Approximate Reasoning using Fuzzy Set Theory

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Lecture - 14
T-Norms: Construction and Representations

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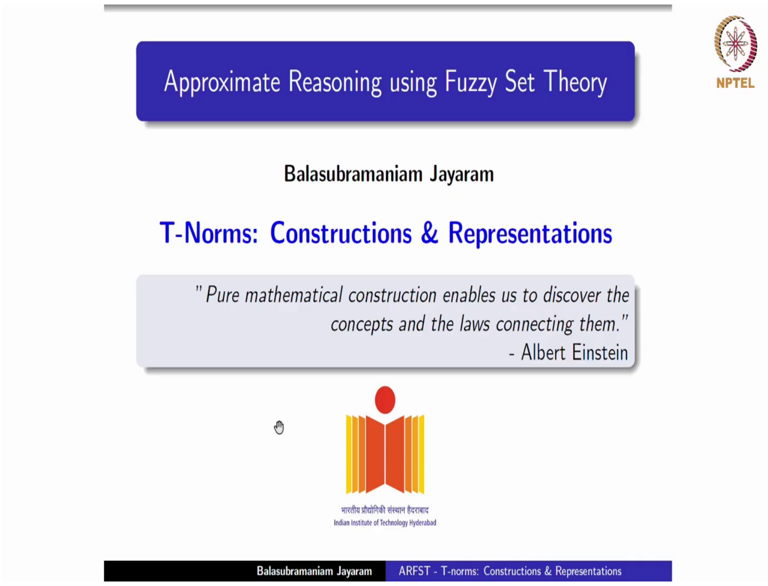

Hello and welcome to the fourth of the lectures this week under this course titled Approximate Reasoning using Fuzzy Set Theory, a course offered over the NPTEL platform.

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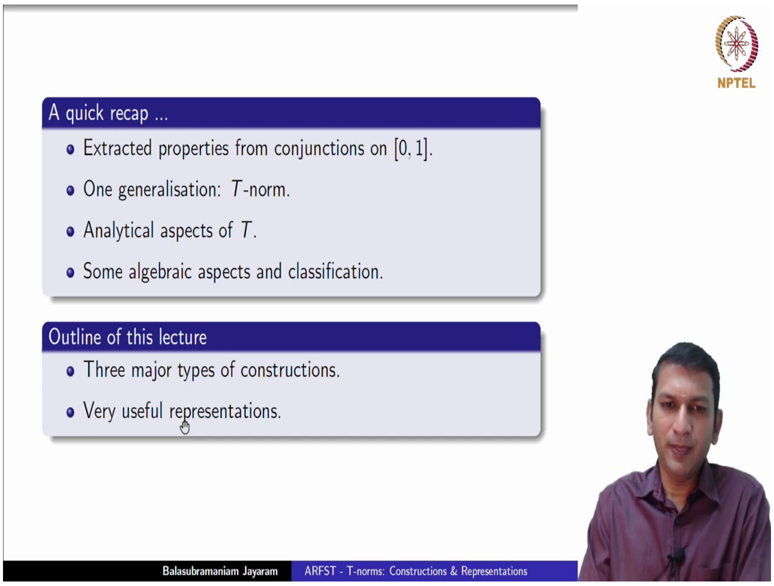

In this lecture so far this week, we have seen one particular generalization of fuzzy conjunction to that of what we call a triangular norm or in brief T-norm. We have also seen some different types of continuities that we can impose or on the set of T-norms. We also seen some algebraic aspects both at the elemental level with respect to a T-norm and also at the level of a function.

And using this we have been able to give some kind of a classification of T-norms into some subclasses. In this lecture, we will look at three major types of constructions; constructions where if you are given a single T-norm or a couple of T-norms, how to construct a T-norm from these and also construction from say first principles. Through these constructions, we will also see that we can obtain very useful and insightful representations for some subclasses of T-norms.

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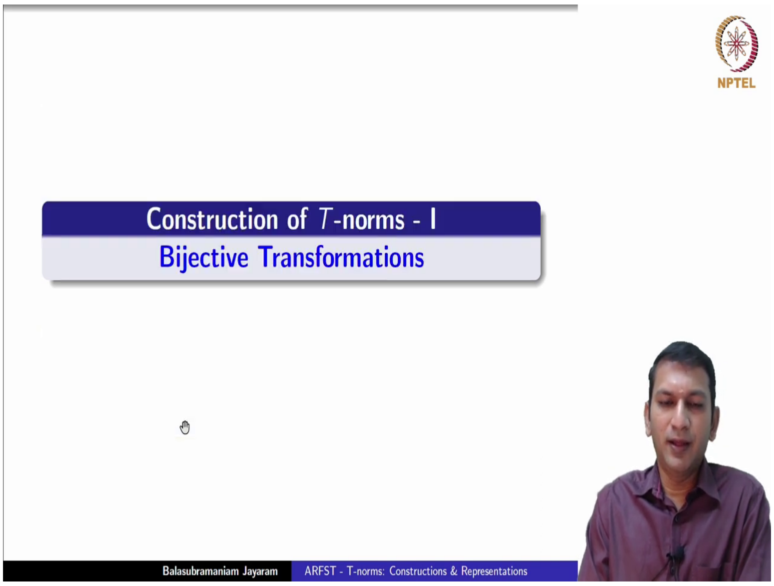

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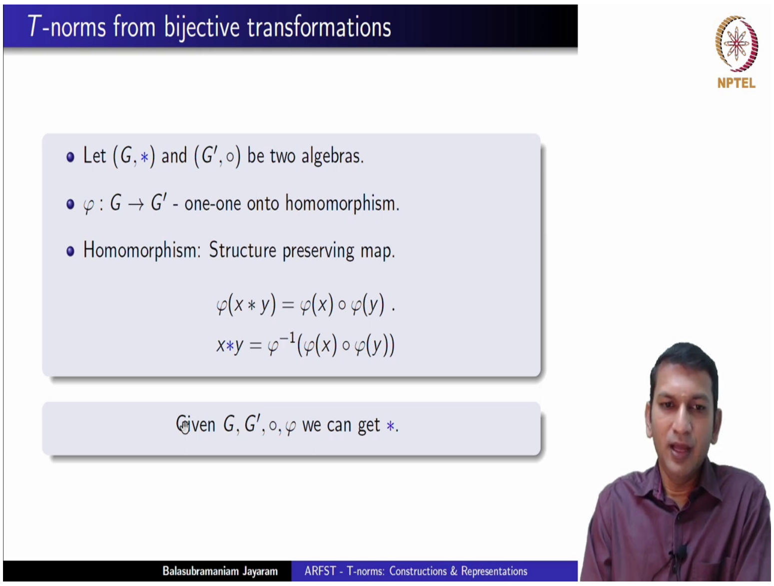

Well, let us take two algebras, as we have seen an algebra is a set with one or more operations of possibly different (Refer Time: 1:58) means, they can be binary or unary operations. In this case let us consider these two algebras, where G is a set and star is a binary operation on G and similarly G' is a set and circle is a binary operation on G'. And let us be given a one to one onto homomorphism between G and G'.

So, one to one onto, we understand it is a bijection that means. What is a homomorphism? Homomorphism is a structure preserving map. What does it mean? Phi satisfies this equality, phi of x star y is equal to phi x circle phi y. Now, in plain English what does it mean? Take two elements x and y from G what this says is; it does not matter whether you first take x and y and operate with star which is the operation on G and then apply phi on it, take the image of this x star y and go to the codomain.

Or first itself go to the codomain through phi; that means given x and y go to G dash as phi x phi y and do the operation there which is circle in G dash, both these should actually be identical. So, this is what it means to say the structure, the algebraic structure is preserved by this map and that is what we call a homomorphism. Notice one thing that since phi is one to one onto which is bi bijection, it is invertible and hence we could actually obtain the operation star as follows x star y is phi inverse of phi x circle phi y.

Note that initially we began with two algebras G star and G dash circle and a homomorphism. Now, what we see is if you are given G and G dash and an operation circle on G dash, which makes it a particular algebra and if we know that phi from G to G' is homomorphism, one to one onto homomorphism. That means it preserves the structure, whatever structure that circle imposes on G dash if it is preserving it, then the star operation on G actually can be obtained as the above formula.

