that each edge will always be in two regions therefore twice of E > =this which proves this result now if on the other hand we put Euler's formula over here if we get F = e – n+ 2 then I get this =3 e - 3n + 6 and which gives me after reduction e <= 3n – 6.

No if I now look at k5 that is complete graph with five vertices here I will have e=n= 5n =5 e= 10 and therefore I will have e 1 side 10 and the other side 3x 5 -6 which gives 9 so I have got 10 < = 9which is a contradiction and therefore it cannot be a planar graph, if I now look at the second graph of we will see that this is K 3, 3 and so I will have e =in this case e= 9 and n = 6 so if I now put in this value and we will get 9 < = 3x6 -6 and this gave me 12 so there is no contradiction over here.

But I think a ,little more I look at the graph again so, I have this graph and we observe that no region in this graph can be bounded by three sides the reason is that it's a it's a bipartite graph so if I start from any vertex if I come to another vertex it is on the other it is on the other set and then I go back it is again the same set and now. I can never have a connection like this so I will never have a face which is bounded by three edge therefore I will have twice e > = 4 times F and which ⇒ that 2 e >= 4 times e - n +2.

Now if I put e = e =9 here I get 18 and if I put e=9 n=6 +2 then I will get here 4 x 5 that = 20 now this is a contradiction and therefore k3,3 also cannot be a planar graph with this I end today is lecture thank you.

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