**INDIAN INSTITUTE OF TECHNOLOGY**
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**NATIONAL PROGRAMME ON TECHNOLOGY**
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**Discrete Mathematics**

**Module-03**
**Mathematical Induction**

**Lecture-02**
**Mathematical Induction (2)**


**With**
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Today we will continue our discussion on mathematical induction.

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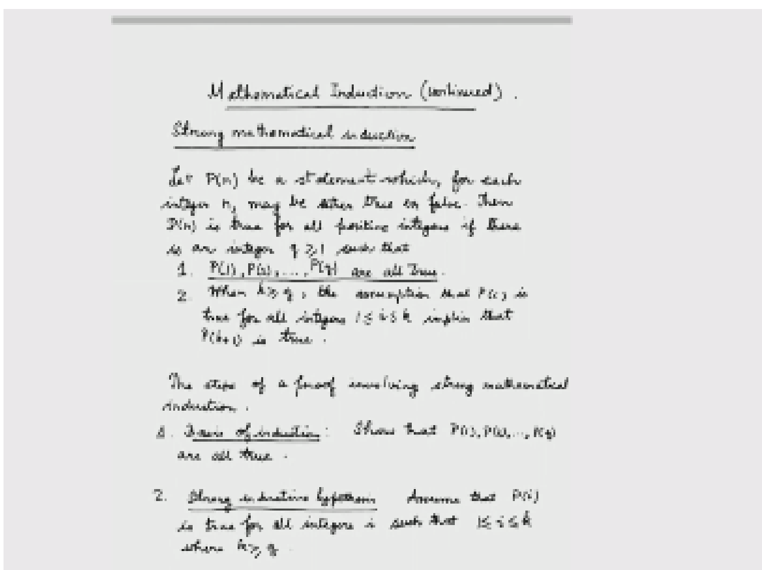


In the last lecture we have studied mathematical induction and some problems related to mathematical induction. Now we will start by giving another alternative and interestingly equivalent version of the principle of mathematical induction which is called the strong mathematical induction, at this point it is worth mentioning that the original mathematical induction that we studied in the last lecture and the strong mathematical induction that we are discussing now are equivalent.

But sometimes depending on the problem the first version is more useful and sometimes the second one that is the one which we are going to study now becomes more useful. Now let us first see what is meant by strong mathematical induction, let P (n) be a statement which for each integer n maybe either true or false then P(n) is true for all positive integers, if there is an integer Q ≥ 1such that 1 P (1) P (2) P (Q) are all true to when k ≥ Q, the assumption that PI is true for all integers 1 ≤ I ≤ K implies that p k + 1is true.

So here we see that our hypothesis and assumptions are slightly different than the previous one what we prove here what we have to stop prove in the beginning is that there is a positive integer Q which is ≥ 1 for which P 1 P 2 and P Q are all true that is we have to show that for a positive integer any positive integer less than that will satisfy P n, then our assumption is that we take a Q sorry if we take a K ≥Q and assume that for all I between 1 to K P I is true and then we have to prove that this implies that p k + 1 is true if we can do this then we claim that the statement PN is true for all N ≥1 and this is what we mean by the principle of strong mathematical induction as I have already said that the two principles are exactly same they have the same power.

But we use either of them depending on the problem under consideration, now let us check the steps of the proof or steps of a proof involving strong mathematical induction the steps of a proof involving strong mathematical induction. So first like before we have basis of induction basis of induction show that $P_1$ $P_2$ $P_Q$ are all true to strong inductive hypothesis, assume that p(i) is true for all integers I such that 1 ≤ I ≤ K where K ≥Q.

Now if you compare the previous version they are we assume that PK is true for some K ≥ n 0 and we prove that n 0 p(n) 0 is true and then we went on to prove that p k + 1 is true in this case we assume that for all i between 1 to k where k ≥ Q P I is true and then we go on to prove this in the inductive step.