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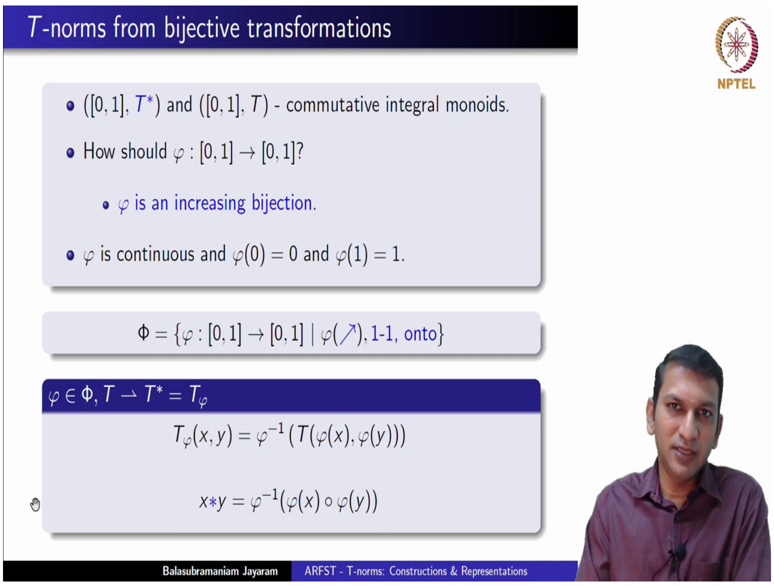

Let us see whether we can put this to good news in the context of T-norms. If we take the set [0, 1]; we know that any T-norm makes it a commutative integral monoid. So, let us consider two T-norms, T star and T; for the moment let us assume that both are known to us. So, we actually have two commutative integral monoids, just like how we started with two algebras G and G dash earlier.

Now, what we need is a structure preserving map; that means we need a map phi from [0, 1] to [0, 1]. How should it look like to be able to preserve the structure? It can be shown it is enough if it is an increasing bijection; so that means 1 to 1 onto increase in which clearly means it is strictly increasing. What is interesting is since now we know the underlying G which is 0, 1; we can show that any increasing bijection phi on 0, 1 to 0, 1 should also be continuous, such that phi of 0, 0 and phi of 1 is 1.

Now, let us consider the set of all such increasing bijections. Now, given any one bijection from the set under T-norm T, we can actually construct T star just like how we saw earlier. In the sense we will denote this T star by T 5, because it tells us that it comes from a particular phi and also particular T-norms. So, these are the two things that are fixed, given a T-norm and an increasing bijection of the unit interval [0, 1]; we see that we can construct a T-norm T phi. How do we construct it?

T_phi of x y is equal to phi inverse of T of phi x phi y is essentially exactly the same as the formula that we have before, where if star were not known, it can be obtained like this. Now, we are coming up with a new T-norm T star, which you will denote henceforth as T phi, so that in through the symbolism we will make it clear what is the T that we are referring to and what is the phi that we are referring to. And it can easily be seen that such T-norms that we obtain T phi, they are in fact T-norms themselves.

That means, they will be commutative, associative, the identity will still be 1 because phi of 1 is 1 and it will also be monotonic, because phi is an increasing bijection.

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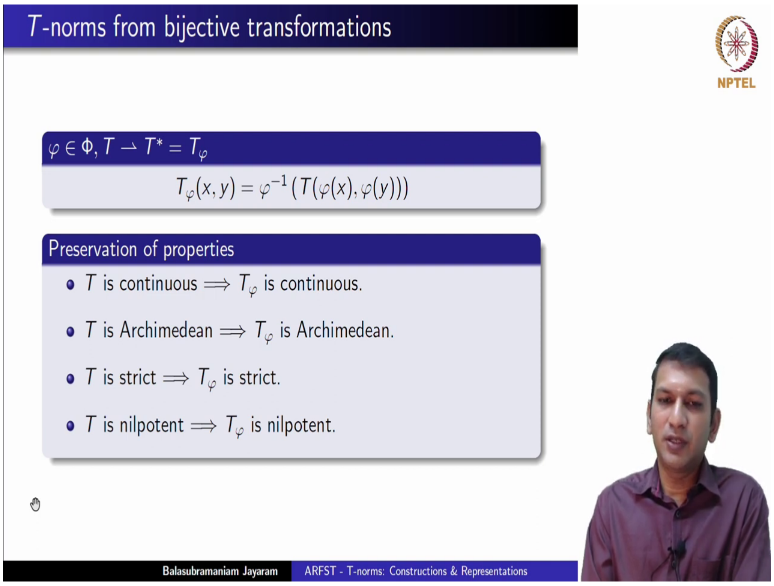

Let us look at what are the properties this transformation preserves. If originally T were continuous T phi will be continuous; if T is Archimedean, please recall that means, if you take any two elements x and y from [0, 1] interval and if you operate you there will exist an n, such that if you operate x on itself n times, it will become smaller than y.

Similarly, there will exist an m, such that if you take this y and operate y on itself with respect to this T-norm T_M times, it will become smaller than x with respect to the usual ordering on 0, 1. So, this is what we mean by, if this happens for every pair of elements we say that the T is Archimedean. Now, if T is originally Archimedean, then this T phi is also Archimedean. If T is strict; that means it is continuous and strictly monotone, then T phi is also strict, continuous and strictly monotone.

If T is nilpotent, T phi is also nilpotent. What is nilpotence? Nilpotence as such means, if you take an element x; then there exists an n such that if you operate x on itself with respect to the T n times it actually becomes 0. And nilpotent T-norm means that every element from open 0, 1 is actually an nilpotent element with respect to the T. So, now these are the properties that this transformation actually preserves. Often in the literature you will see this as T phi is referred to as phi conjugate of T, that is a nomenclature that is very common and popular. So, we will also use this often.

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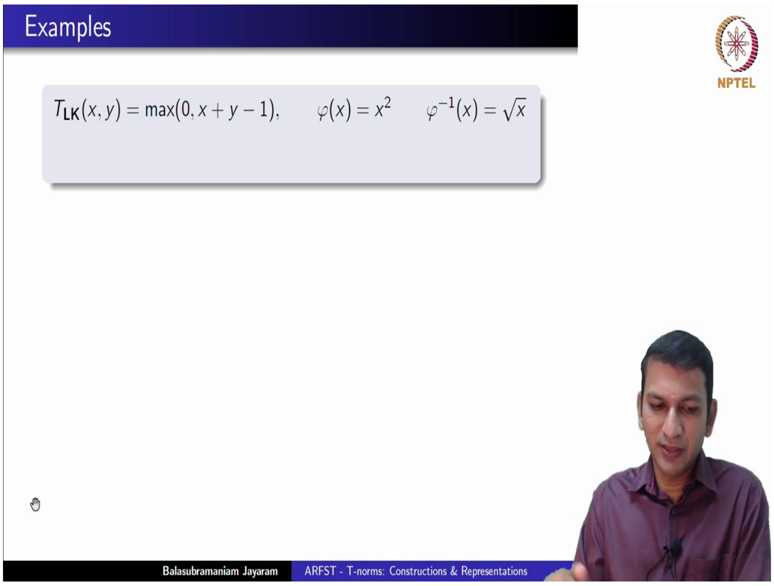

Let us look at some examples. Let us take the Lukasiewicz T-norm and for the increasing bijection, let us look at x square, clearly phi inverse x is root x.

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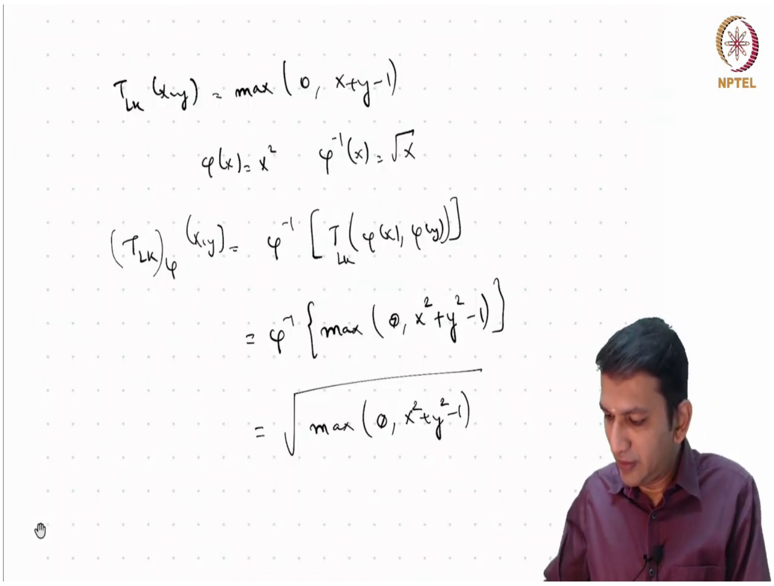

Now, if you actually work out what it will turn out to be, so we have T L of T L K x y is this maximum of 0, x plus y minus 1, we have phi of x to be x square. So, phi inverse x into root x. Now, what we want is T_LK of phi that x y this is phi minus L T of T L K of phi x phi y. So, this will be phi inverse of T_LK of phi x phi y is maximum of 0,, phi x we can substitute as x square plus y square minus 1 phi minus x square root of this. So, it is essentially square root of maximum 0.

