Approximate Reasoning using Fuzzy Set Theory
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Lecture - 08
Lattice of Fuzzy Sets

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Hello and welcome to the next of the lectures under this course titled Approximate Reasoning using Fuzzy Set Theory, a course offered under the NPTEL platform.

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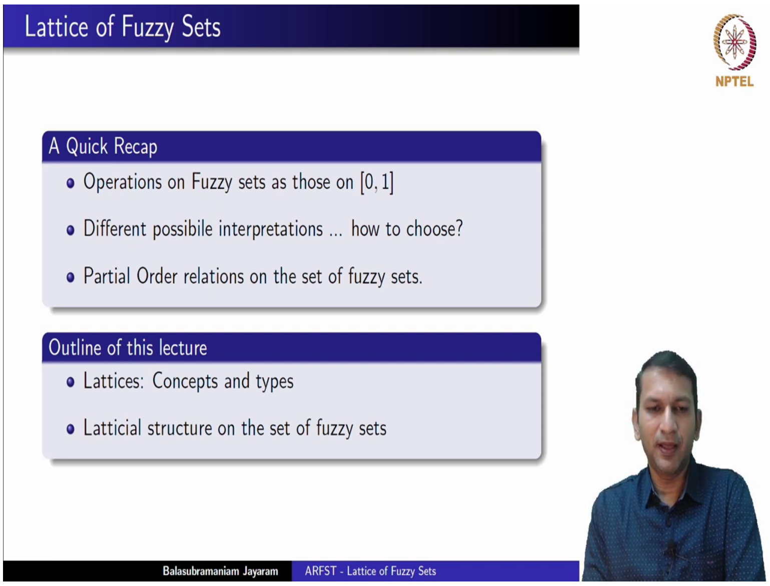

In the previous lectures of this week we have seen how operations on fuzzy sets can be seen as performing operations on the [0, 1] interval. We have seen that there are different possible interpretations of the operation themselves and we were left wondering on how to choose a particular operation.

Towards this end we have made a beginning by looking at a partial order relation that could be defined on the set of fuzzy sets, thus making the set of fuzzy sets a poset. In this lecture we will look at a special type of poset called lattices and go on to show actually we could have a lattice structure on the set of fuzzy sets.

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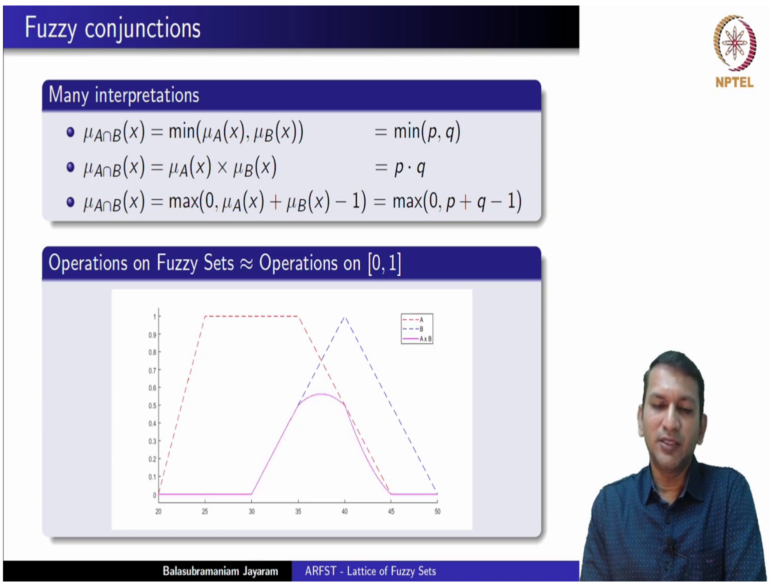

Recall that when we discussed that operations on fuzzy sets can be seen as operations on a unit interval [0, 1]. We had many interpretations for the fuzzy conjunction. Given these two fuzzy sets A and B as denoted by the red and blue curves we saw that if we were to use the min operation then the fuzzy set that obtained is given in black.

If instead if we used the product operation we would use we would get the fuzzy set as in magenta or if we did use the third operation which later on we will see has been proposed by (Refer Time: 02:15) we would get the conjunction of these two fuzzy sets to be the curve as given in magenta curve. We have seen that as the operations change. So, do the shapes of the curve.

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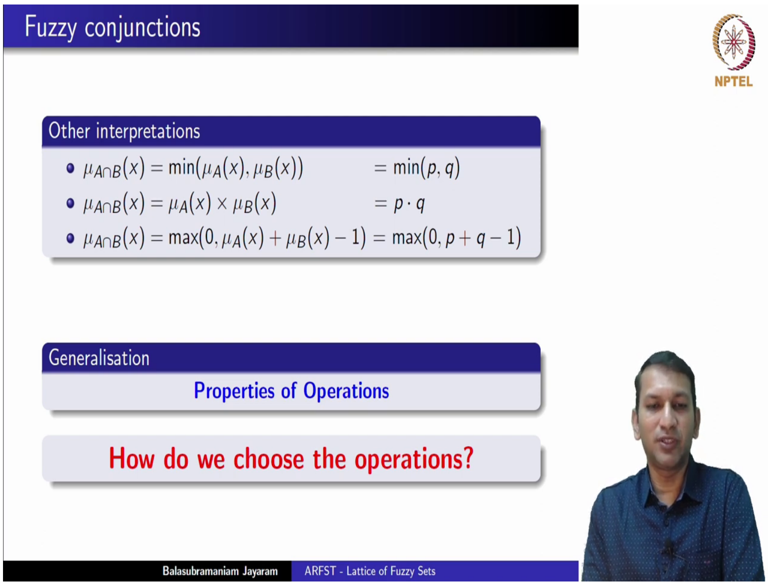

As an important outcome of this we have seen that it is possible to generalise this. These operations can be generalised by extracting the properties that we want them to hold in that sense it was very positive outcome. However, we are still left wondering given these many operations or interpretations of conjunction how should one choose these operations? Is there something special about each one of them?

We hope that by the end of this lecture you would get some partial answers at least for this query.