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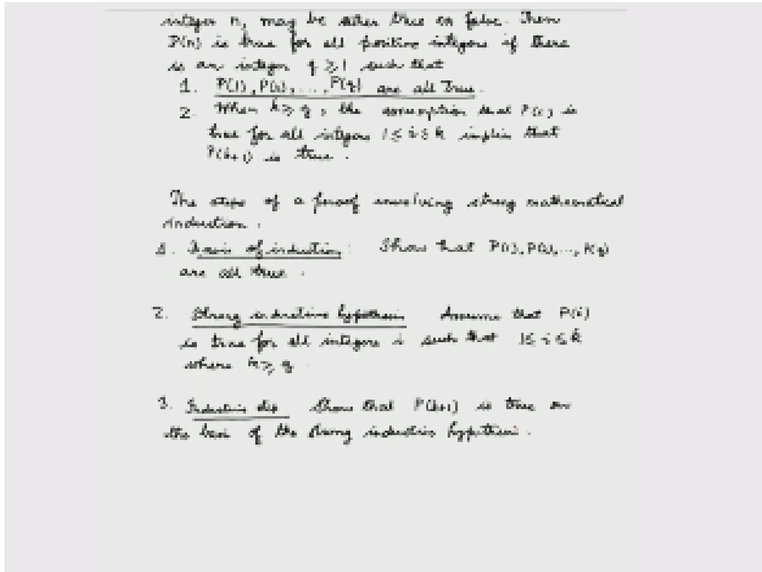


Show that p k + 1 is true on the basis of the strong inductive hypothesis. Now before we go on to discuss some examples where strong mathematical induction is useful we will quickly have a look at recursions and how we define recursions.

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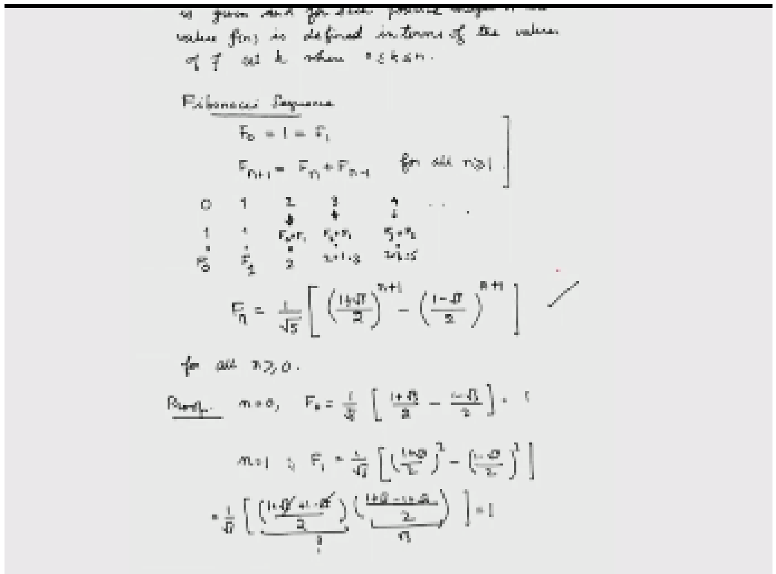

Let n be the set of non-negative integers if function from the set of non-negative integers is defined recursively if the value of F at zero is given and for each positive integer n the value FN is defined in terms of the values of f at k were $0 \leq K \leq n$, now we look at some recurrence relations and the most probably the most famous recurrence relation that is a Fibonacci sequence f-zero is defined as one which is equal to f1 and f n +1 is f n+ f n - 1 for all $N \geq 1$ this is how we define Fibonacci sequence recursively.

So let us start calculating the elements of the sequence so as we see for 0 F 0 = 1 then for 1 if 1 = 1 then for to this value is F 0 + F 1 which gives me 2 then for 3 this value is F 2 +F 1 that is 2 + 1 which is equal to 3 for 4 this is F 3 + F 2 = 2 + 3 = 5 and so on. Now it is known that the inert Fibonacci number that is F n is given by 1/ $\sqrt{5}$ then 1 + $\sqrt{5}$ / 2 whole raised to the power n + 1 -1 - $\sqrt{5}$ over 2 whole raised to the power n + 1 for all $N \geq 0$ and we would like to prove this relation by using strong mathematical induction proof.