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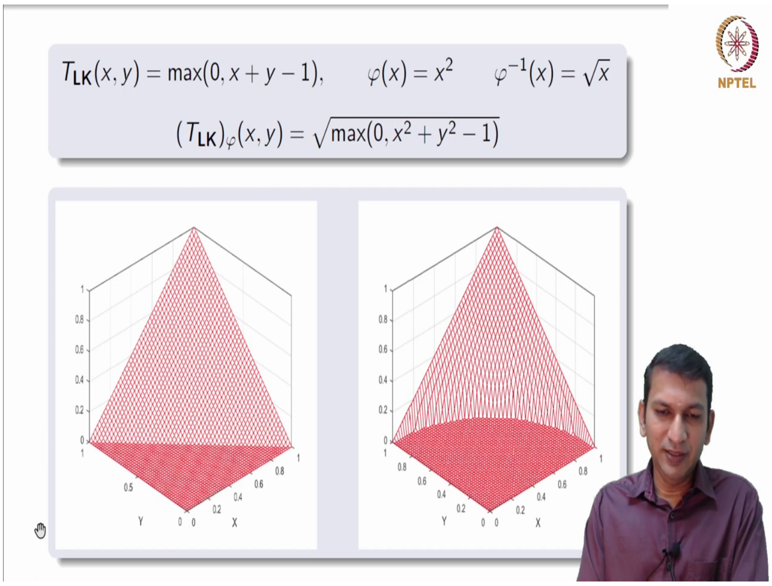

So, this is what you will get as the phi conjugate of T_LK. Now, how will it look like the graph of it? We know this is the usual Lukasiewicz T-norm; the modified the phi conjugate of Lukasiewicz T-norm with respect to X square looks like this. You will see that all the properties are valid; it means if it is nilpotent it will continue to be nilpotent, it is Archimedean, it will be Archimedean, it is also continuous.

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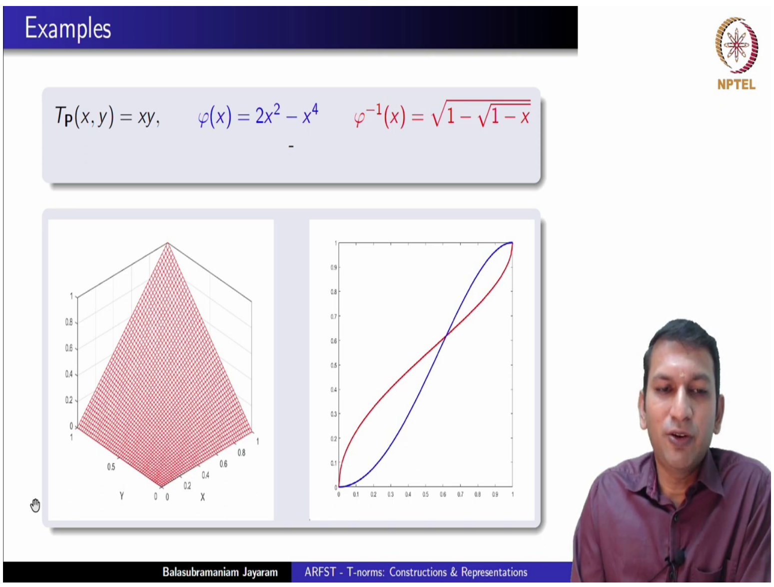

Now, let us look at T_P of x y and let us for the moment consider phi x to be x square, which means phi inverse x is root x again.

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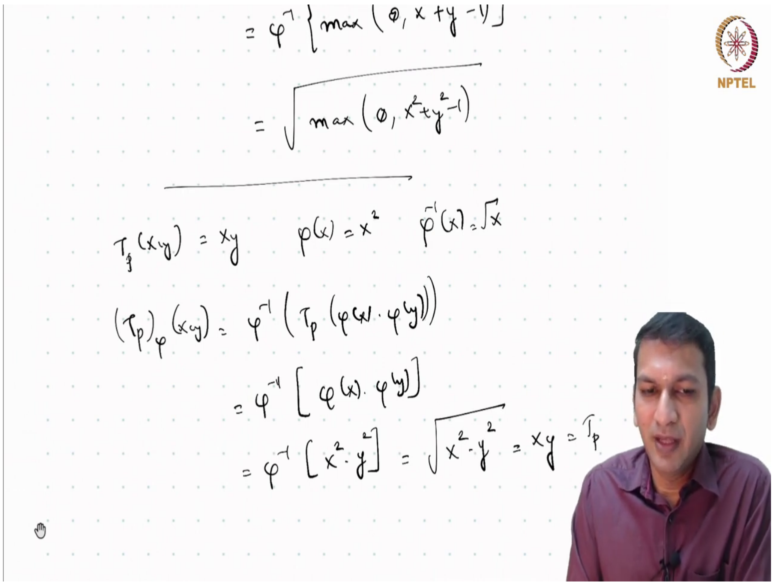

Now, if you want to find out what is T_P of T_P_phi of x y, this should be phi inverse of T P of phi x phi y; this is phi inverse of phi of x into phi y, this is phi inverse of phi x is x square y square. Now, phi inverses root of this x square y square and what we get is actually x y, which is T_P.

So, you see here there are some bijections which actually do not their conjugate turns out to be the original T-norm itself. And you can easily see that this will happen for any x power m and m comes from n. So, let us consider a different bijection. So, let us consider this bijection phi of x, which is 2 x square minus x square 4; clearly this is increasing at 0 it is 0, at 1 it is 1. If you look at what is the inverse of this, without much difficulty you can get this root 1 minus root 1 minus x.

This is the product T-norm and if you are wondering how these two bijections would look like, this is how they would look like. The blue curve is the original phi 2 x square minus x bar 4, the red curve is the phi inverse; note that both of them are increasing as you can immediately visualize. Now, what is the phi conjugate of product T-norm with respect to this phi.

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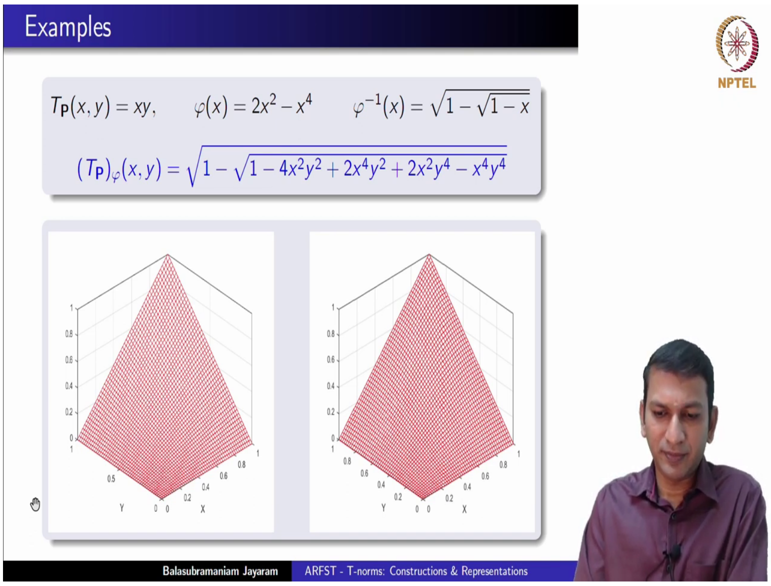

It looks like this, little scary formula; but it is not very difficult to get, because you know how phi inverse looks like. So, on the left you see actual product you know. Now, let us graph the phi conjugate of this product T-norm, this is how it looks like. So, visually there is not much of a difference; but you will see that both of them are Archimedean, both of them are continuous and they are strictly monotone.

But however of course, they are not identical. So, for the product you know they are bijections, which will give you different T-norms when you look at when you apply the phi conjugate on the product T-norm.

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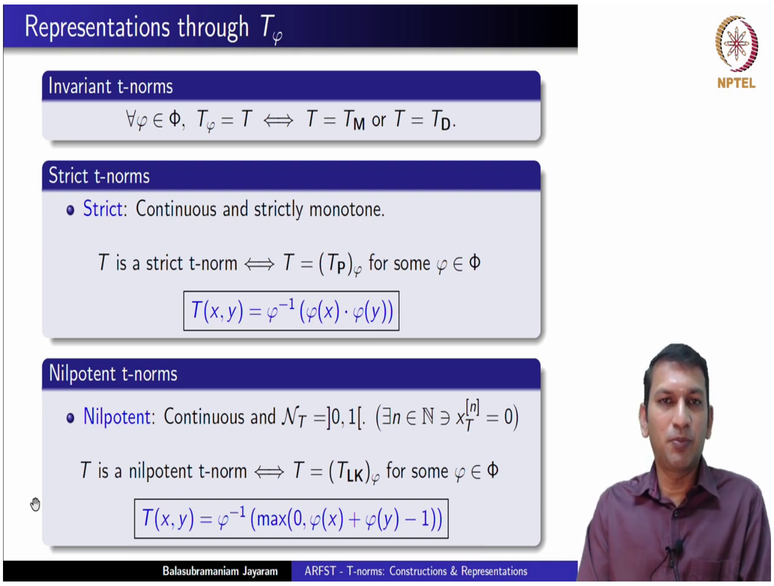

Well let us look at obtain some representations through this. What do you mean by representation? So, knowing the properties of T-norm can you say how it would look like; is there a clear formula that we can give for it? Firstly, if you have a T-norm T, such that for every bijection that you can think of, increasing bijection on 0, 1 that you can think of with the phi conjugate of that T happens to be T itself. Like in the case of product, you know we took phi to be X square and we found that the phi conjugate of T P was actually T_P.