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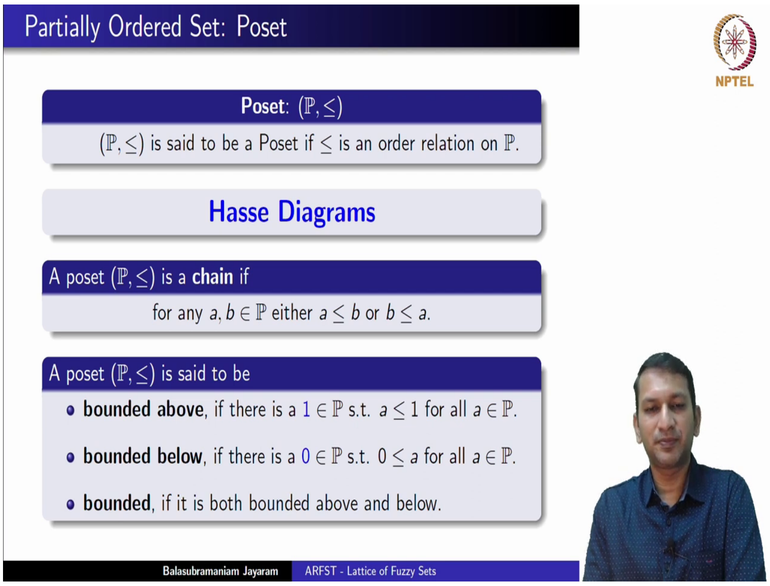

Let us quickly recall what a poset is. We have P to be a non empty subset non empty set and a binary relation denoted by the symbol less than or equal to. We say this tuple P along with binary relation is a poset if this binary relation turns out to be an order relation on P that is it is reflexive, anti symmetric, and transitive.

And we know that typically if the set P is finite towards easier understanding and a way of graphically illustrating it we take recourse to Hasse Diagrams. We have also seen that a poset can be a chain; that means, totally or linearly ordered we say poset is a chain if any two elements are relatable under that order or relation.

We have also seen a poset can be bounded above, bounded below and perhaps both bounded above and below in which case we say it is bounded.

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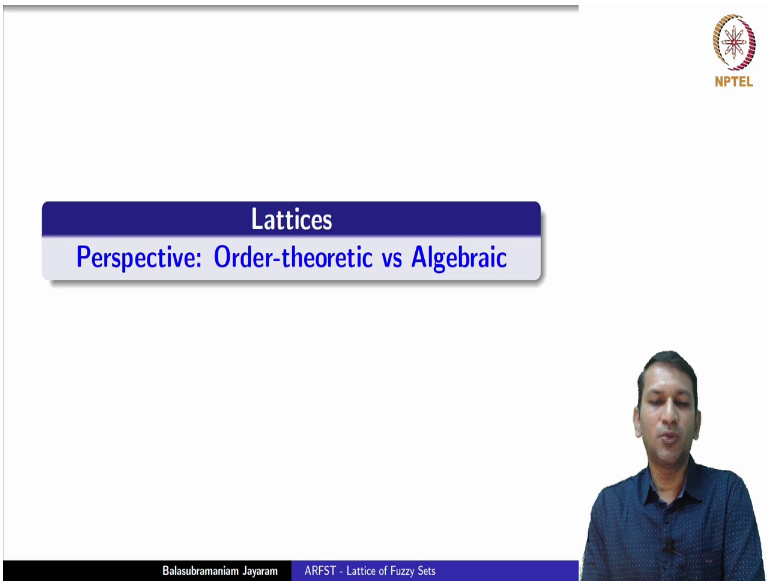

In this lecture let us touch upon lattices. We could look at lattices from two different perspectives an order theoretic or an algebraic perspective. In this lecture we confine ourselves to looking at lattices from an order theoretic perspective. We will take up an algebraic perspective of lattices in the next lecture.

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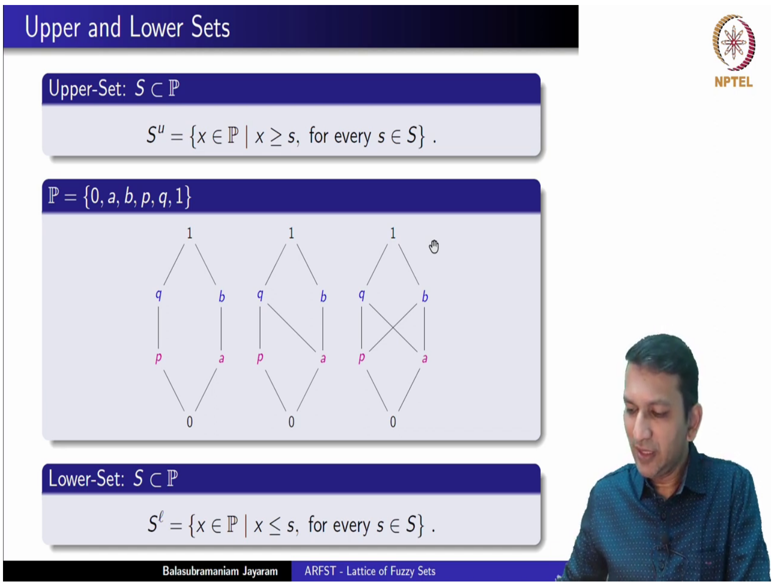

We need to set the stage to discuss what lattices are, how they become special types of posets towards this end. Let us introduce a few concepts, let us consider a subset S of P where P is a poset, we introduce the notion of a upper set. So, given a subset S of P we denote the upper set of S by S^u. This consists of all the upper bounds of the set s; that means, all those x element of P such that x is greater than or equal to s for every S. Its bigger than every s in S.

Let us consider this now familiar set with different possible orderings. So, essentially on the same set we have three different orderings and hence we can consider them as three different posets.

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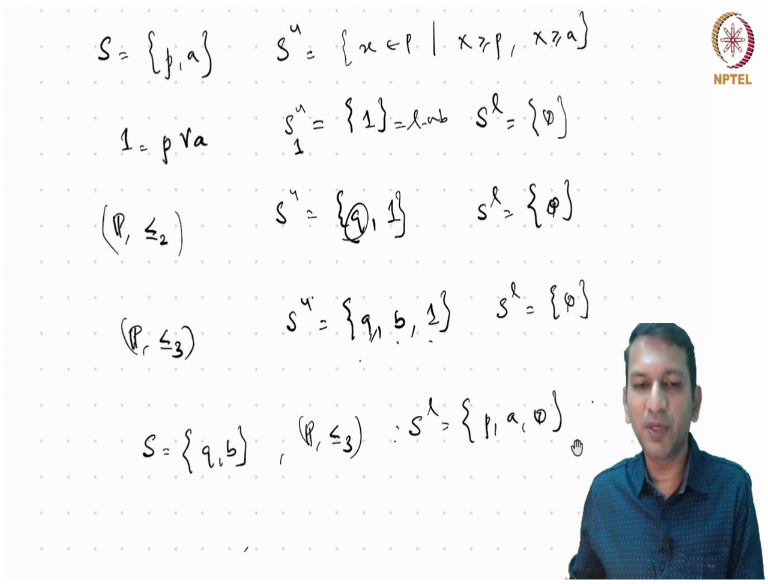

Let us now take the first poset, if we consider the set S to be p comma a then S u is set of all x in p such that x is both greater than p and x is greater than a.