Now let us put n = 0 if we do that then f 0 is 1 / $\sqrt{5}$ and here it will be 1 + $\sqrt{5}$ / 2 - 1 - $\sqrt{5}$ / 2 = 1 for n = 1 we have F 1 = 1 /$\sqrt{5}$ and this is 1 /$\sqrt{5}$ /2 and $\sqrt{\ }$ - 1 - $\sqrt{5}$ / $2^2$ now this is equal to 1 /$\sqrt{5}$ and 1 +$\sqrt{5}$ + 1 - $\sqrt{5}$ / 2 x 1 + $\sqrt{5}$ -1 + $\sqrt{5}$ /2 yes and therefore here it will get cancelled and this factor is equal to 1 whereas this factor is equal to $\sqrt{5}$ and this gives me 1 so we have checked that our formula given here works for n = 0 and n = 1.

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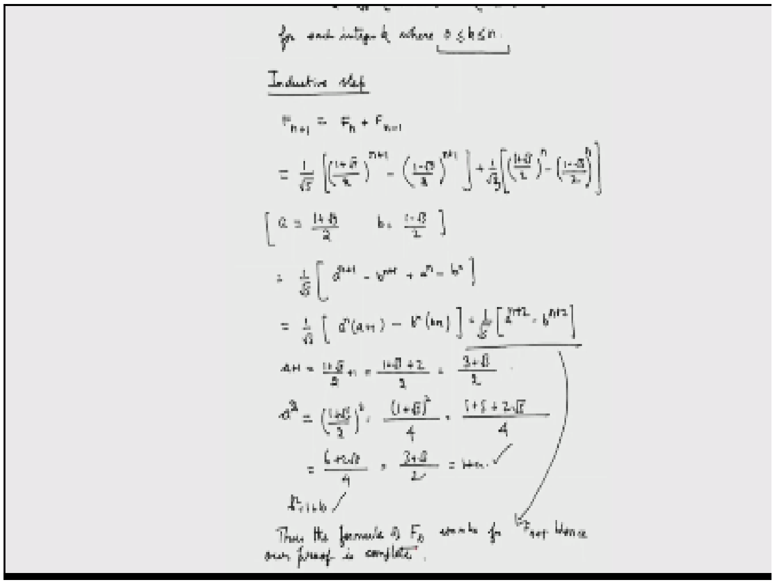


Now we go for the induction hypothesis which in this case becomes strong induction hypothesis for N ≥ 1 assume that FK = 1 /√ 5 x within bracket 1 + √5 / $2^{k+1}$ -1 - √5 / $2^{k+1}$ yes for each integer K for each integer K ≤ K ≤ M, now once this is my induction hypothesis I go for the induction inductive step inductive step, we start with if n + one the question is why if n +1 and not if k + one so we go to the induction hypothesis and we see that here we have assumed that I have an N and for all K from 0 to n this formula works and I am now checking for n + 1 if I am successful in proving that this formula works for if n +1 that means for all K between 0 to n +1 the formula works and therefore we can start checking for n + 2.

But that is of course that will work by using the same argument because we have we are able to choose n any positive integer > 1 and fix it so we now have if n + 1 and if we use the basic recursion formula we know that if n + 1 = FN + FN - 1 and since in both these cases the subscript n and n - 1 ≤ n we can write by using the induction hypothesis that 1 / √5 1 + √5 / 2 n +1- 1 - √5 / 2 n + 1 +1 + √5 1+√ 5n-1-√5 / 2 n.
Now for convenience we replace 1 +√5 upon 2 / a and 1 - √5 / 2 / B and the expression will become $a^{n+1}$ - $B^{n+1}$ + $a^{n}$ - $B^{n}$ which in turn becomes 1 /√ 5 $a^{n}$ a + 1 - $B^{n}$ B +1, now if we start with a + 1 and we will see that a + 1 = 1 + √5 / 2 + 1 which is of course equal to 1 + √5 / 2 and 1 +√ 5 + 2 which after simplification becomes 3 + √5 / 2 and by a sudden streak of imagination if we are able to check a square that is 1 + √5 upon 2 2 which is equal to 4 in the denominator and the numerator 1 + √ 5 whole square which gives us 4 1 + 5 + 2 √5 which brings us to 6 + √ 5 / 4 which is 3 + √5 / 2.

I said that we need imagination somehow we have to guess the right thing and to realize that what we get for a + 1 is same thing as a2 so this is of course a + 1 similarly we will see that B 2 = 1 + B and if we replace these 2 expressions in the calculation of FN + 1 then I get $1 / \sqrt{5} = a\ a^{n} . a2$.