But for a different phi, it was not so. However, if for a T-norm no matter which increasing bijection on the unit interval you consider; if T phi is equal to T, then what we know is it either has to be the minimum T-norm or the drastic T-norm. So, you see here we have some kind of an idea of how what characterizes the minimum or a drastic T-norm; but this does not give us any more information. Of course both minimum and drastic themselves they are actually very easy to represent also, in terms of the formula they are simple, they are not very difficult.

But this tells you if you have a T-norm, no matter what phi you apply on it, it resonance it remains the same; then you know that it has to be either minimum or the drastic T-norm. Now, what about strict T-norms? It is interesting to note that we can show that if T is a strict T-norm, then T is actually a phi conjugate of the product T-norm. Now, here is where we can look at this is a T strict T-norm in terms of this phi conjugacy of product T-norm.

Now, it is not enough to know this, we in fact even know the formula how it would look like; that means any strict T-norm, T would look like this phi inverse of phi x dot phi y. So, the moment you come up with the phi and substitute in this formula, you know it has to be a strict T-norm. Not only that if you take any strict T-norm; that means a continuous and strictly monotone T-norm, you know that it will look like this and all you need to do is find that particular phi for which you can give this representation.

Now, this is with respect to strict T-norm. What about nilpotent T? What are nilpotent T-norms? Continuous and every element has an nilpotent element. What is an nilpotent T element? There exists an n, such that x T of n is 0. Now, T is a nilpotent T-norm if and only if T is a phi conjugate of the Lukasiewicz T-norm. Once again what it means is, if you take any nilpotent T-norm; it is sufficient to think of it as some phi conjugate of the Lukasiewicz T-norm.

So, that means any nilpotent T-norm the moment you think of it, you are only searching for an increasing bijection of 0, 1, such that it can be easily constructed. Now, when we introduced the product T-norm and the Lukasiewicz T-norm, you saw that there were not giving us fantastic algebras or what we expected as in the case of classical set theory; however you see here when we are dealing in the space of triangular moves, they actually hold a very unique position. Let us continue.

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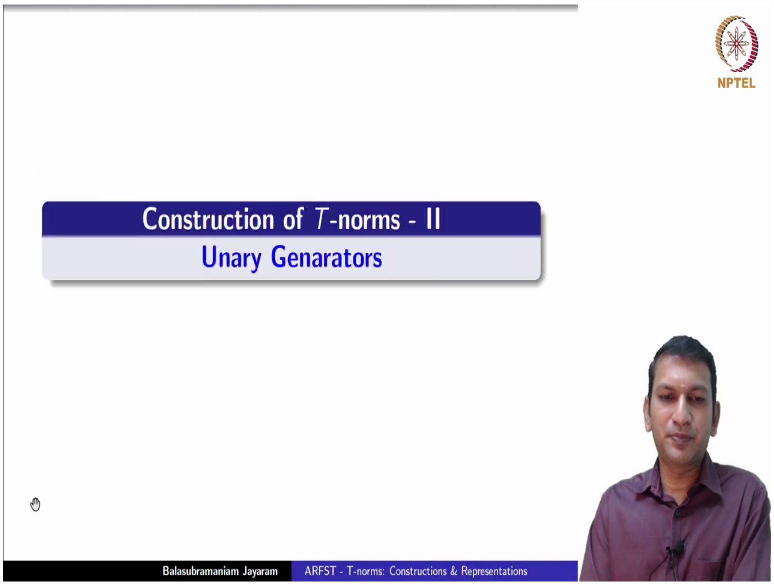

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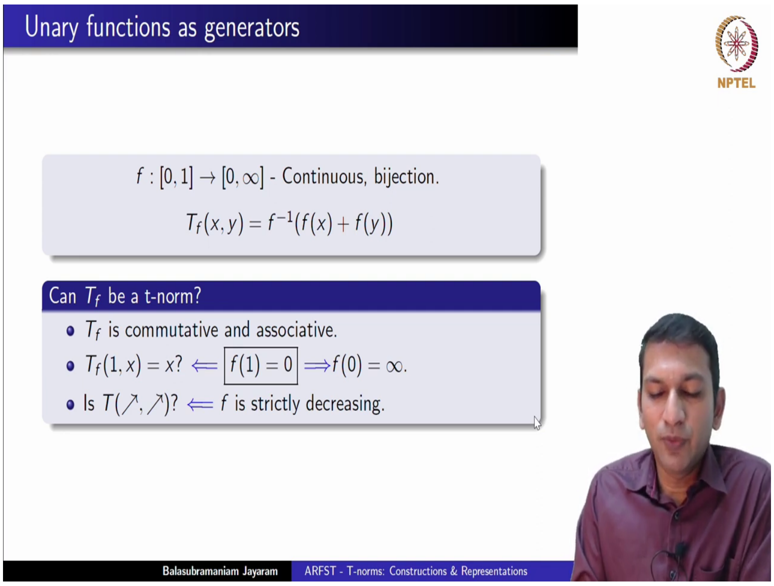

Well, in the previous case given a T-norm, we picked up an increasing bijection of [0, 1] and modified the given T-norm into the new T-norm. Now, let us start with just a function, function from 0, 1 to 0, infinity which is a continuous bijection. And consider this function T f, the f where denotes that this T has been constructed from the function f. Let us define this T of x, y as f inverse of f of x plus f of y. Now, the question is can _ f be made a T-norm? Now, it is immediately clear that this T f is commutative; is it also associative?

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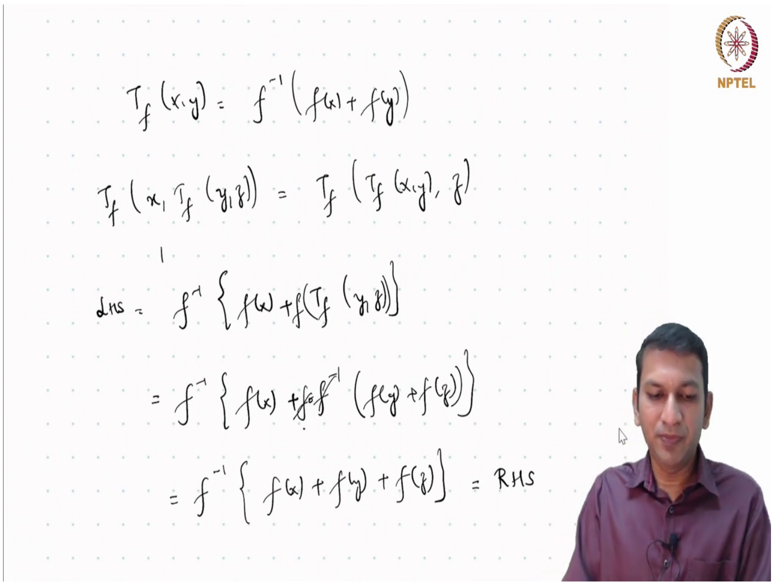

It is easy to see, it is the case look at this you are defining it as f inverse of f of x plus f of y. Now, associativity means what we want is, this should be equal to T f of, let us consider this part LHS. By definition this will look like f inverse of f of x plus T f of y, z. Now, this is nothing but f inverse of f of x plus f inverse of should have been f of T f x square f circle f inverse of f of y plus f of z.

Now, f and f inverse they are inverse of each other and cancel this and what we are left with is f of x plus f of y plus f of z. And from the way we have expanded, it is clear that if you expand the right hand side that will also be turned out to exactly the same formula; that means we have a function which is both commutative and associative.