Now, from the poset we see that this is actually the set the singleton set 1. Now recall the way to read a poset is like this. We see that it is a graph the vertices are its elements and if two elements are connected by an edge then they are relatable under the order and as we move from the bottom to the top it is an increasing order; that means, we read it as 0 is less than or equal to p, p is less than or equal to q and so on.

While there may not be an edge between p and 1 by transitivity we know that p and 1 are also related and that p is actually less than or equal to 1. So, if we consider the set p a which is a subset of p with respect to the first poset that is with respect to the first partial order on the set p we see that the set of all upper bounds of this set s is nothing, but just the singleton 1.

Now, if we consider the same set if we consider the same set under the second ordering we would get a different set of super set upper bounds. So, now we are looking at p this with the second order. We see here q is an upper bound for p both p and a. So, q comes into picture and so its 1, 1 is an upper bound for p, it is also an upper bound for a that is it is bigger than both p and a. So, S u will consist of q and 1.

On the other hand if we consider the third poset; that means, let us take p with the third ordering and for the same S that we have considered {p, a} the set with p, a. If you consider the set of all its upper bounds we see that p and a both are smaller than q. So, q will be here p and a both are smaller than b. So, b also will be here and of course, 1 we know is the top element is the upper bound for the entire set, so obviously 1 also will come.

So, you see here that when we consider the same subset S depending on ordering the upper set of S varies. Similarly we can talk about the lower set of S this is nothing but set of all lower bounds of the given set S. So, if you consider the same set S, then we could also write the corresponding we see here for the first poset the only lower bound of p and a is the set 0.

In this case in this case of second poset once again the only lower bound for both p and a is 0. Now let us consider the third order. So, we have p and a once again the lower bound for p and a is just the set of 0. However, if we consider S to be the set q b and let us consider p to be the third poset that is q and b and if you consider this as a subset S and we want to consider the lower set of S we see that for q and b p is smaller than both of them a is smaller than both of them and 0 is smaller than both p and a which means by transitivity 0 is also smaller than both q and b.

So, that means, S_l in this case would look like p a and 0. It is clear now then that given any subset S of p we can find the set of upper bounds and the sets of the lower bounds which are called the upper set and lower set of S respectively.

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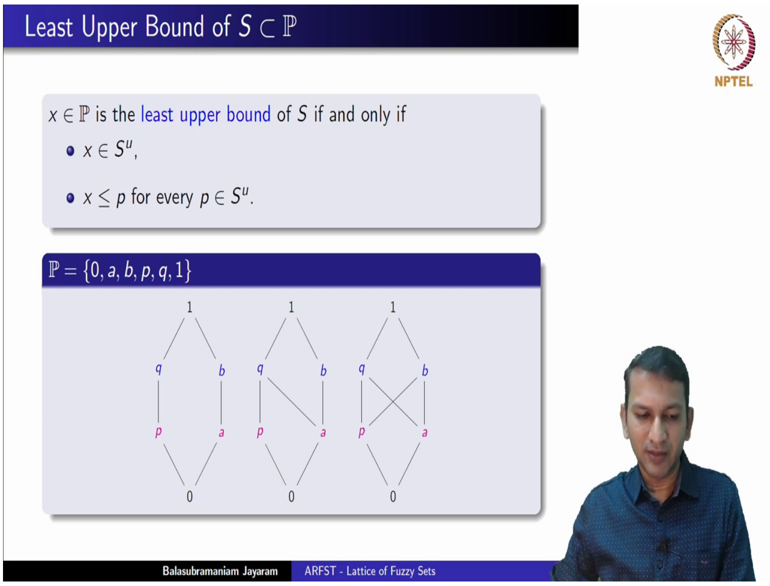

Let us move on to the concept of least upper bound of subset S. An element x from b is said to be the least upper bound of S if firstly, x should be an upper bound of S and it should be the least upper bound of all the upper bounds of S; that means, it should be smaller than every other upper bound in of S with respect to the ordering that we are consider.

For instance, once again let us consider the same three posets that we have. We have seen that S u in the case of the very first poset that we consider it has only 1 element. So, there is this turns out to be also its least upper bound. In the case of the second poset we know that both q and 1 they are the upper bounds of the set {p, a}.

Now, the least upper bound is that upper bound which is smaller than every other upper bound clearly; that means, q is the least upper bound. Now come let us go to the third poset. Here we saw that for the set {p, a} we have three upper bounds those of q b and 1; however, we see that q while it is smaller than 1 it has not smaller than b. Similarly b while it is smaller than 1 it is not smaller than q which means this set does not have a least upper bound.

So, what it means is with respect to the third ordering the set S consisting of p and a does not have a least upper bound. This is not to say that no subset S of p will have a least upper bound.

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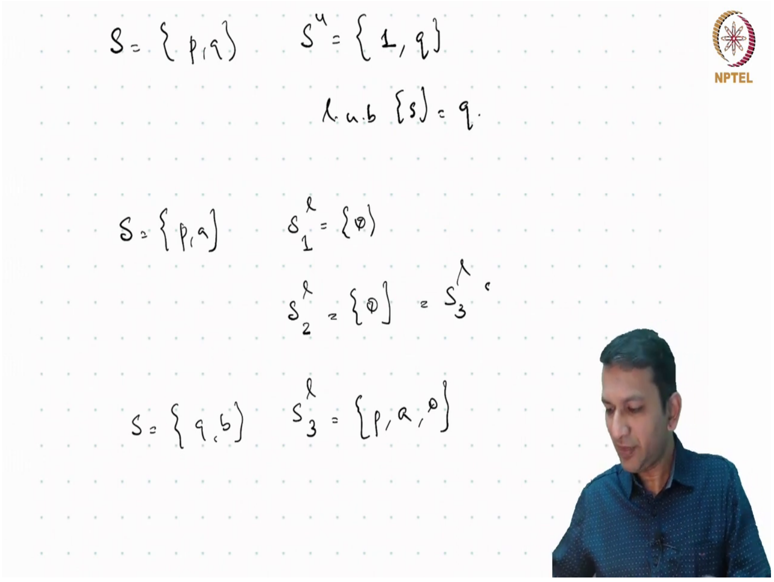

Let us look at the set S with p and q. Now if you look at S u the set of all upper bounds of p and q you will see from here that 1 definitely is bigger than both p and q, but by the definition here you will see that q is less than or equal to itself and q is bigger than p.