So that gives me a n + 2 and - B n + 2 and which is which shows me that the formula works for n + 1 and therefore our proof is complete from this we can say thus the formula of FN works for F n + one hence our proof is complete thus we have seen one example using the principle of strong mathematical induction.

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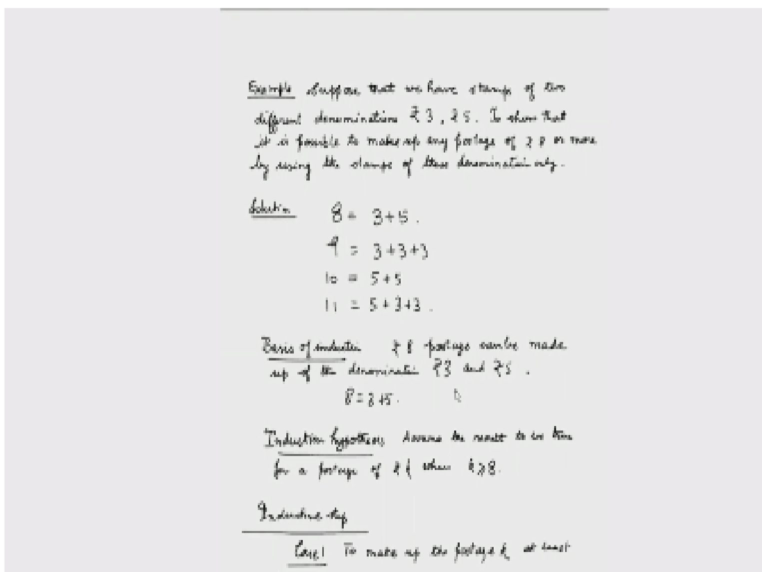


Now let us look at another problem which uses the principle of mathematical induction and it is a problem related to postage stamps suppose that we have stamps of two different denominations rupees three and rupees five to show that it is possible to make up any postage of Rufus eight or more by using these two stamps to show that it is possible to make up any postage of rupees eight or more by using the stamps of these denominations only now to solve this problem we start from the beginning.

If we have a postage of rupees eight of course eight can be written as 3+5 now let us check nine, nine can be written as 3 + 3 + 3 10 10 can be written as 5 + 5 now if we come to 11 11 can be written as 5 + 3 + 3 so we see that at least for first few consecutive possible post stages we can make up those post stages by using the denominations of rupees 3 & 5.

Now the question is that is it always true and we can say that let us use mathematical induction we have already proved the basis of induction so basis of induction rupees eight postage can be made up of the denominations rupees 3 rupees 8 and no rupees 3 this is rupees 3 and rupees 5 of course because 8 equal to 3 + 5 now we go to the induction step no if we go towards induction hypothesis assume the result to be true for a postage of rupees K where K is greater than or equal to 8 now we come to induction step in the inductive step we have two cases case 1 to make up the postage kay at least 1 rupees 5 stamp is required what do we do.

Then replace the rupees 5 stamp by 2 stamps of rupees 3 if we do that then we see that five gets replaced by six and therefore the total postage will be k + 1 the postage will be k + one now we come to case two now in the case two all the stamps required to make up the postage of K is of denomination three so there is no five denomination five stamp so in this case all the stamps are of denomination rupees three now what we realize over here is that this postage cannot be eight so it can be nine in which case it is 3 + 3 + 3 9 and if it is something more than nine.

Then also it will have at least three stems of denomination three in this case there must be three stamps of denomination three rupees three now our strategy will be to replace these three stamps by two stamps of denomination five so replace them that is all the three stamps by two stamps of denomination five rupees five to obtain a postage of rupees k + 1 now we see that this is exactly what we wanted we wanted to show that if we have a postage of K and if we can make up a postage of K by using stamps of denomination three and five.

Whatever the case may be I can make up a postage of rupees k + 1 once we approve that and we know that we can make up postage of eight therefore by mathematical induction we know that we have got the complete proof the last problem in this lecture on mathematical induction is involving chess boards now let us consider an eight by eight chess board well this is an 8x8 chessboard and suppose we take out one block from it suppress one block or square from it then we will call this a defective 8 by a chess board defective 8 by 8 chessboard.