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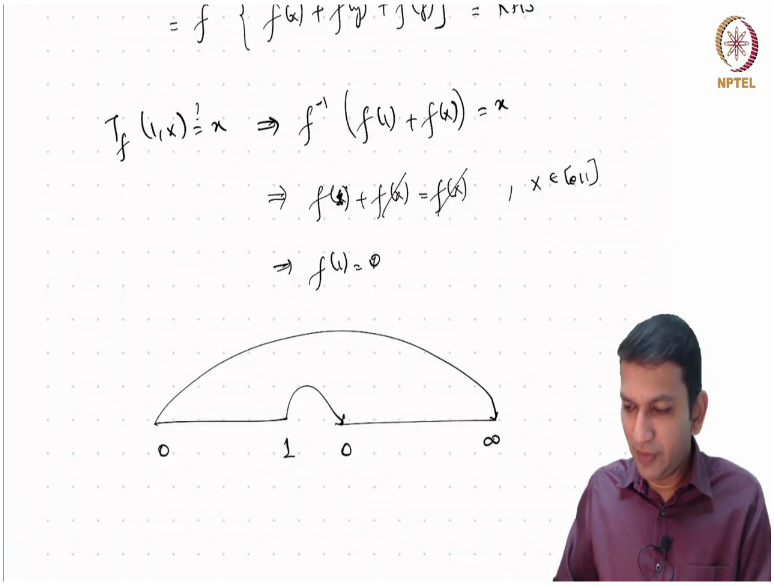

Now, let us ask the question is T f of 1 x equal to x? T of 1, x equal to x means you are asking this question; it means f inverse of f of 1 plus f of x should be equal to x. And since we know that f is bijective, this implies f of 1 plus f of x is actually equal to f of x for any x in 0. Now, since these are real valued number functions, so these are real numbers, what we get is f of 1 has to be 0.

So, if you want 1 to act as an identity, then we want that in the function f at 1 it should take the value 0. But now look at it this also means something else; you can look at this function like this, it is essentially a map from 0, 1 to 0, infinity we are making 1 go to 0 and we want it to be a bijection, which means 0 has to go to infinity. So, this means f of 0 has to be an infinity, the only thing that is left is it should be monotone.

When can we make T f to be monotone? It is clear that f has to be strictly decreasing, because 1 goes to 0 and 0 goes to infinity; that means with respect to the usual order on [0, 1] we see it is a decreasing function. So, we have all these things; we have monotonicity, identity, commutativity and associativity, which means we can get a T-norm.

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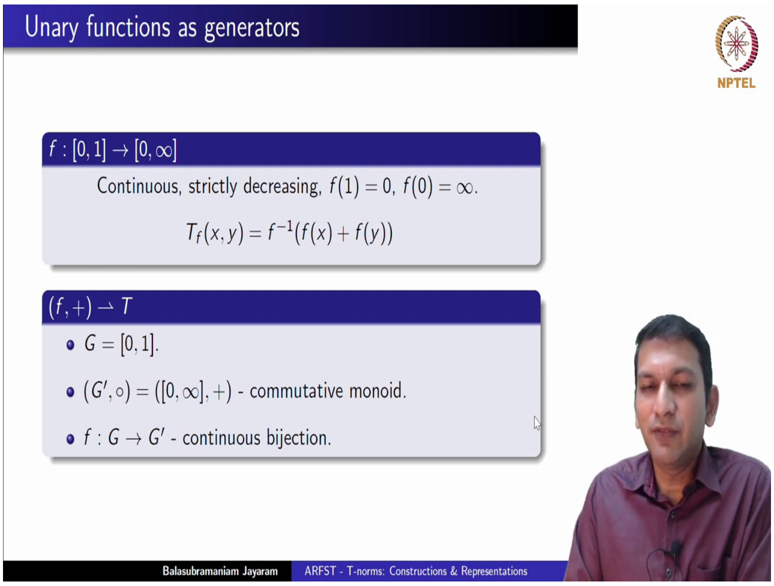

So, we start with the function f which is continuous, strictly decreasing, f of 1 is 0 and f of 0 is infinity and we define the function T_f like this. Now, notice one thing, compared to the previous construction, how does it differ or what are the similarities? Here we start with the function f and the operation plus. So, here G we can look at it as just the 0, 1 interval; G dash circle is essentially the commutative monoid that we have on r plus set of non negative real numbers with addition it becomes a commutative monoid, 0 is the identity there and now we have f to be continuous bijection.

And so, in this sense, it is comparable to the previous construction that we have; of course there we start with the T-norm, but instead here we are starting with a commutative monoid on a different domain.

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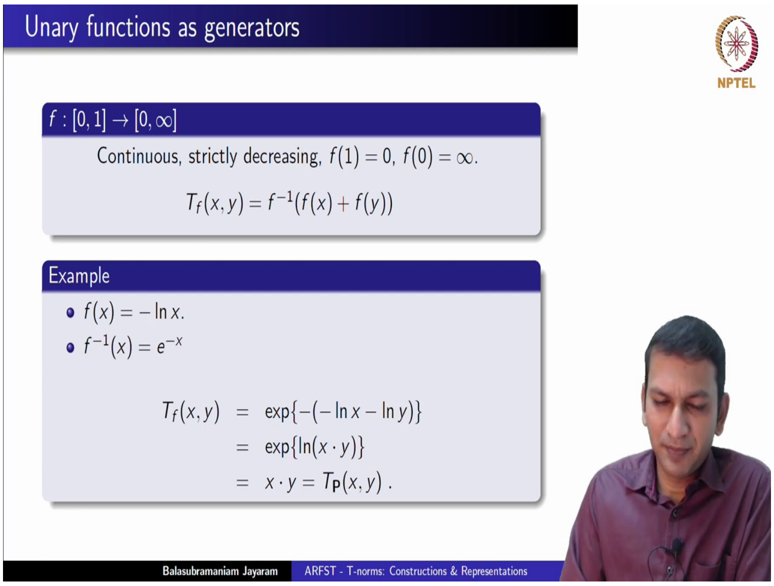

Let us look at some examples; is there really a function f from 0, 1 to 0, infinity which is a bijection? Well, look at this function minus 1, x; we know that immediately f inverse of x is e power minus x. And if you apply this formula T f of x, y is f inverse f of x plus f of y, this is what we get T f of x, y is this, which on simplification looks like this. And we know that e power log we are inverse of each other, which means we get X dot Y which is essentially the product T-norm.

So, that means through this construction also for a particular f, we are able to get product T-norm. And remember recall that any strict T-norm is actually phi conjugate of T_P. So, immediately you should see some connections, but going forward it will become amply clear.

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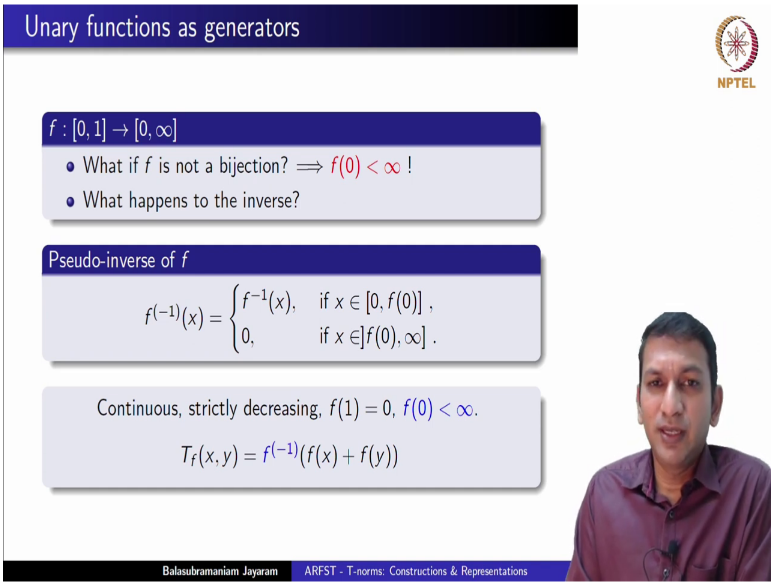

Now, the question is what if f is not a bijection; that means f of 0 is less than infinity. What do we do? Because we do not have a bijection, we do not have an inverse; how do we handle this case? We look at what is called a pseudo inverse of f. Now, this is not a very difficult concept; you see here before for a bijection we had the entire [0, 1] being mapped on to the [0, infinite) interval.