So, the set of all upper bounds of S u of S would be 1 comma q. Now among these if you are choosing the least of them clearly the least upper bound of s is going to be q.

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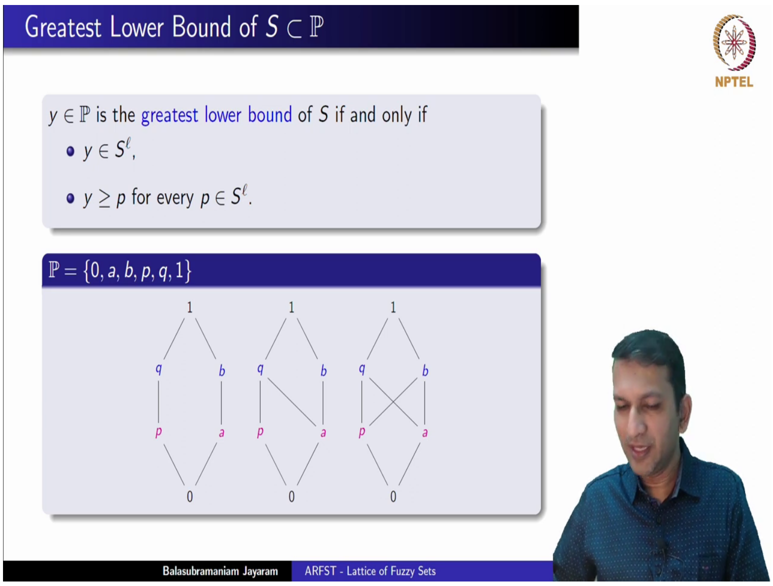

Now, similarly we can talk about the concept of a greatest lower bound by definition it is a lower bound of the set S and among the lower bounds it is the largest of them the biggest of them.

Once again let us look at the same three posets, we have seen that if we consider S as p comma a then S l with respect to the first ordering is actually only 0. S l with respect to the second order is also 0 and so is the case with the third order. However, we actually consider the set {2, 1}, now once again if we consider the third poset and consider the lower side of the third poset we know that we obtain p is smaller than both q and b a is smaller than both q and b.

So, o so we get this set as the set of lower bounds of S with respect to third order. Now to find the greatest lower bound means to give an ordering on these three elements clearly 0 is smaller than both p and a. However, neither is p smaller than a nor is a smaller than b; that means, we are not able to find a single unique element which is smaller than all of these lower bounds which is greater than all of this lower bounds. Thus in this case we not have a greatest lower bound for the set p comma.

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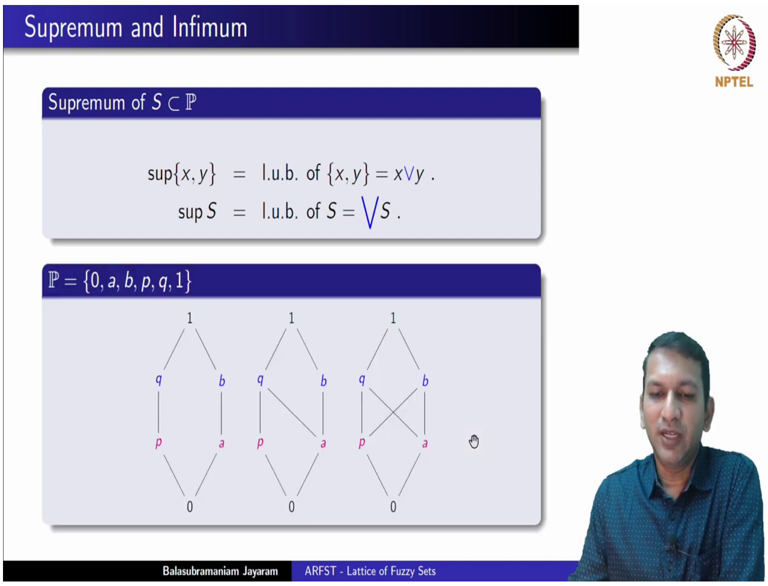

Now, let us come to the concept of supremum and infimum, these are nothing, but the greatest lower bound on the least upper bound repackaged as this in terms of each terminologies. So, given a pair of elements x comma y the supremum of x comma y is nothing, but the least upper bound of x comma y.

Typically it is denoted by this symbol which we call the join. So, when we it is read as x join y when we say x join y we understand it as supremum of x y. Now if you have a set S then the supremum of S is nothing, but the least upper bound of S. It is also denoted like this and it is simply read as join of S. So, if we consider the same posets we know that we have actually considered.

So, if you consider the set S is equal to p a and we have found out the supremum of the set p a with respect to the order in 1 the least upper bound is 1. So, this is essentially the supremum, thus you could also write 1 is p of a with respect to the r in 1. So, similarly we see that for other posets also we can find this.

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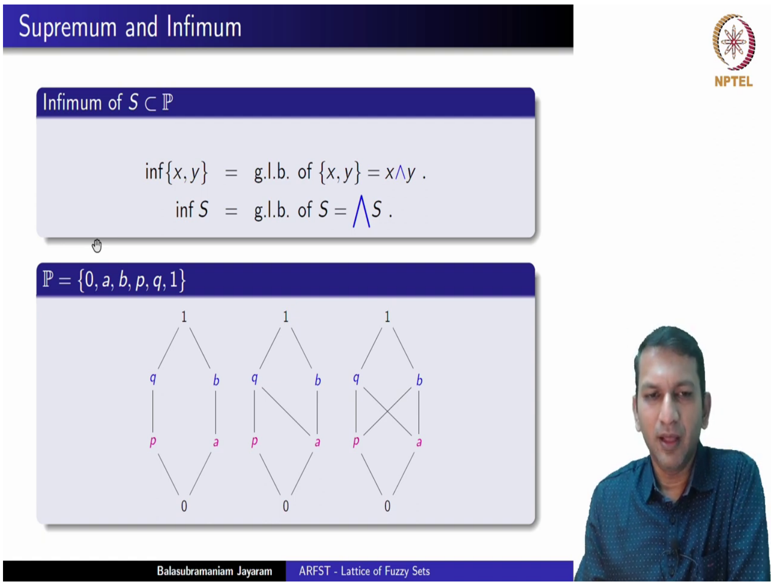

Similarly, as supremum we could also define the infimum of a subset S of P, if you have a pair of elements the infimum of this pair of elements x y is nothing, but the greatest lower bound of the set consisting of x and y.