Now there is another object that we would like to introduce which is called a tri amino it looks like this the question that we are asking is that can I can I cover this egg by a chess board by using try minerals of this type let me correct myself my question is that can I cover a defective 8x8 chess board.

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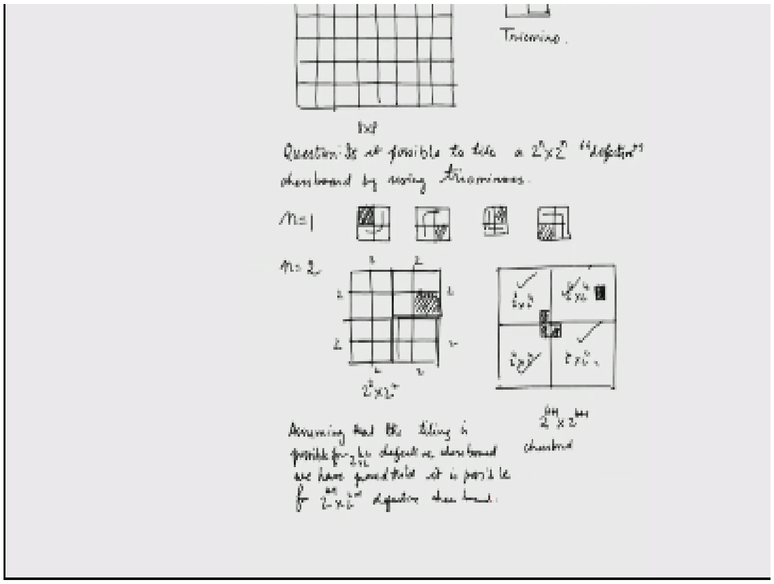

By try - or in general is it possible to tile an N /2 to the power n / 2 to the power n / n defective chess board by try - question is it possible to tie - to the power n / 2 to the power n defective chessboard by using Trevino's now we will try to use mathematical induction suppose we take n = 1 then I get a 2/ 2 chessboard and I know that I can tile it by try amino suppose because of being defective this shaded square is excluded then whatever is left is a try me know it does not matter which square I leave out if this square is this then I can place the try me know like this or if the square is like this.

Then I can place the try manner like this or if the square is over here which is removed and the try Mueller will be kept like this so I can I can put any time you know one time you know and by tiling I mean that I want to cover the chessboard by try minnows and distinct try minnows must not intersect that is they must not overlap that is something that we have to be careful and for n = 1 I say that I have only single try - oh I can do this without any overlap now the question is that what about a 4x4 tromp you know sorry what about a 4x4 chess board defective chess board so n = 2.

So I have got a situation where I have got 4x4 so this is $2^2$ by $2^{2}$ chessboard and let us suppose that I put a defect over here now we see over here is that four by four chessboard is made up of four two by two chess boards one here this is 2 2 and this is 2 so we have got 2 by 2 chess boards over here for 2 by 2 chess boards are giving me 1 4 by 4 chess board and if what we want to go in general you will see that 2 to the power k + 1 / 2 to the power k + 1 chess board is made up of 4 chess boards each of which are 2 to the power K by 2 to the power K alright.

Now we know that there is a defect so the defect in the 2 to the power K + 1 / 2 to the power k + 1 chess board has to lie in one of these smaller chess boards suppose it is here then what we can do is that we can put a tribunal around the center here making the remaining three chess boards defective now let us assume that $2^{K\ 2}$ to the power K defective chess boards can be tiled by try - then this one can be tiled this one can be tiled this one can be tiled this one can be tiled and of course this one can be tiled.

Because this is defective from the beginning and therefore the whole chess board can be tiled so if we assume that tiling is possible for K we have proved that tiling is possible for K + 1 assuming that the tiling is possible for K we have that is to be more precise to three power K by 2 to the power K defective chess ball let me let us write more precisely so 2 to the power K by 2 to the power K defective chess board.

We have proved that it is possible for $2^{K+1/2}$ to the power k + 1 defective chess board and we have already seen that tiling works for n equal to 1 therefore we know that if we have a 2 to the power n / 2 to the power n defective chess board no matter how large n is we can tile it and of course in that case we can tile an 8x8 chess board and that is all for today thank you.

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