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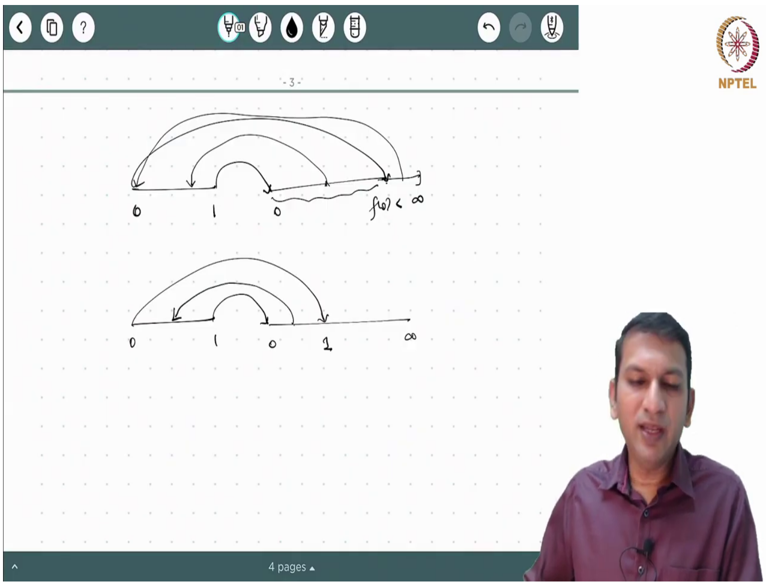

But now what we have is 0, 1, 0 of infinity, 1 is being mapped to 0, but f of 0 perhaps is some value which is less than infinity. Now, how do we define the pseudo inverse? For anybody here between 0 and f 0, you can always find an inverse; but anybody above f of 0 to infinity what we do is, we map them to again 0. So, this is how we define the pseudo inverse of f. So, if x belongs to 0 f the interval 0 f 0, then it is the usual inverse; if it is greater than f of 0, then we map it to 0.

So, now, consider an f which is continuous strictly decreasing f of 1 is 0 and f of 0 is less than infinity. And use the same definition of T f, except that instead of inverse, let us use the pseudo inverse. Now, once again it can be shown that this turns out to be a T-norm.

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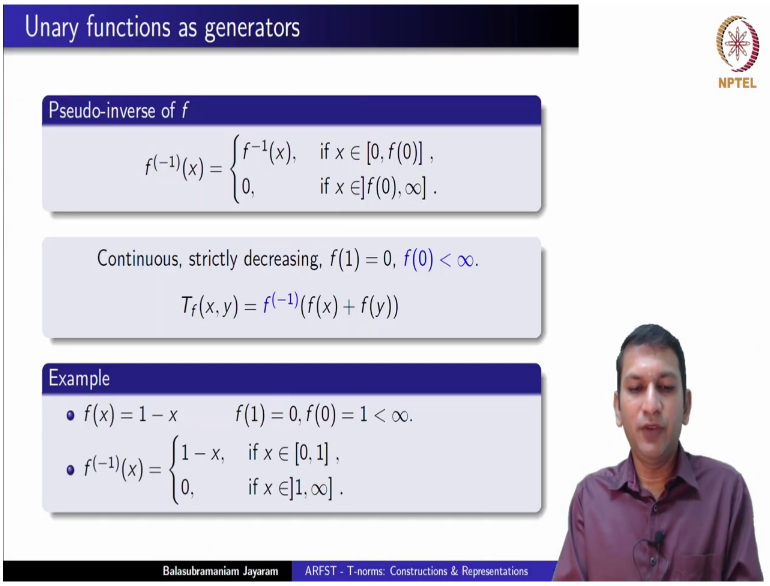

Let us look at an example first let us consider f of x is equal to 1 minus x; that means the function you have looks like this 1 minus x f of 1 goes to 0, f of 0 actually goes to 1. So, you see that there is not a bijection from 0, 1 to 0, infinity. What is a pseudo inverse? It looks like this, which is again clear; because between 0 to 1, we are using the usual inverse, between 1 and infinity, we use the value 0.

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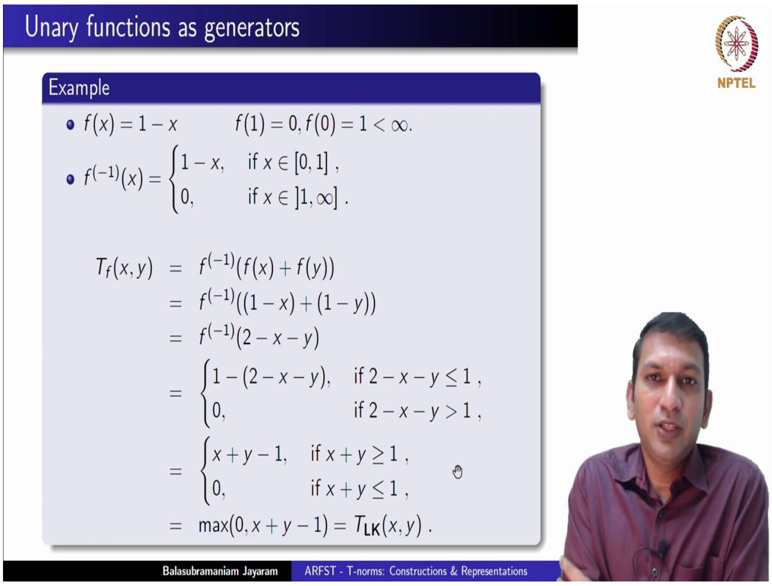

Now, let us construct the T_f from this function. So, this f x this f inverse x, f pseudo inverse x. And how do we construct? As f pseudo inverse of f x plus f x. Now, in this case f of x is 1 minus x. So, let us substitute that 1 minus x plus 1 minus y; this turns out to be 2 minus x minus y, which is clear. Now, comes the question of applying the pseudo inverse. What do we know? We will apply 1 minus x the actual inverse only if the argument lies between 0 and 1; that means we want that 2 minus x minus y should be less than or equal to 1 between 0 and 1.

So, that means we have two cases to consider; if 2 minus x minus y is less than or equal to 1, then it is 1 minus that argument, in which case it is 1 minus 2 minus x minus y. If 2 minus x minus y is greater than 1, strictly greater than, then it is 0. Now, let us simplify both the equations on under the cases and also that we get as the work value; this is simply x plus y minus 1 and this if you take minus x minus y, that is again on this side all you see is that x plus y is greater than equal to 1.

And similarly this is the case where x plus y is strictly less than 1, but in this case you will see that they actually coincide, so it is not a problem we have to use an inequality. But what is this? You could also write it like this max of 0, x plus y minus 1, which is essentially your Lukasiewicz T-norm. So, you see here even through this construction, you are able to get both product and the Lukasiewicz T-norm.

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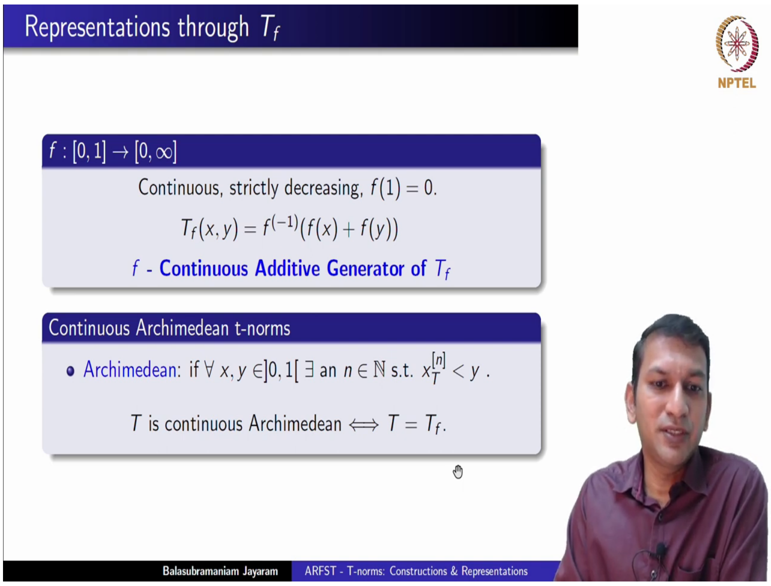

Now, through this construction, do we get some interesting representations for some subclasses of T-norms? Look at this what we have is an f from 0, 1 to 0, infinity which is continuous strictly decreasing with f of 1 is 0. Now, we do not talk about whether f of 0 is infinity or less than infinity; let us only talk about what happens to f at 1 and consider this particular construction using the pseudo inverse of f.

Any such function f, we call it a continuous additive generator of the function T f and we know that this actually is a T-norm, T_f is a T-norm. So, we say it is a continuous additive generator for a T-norm which we denote by T f. What is interesting is if you consider the class of continuous Archimedean T-norms that is mean; that means it is both continuous and also Archimedean. Note, Archimedean means for any pair of filaments there exists an n, such that x power n with respect to T is less than y.

Similarly, there will be an m, such that x power m with respect to T is smaller than y; there exists an m such that y power m with respect to T is smaller than x. Now, what is interesting is this; if you take any continuous Archimedean T-norm, you can see that there will exist a continuous additive generator f, such that this key is obtained from it.

Remember when we thought of a strictly T-norm, we always were searching for an increasing bijection on 0, 1 to 0, 1 such that we could just use that and then get the T-norm as a phi conjugate of product.

Similarly, if you know it is an nilpotent T-norm; look for a phi and get the T-norm as a phi conjugate of the Lukasiewicz T-norm. Now, what we are saying is, if you consider any continuous Archimedean T-norm; all you need to do is look for the corresponding continuous additive generator.