We will also denote this by the met symbol and read it as x meet y. If you have a subset S of p the infimum of this S is nothing, but the greatest lower bound of S and it will be denoted as inf of S. Once again we could calculate the infimum of any subset of S with respect to any of the ordering that we have defined on p.

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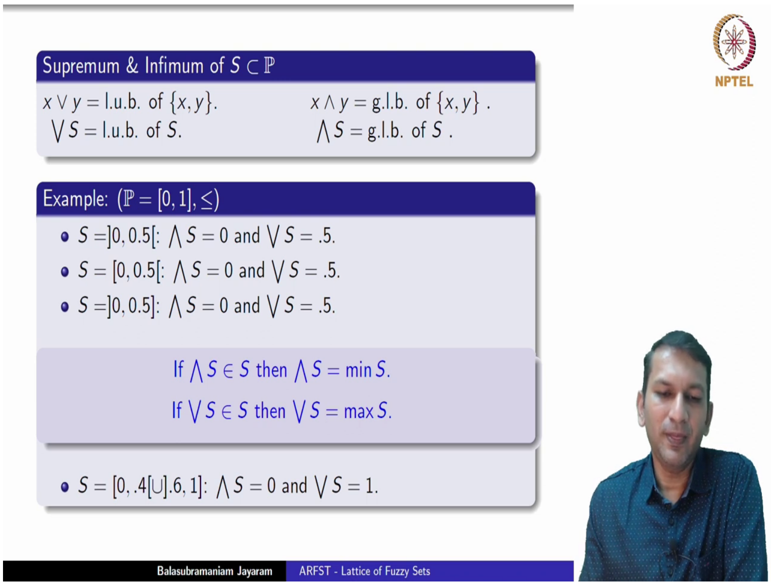

Well, in a nutshell this is what we have given a pair of elements or a subset S of p we can define and determine its least upper bound or greatest lower bound if they do exist. Let us look at some examples: consider the set p to be the unit interval 0 1 with the usual order. If we consider the subset S which is let us take this to be the open interval 0 to 0.5 and we ask the question what is its infimum? And what is its supremum that is what is meet S and what is join S?

It is clear that 0 is the only lower bound; however, it does not belong to this still the infimum of S is 0 and the supremum of S is actually equal to 0.5. Please note here in the definition of the infimum of the greatest lower bound the y need not belong to S. It is an element coming from p; hence 0 or 0.5 even though they do not belong to the set S, they still can serve as the infimum and supremum of S.

Let us slightly tweak this interval, let us push 0 inside the set and consider the closed to 0 open 0.5 interval. Once again the infimum of S remains to be 0 and the supremum of S remains to be 0.5. This would be the same if you alter the interval S dually that is push 0.5 inside the interval and still keep 0 outside.

The infimum of S will remain as 0, the supremum of S will be 0.5, in fact if the infimum of the set S belongs to S that is when we call that infimum to be the minimum of S. Similarly if the join S the supremum of S belongs to S then we say that join of S is actually the maximum of S. Let us finally, consider this particular interval.

So, this is an union of two disjoints intervals 0 to 0.4 where 0.4 is not included and 0.6 to 1 where 0.6 is not included. It is now clear then that if you look at the infimum the infimum of this set S is again 0 and the supremum of set this set S is actually 1.

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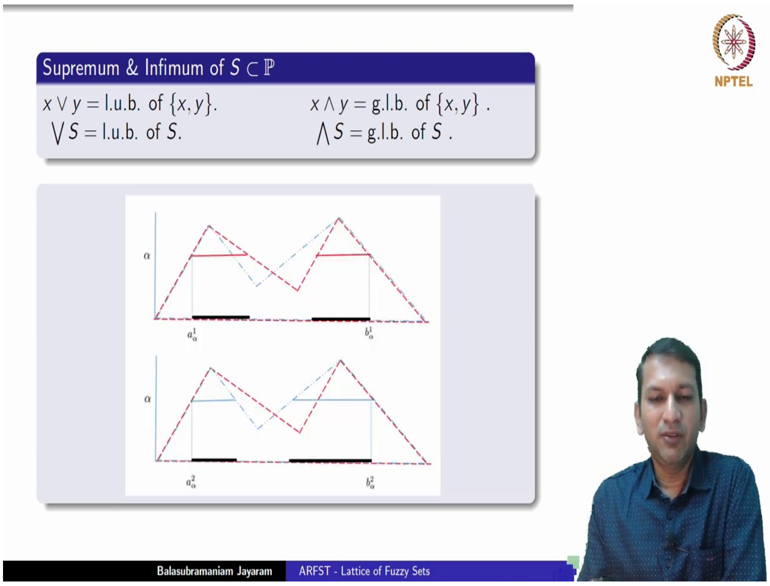

You might recall from the previous lecture when we discuss the alpha cut based ordering of fuzzy sets defined on r. We have seen these two fuzzy sets which are given in red and blue curves and we have seen that for a particular alpha the alpha cut looks like this. So, for a particular alpha the alpha cut for this fuzzy set a 1 is union of these two intervals and the same and the for the same alpha the alpha cut of the blue curve the second fuzzy set is given by the union of these two disjoint intervals.

We will see that even though these two intervals are different, the infimum of this these two intervals is essentially the same the left end points of the interval and the supremum of these two intervals is the right end point of these two intervals which are identical.

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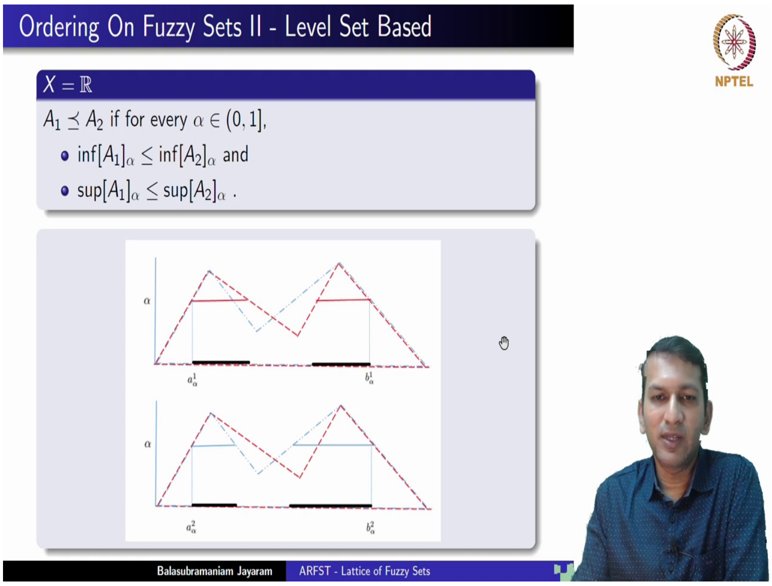

Let us recall the alpha cut based ordering of fuzzy sets that are defined over the real line. We say that two fuzzy sets A 1 and A 2 are ordered under this relation if these two inequalities are there.

Clearly if you consider A 1 and A 2 as the fuzzy sets, we see that for every alpha even though we have only denoted for one particular illustrated for one particular alpha for every alpha we will see that while for some alphas the alpha cuts themselves may differ the actual these two inequalities will still be there. In fact, the infimum of A 1 alpha will be equal to infimum of A 2 alpha for almost for every alpha here.

And similarly the supremum of A 1 alpha will be equal to supremum of A two alpha for every alpha coming from 0 1 interval which means both these fuzzy sets are orderable with respect to this relation. In fact, we get A 1 is less than or equal to A 2 and A 2 is less than or equal to A 1..

However, we have seen that these two fuzzy sets are not equal and hence this relation does not satisfy anti symmetry and hence we have seen that if you only consider arbitrary fuzzy sets the set of all fuzzy sets on R then with respect to this relation it does not become a order relation. We do not get a partially order set.

So, this is something that we have seen earlier, but perhaps now with after having discussed the concepts of infimum and supremum perhaps now this will make more sense to you.

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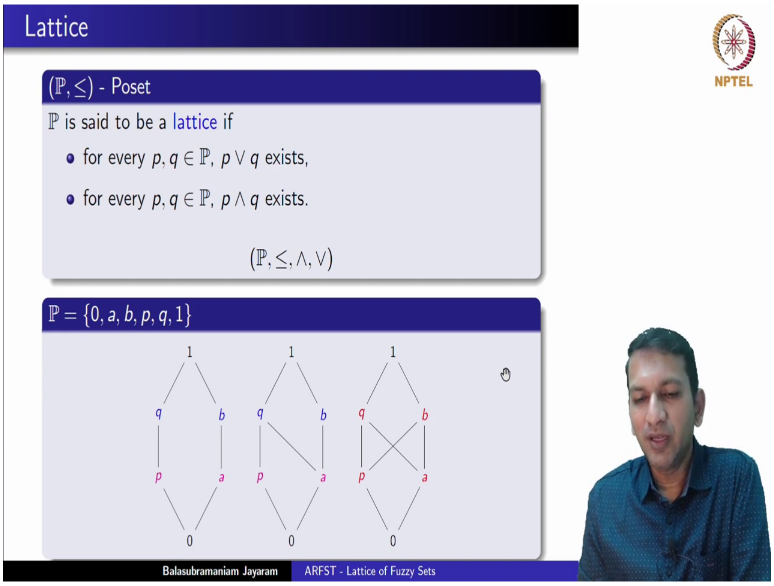

Now, we are in a position to discuss lattices which are special types of posets. Let P be a poset we call it a lattice if for every pair of elements the supremum exists. Similarly for every pair of elements the infimum also exists that is for every pair of elements both the infimum and the supremum exist

In fact, it can be proven that it is sufficient for one of them to exist and by the principle of duality the other one will also exist. However, for sake of completeness let us have this as a definition; that means, for every pair of points we want both the supremum and the infimum to exist. Towards amplifying this and emphasizing the relationship between the supremum and infimum and the exist and their existence we actually indicate a lattice not just only with the set and the order relation, but also with the meet and join operations.

Let us look at the posets that we have been discussion so far. It must be clear to us that the first and the second posets all of them turn out to be lattices because for any 2 elements any pair of elements you will be able to find the infimum and the supremum; however, the last poset is not a lattice.

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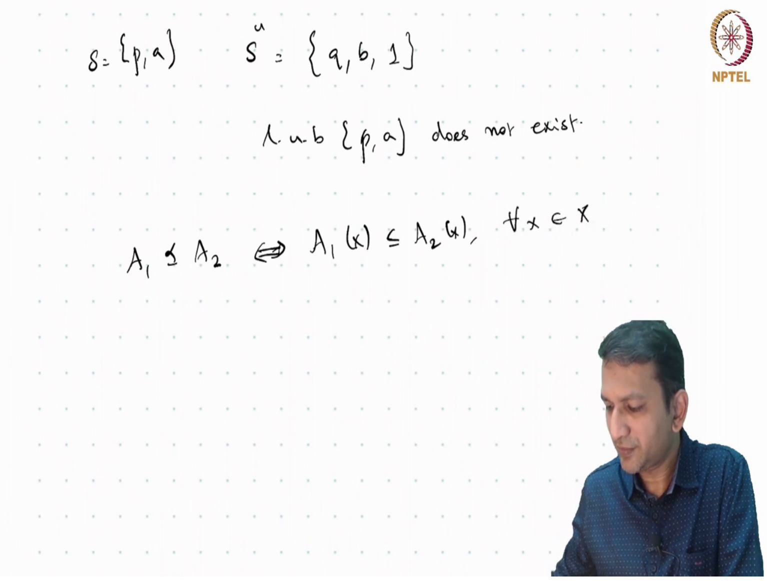

To see this consider the set consider the pair of elements p comma a, we want to see what is its least upper bound. So, let us find out for this set S the upper set and we have seen the upper set consists of q b and 1. However, while the set of upper bounds is actually the set with q b and 1, we are not in a position to find the least upper 1 as we have seen earlier.

So, the lub of p comma a does not exist. Remember by definition for a lattice any pair of points should have both the infimum and supremum clearly this the pair of points p and a they do not have a least upper bound. Note that they do have an infimum which is 0. Similarly if you consider the pair of points q comma b then they definitely have a supremum which is 1 as you can see on the screen.

However, it can be shown that it does not have an infimum. The point clearly arises from the fact that if you consider the lower set of these two points q b then it consists of p a and 0 and you will not be able to find the largest of these lower bounds well.