Once you find that f, immediately you can find the t or if you want to think of a continuous Archimedean T-norm, you can always think of it as having a continuous additive generator and this is exactly the formula that you have with respect to that f, that is the power of representation results, it does not stop there.

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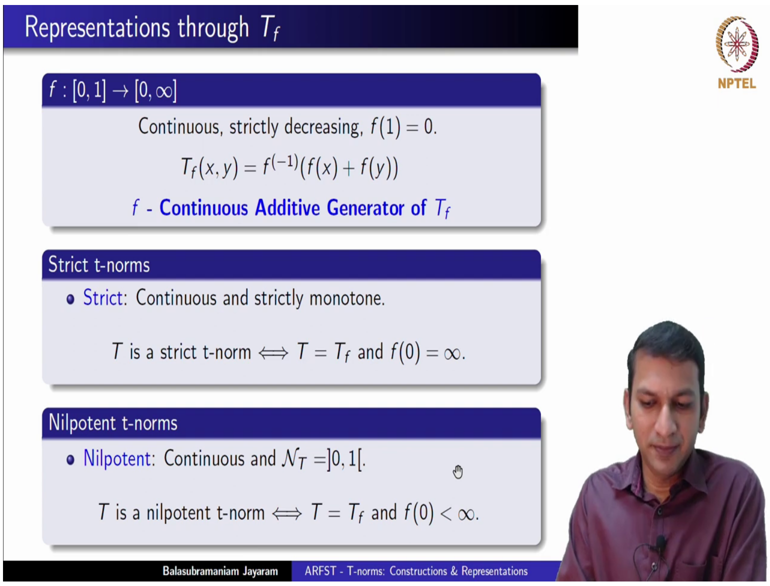

What about strict T-norms? Because, we know that, strict T-norms are also continuous and Archimedean, just that they are also strict T-norm. What we can say is T is strict t-norm if and only if T is generated by a continuous additive generator and the generator is such that f of 0 is infinity, just like -log X.

And you must now immediately be in a position to guess that, nilpotent t-norms can also have a similar representation; yes they do have. Any nilpotent t-norm can be thought of as being generator from continuous additive generator f, such that f of 0 is less than infinity, means f of 0 is finite.

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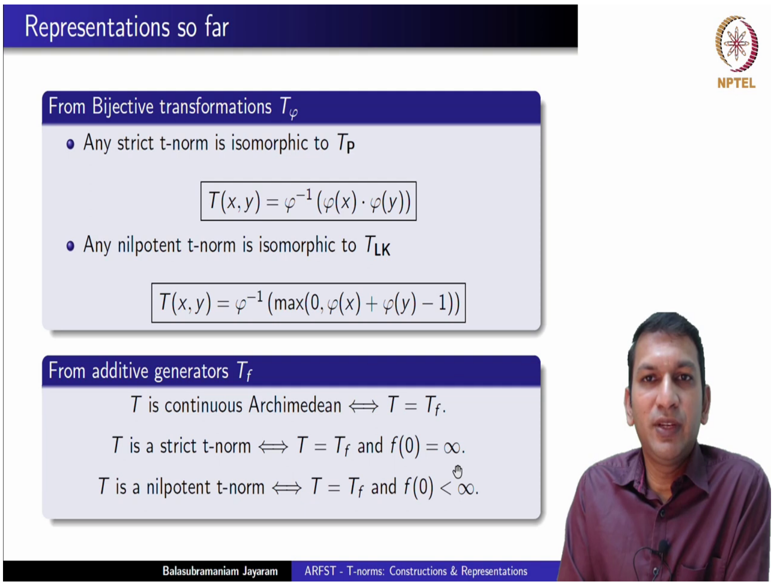

Well, let us look at the two constructions that we have seen so far and the representations that they have given us. From bijective transformations what have we got; that any strict t-norm can be looked at as isomorphic or phi conjugate of product and similarly nilpotent t-norm is isomorphic to Lukasiewicz t-norm. From additive generators what we got was that any continuous t-norm can be part of as being generated by m f and in the same way is a strict or nilpotent it is still generated by a function f, it all depends on whether f of 0 is infinity or finite.

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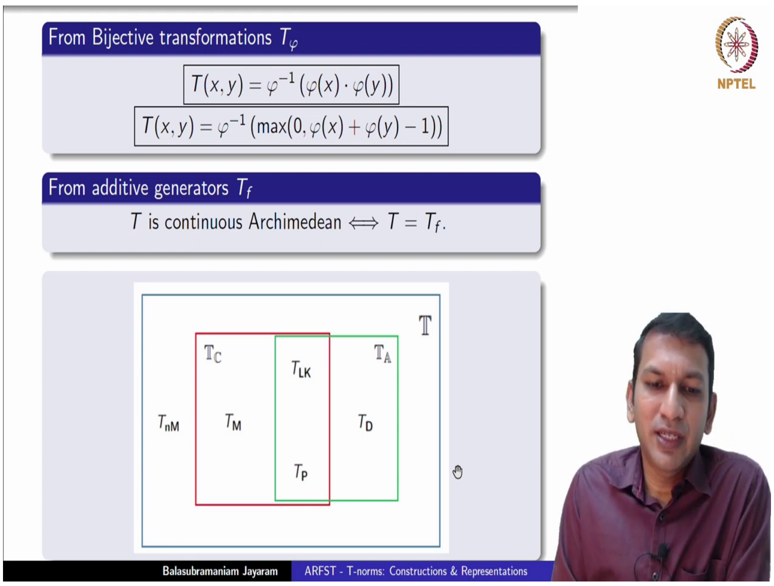

Now, if you consider the class of T-norms themselves, you can have a sub class which is the class of continuous T-norms also the class of Archimedean T-norms. The question is do they intersect? Yes we have seen, we have some examples specifically that of Lukasiewicz and the product T-norm. Now, is there a T-norm which is continuous, but not Archimedean? Yes the minimum T-norm, note that we have an it is an idempotent T-norm; the moment you have an idempotent element, it will not how many of times you operate, it will not go below itself, it is just resolutely stand there.

Is there a T-norm which is Archimedean, but not continuous? Yes, the drastic T-norm. Is there a T-norm which is neither continuous nor Archimedean? Yes, the nilpotent minimum; because it is not continuous and it has an idempotent element, which means it is not an Archimedean. So, you see here the choice of example that we have taken, five example that we have taken; they actually capture different parts of the space of T-norms. But this diagram was not only to illustrate that; this diagram is to ask a question.

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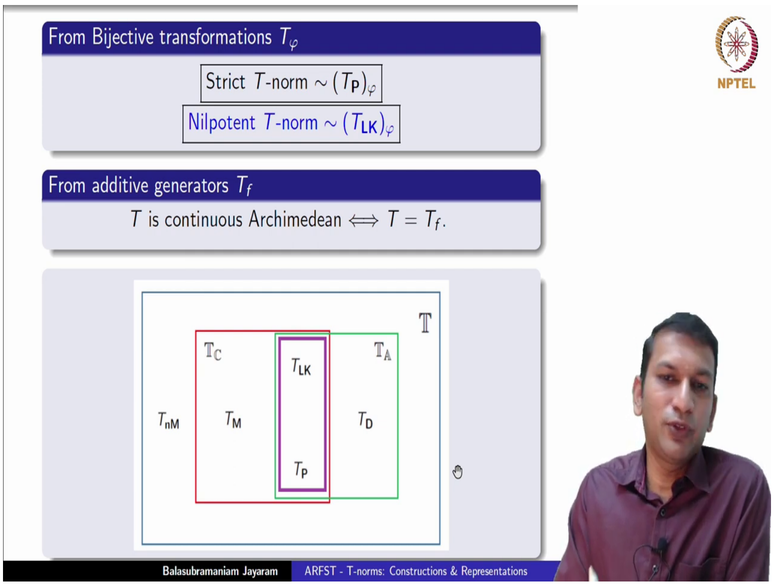

Now, we know that we have a characterization for continuous Archimedean T-norm; that means essentially either they are strict or nilpotent in this form. This is what we have now known; that is if it is continuous Archimedean, then we know that it has such a representation. The question now is, what if it is only Archimedean, can we give a representation? Well, that is not known; but however it can be said that if you have an additive generator either continuous or not, it will turn out to be a Archimedean.

Now, let us ask the same question for the set of all continuous T-norms; can we have a representation for this?