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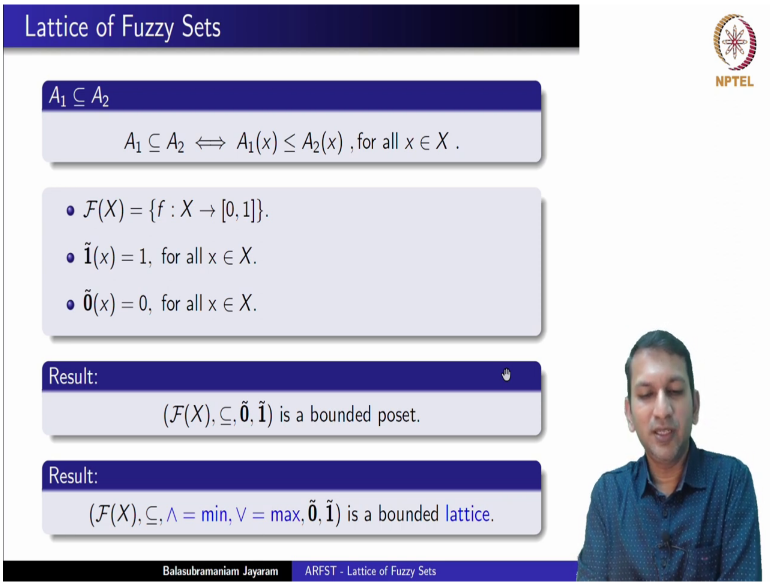

Now, it brings us to discussing the lattice of fuzzy sets. Recall the usual ordering on the set of fuzzy sets which is looking at them as functions we assign the point wise ordering between any pair of fuzzy sets.

We have seen that if you consider F of X to be a set of all fuzzy sets defined on the point X and by 1 tilde we denote the function which takes the membership value 1 throughout x and by 0 tilde the function whose membership value at every point of x is 0. We have seen that considering this set of fuzzy sets and the usual point wise ordering and these two functions 0 tilde and 1 tilde it is a bounded poset.

However, we can go one step further and say that this actually becomes a bounded lattice. So, the question immediately arises if it were a lattice given in any pair of elements from this set of fuzzy sets what would be the meet and what would be the join. Let us calculate this. Remember we have that A 1 less than or equal to A 2 if and only if A 1 of x is less than or equal to A 2 of x for every x in x.

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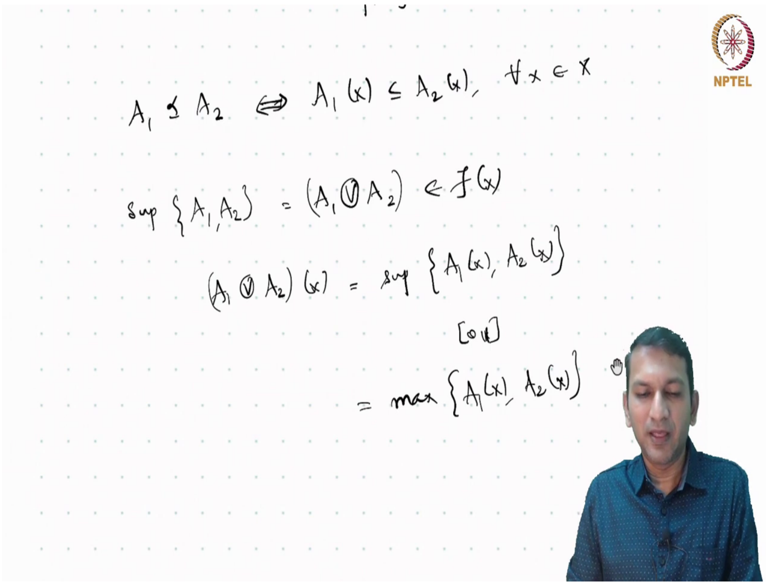

Now, we want to find the supremum of a pair of elements. The supremum of pair of elements means we are looking at A 1 join just for the sake of symbolism let me just put a circle. So, that we do not immediately confuse it with the math's that we are thinking of this is, what we need to find..

However, this is also a function again; that means, this is an element of f of x which means this has to be defined on every x; that means, we are looking at supremum of A1(x) and A2(x) and we know that these are elements coming from [0, 1] interval and with the usual ordering the supremum on this turns out to be the maximum for a pair of elements or if it is set of elements it would be the supremum.

Since, we are considering a pair of fuzzy sets A_1(x), A_2(x) is a pair of elements coming from 0 1 level and this turns out to be max of A 1 of x comma A 2 of x plus the supremum of any two elements in f of x turns out to be the max of the corresponding fuzzy sets.

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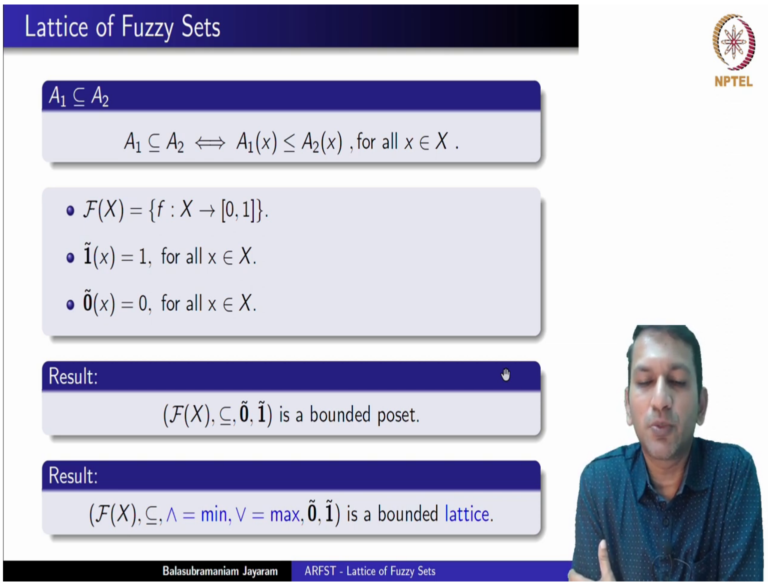

It can be shown similarly that the infimum of a pair of fuzzy sets is actually doing the point wise minimum on the corresponding membership functions. Thus min and max the operation that we have seen on fuzzy sets they turn out to be the lattice operations of meet and join on the set of fuzzy sets with respect to the point wise order.

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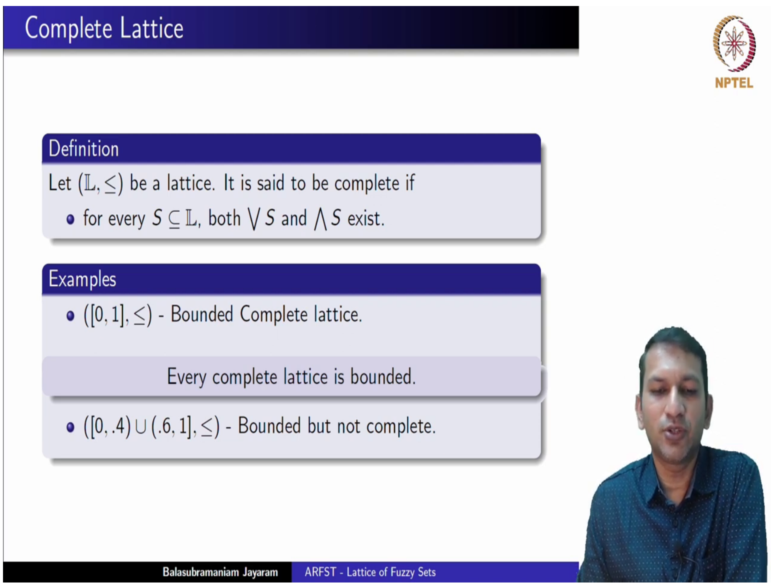

Let us look at a special type of lattice or the complete lattice. A lattice is said to be complete if every subset S of L has both the supremum and infimum; that means, both the join S and the meet S exist.

Let us for the moment consider the unit interval [0 ,1] with respect to the usual order. You already know it is bounded it is bounded above by 1 and bounded below by 0 of also it is a complete lattice; that means, any subset of [0, 1] we take it has both the supremum and the infimum. Please note the supremum and the infimum of a set S a subset of [0, 1] need not belong to S itself, it has to belong to the larger subset the super set of it.

So, as subsets of the [0, 1] interval any subset of [0, 1] interval will have both its infimum and supremum thus the [0, 1] interval with the usual order is not just a bounded poset it is a bounded lattice, in fact it is a bounded complete lattice. In fact, from the definition itself it follows that every complete lattice is bounded. This can be immediately seen by taking S to be L itself in which case the join of L and the meet of L will exist and they will turn out to be the upper and the lower bounds of L.

Look at this example it is a disjoint union of two intervals 0 comma 0.4 where 0.4 is not included and 0.6 comma 1 where 0.6 is not included. While this is bounded with respect to the usual order it is not complete. Consider the interval 0.3 comma 0.4 perhaps even open interval 0.3 open 0.4. We see that infimum of this interval open 0.3 0.4 is 0.3 which definitely belongs to the set.

However the supremum of this interval does not exist. Thus we might have a lattice which is bounded, but not complete; however, every complete lattice is actually bounded.

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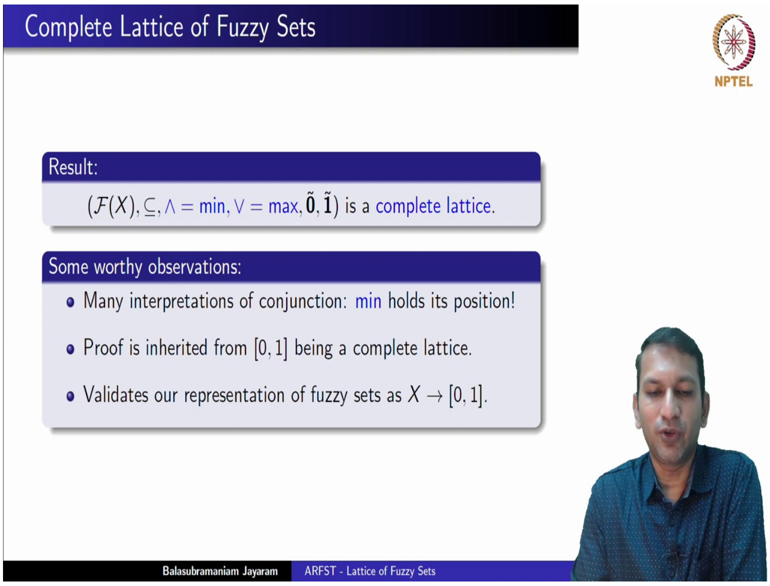

Let us wind up this lecture with this very important result. We have seen the set of fuzzy sets on X with respect to the usual point wise ordering is a bounded poset. In fact, a bounded lattice where the lattice operations were given by the min and the max operations on the fuzzy sets. It can also be shown that it is in fact a complete lattice.

A few noteworthy observations can be presented. We have seen that there are many interpretations of conjunction; however, min seems to hold its position. The proof of the above result is in fact, inherited from the fact that [0, 1] interval is a complete lattice. We have seen this a few slides back that the [0, 1] interval is in fact, a complete lattice..

The proof of the above result can be easily inherited from this factor and what this shows is that the representation of fuzzy sets that we have come up with as a function from X with the co domain as 0 1 which is a complete lattice is a useful and an appropriate one for us.

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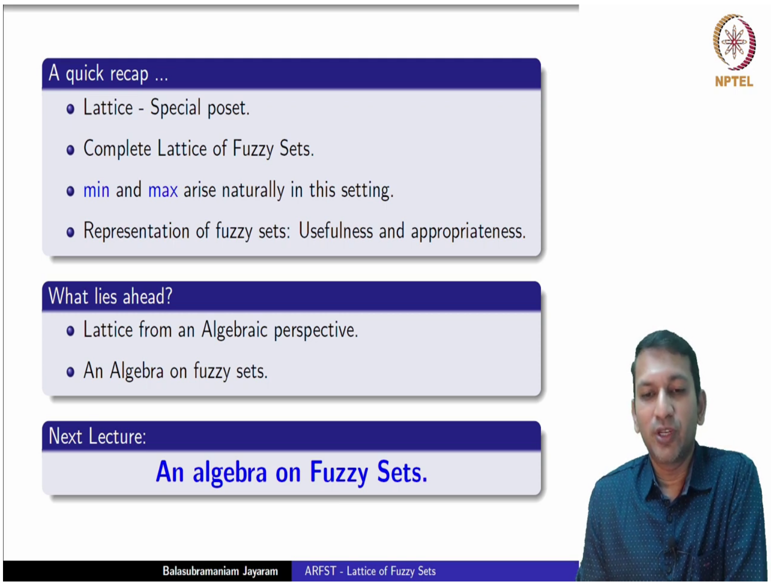

In this lecture we have seen lattices which are a special type of poset. We have seen that the set of fuzzy sets can be given a lattice-al structure in fact, they can be made a complete lattice and also the fact that min and max these operations arise quite naturally in this setting and the representation of fuzzy sets that we have chosen seems to be both useful and appropriate. What lies ahead? We have seen lattice from an order theoretic perspective.

In the next lecture we will see lattice from an algebraic perspective and it is through that we will come up with an algebra on the set of fuzzy sets itself. So, the next lecture we will deal with one particular algebra on the set of fuzzy sets. Thank you for joining me in this lecture and hope to meet you soon in the next lecture.

Thank you.