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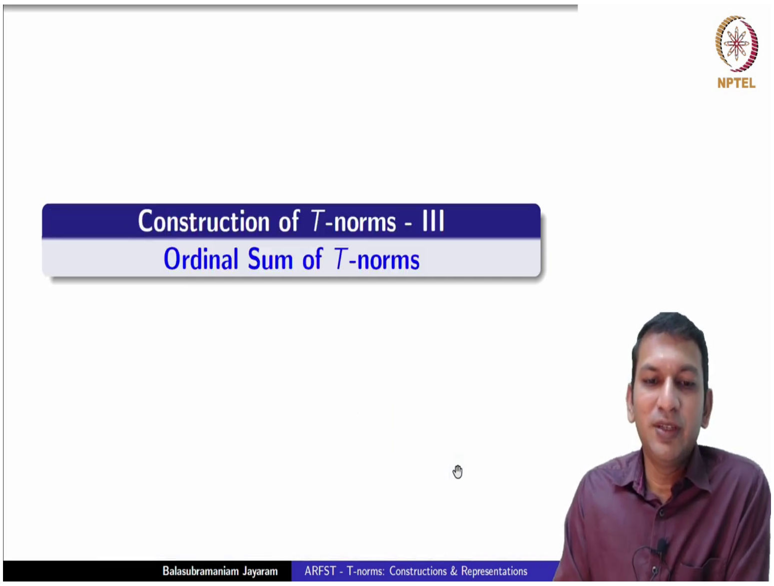

Well, hopefully soon, that is what takes us to the third type of construction.

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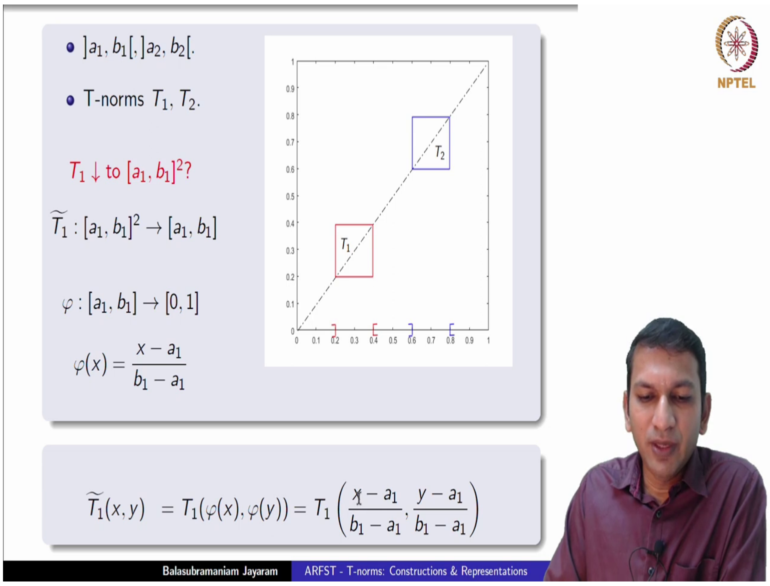

What do we have here? Excuse me. So, let us take the unit interval and you see the diagonal plotted there; let us for the moment take two intervals a 1, b 1, a 2, b 2 and we are given two T-norms T1 and T2. So, in this case a 1, b 1 is just 0.2, 0.4 and a 2, b 2 is 0.6, 0.8 on the figure. So, note that these are open intervals and we are given two T-norms T 1 and T 2. What do we want to do with this? We want to now construct the square a 1, b 1 square and a 2, b 2 square and what we want to do is, we want to restrict T 1 and T 2 only to these two squares.

That means only when the values come from these two squares from a 1, b 1 square, T 1 will act on it and if it comes from a 2, b 2 square, T 2 will act on it. So, essentially what we want is a way of reducing t 1, which is originally defined on 0, 1 square; unit square to just this a 1, b 1 square. And what we want? We want this let us denote this restricted T 1 as T 1 theta. So, we want it only on this term restricted to 1.

How do we do this? Well, from what we have seen so far, it appears that we could just make use of a transformation a phi from a 1, b 1 to 0, 1; one such phi is just the usual linear transformation, using this we could define T 1 tilde of x, y like this. Remember T 1 tilde x and y are coming from a 1, b 1. Now, we want to apply T 1, but t 1 acts only on 0, 1 square; but that is what phi does, phi takes x which is in a 1, b 1 to 0, 1.

Similarly, phi of y is also in 0, 1 and this is how it is different. Now, if you look at it, this is just a linearly transformed form of the arguments and T 1 is active on them. And what is interesting about this T 1. If x is a 1, then this is 0, so then this is 0. If x both x and v are x and y are b 1, then this is T of T 1 of 1, 1; that means at this point b 1, b 1 this value is 1. So, within this it still looks like it is being mapped between 0 and 1, which is what T 1 will do. Now, what we want is, we want to construct a T-norm on the entire domain 0, 1 square using these two. How do we do this?

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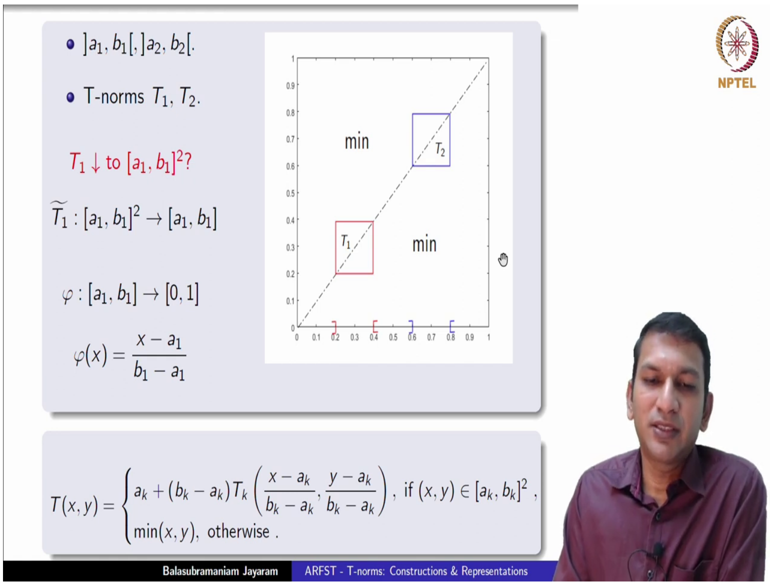

We do it like this, look at this; if you are given many such T-norms, in this case it was just only T 1, T 2 many such T-norm, let us look at this as T-norm. So, that means whenever x and y come from a 1, b 1 square; this is how T of x, y will be defined a 1 plus b 1 minus a 1 T 1 of this value. Now, once again at a 1 what happens, at a 1, then one of them is a 1; that means we are looking at either this boundary or this boundary. If this is a 1, then there is 0; so this entire thing is 0, b k minus a k into that is 0, it is only a 1.

So, at this point on this entire interval, it will be just a 1. Now, what happens at this point when x both x and y are b? That means this is 1 a 1 plus b 1 minus a 1, which is b 1. So, at this point it is b 1. So, here it is a 1 and here it is b 1. So, based on this what we do is, we can actually a for the rest of the domain we can allocate it to the minimum T-norm. Now, it can be verified that this is a T-norm; means commutative, associative, increasing in both variables and one will still remain as the identity element. So, such a construction is called the ordinal sum construction of T-norms.

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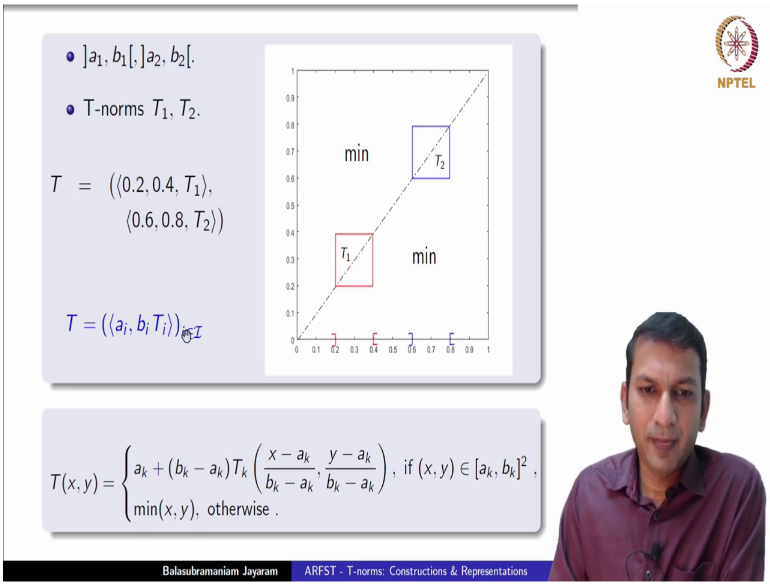

And in this case for this particular figure, we would be writing it like this. So, you see here we denote that there are two T-norms and accordingly two intervals a 1, b 1 is 0.2, 0.4 a 2, b 2 is 0.6, 0.8 and in the rest of the places it is the minimum T-norm. So, that is all we understand. Now, this is the case for two T-norms, but we could have many countably many T-norms T i and this is what we get, this is how we represented.

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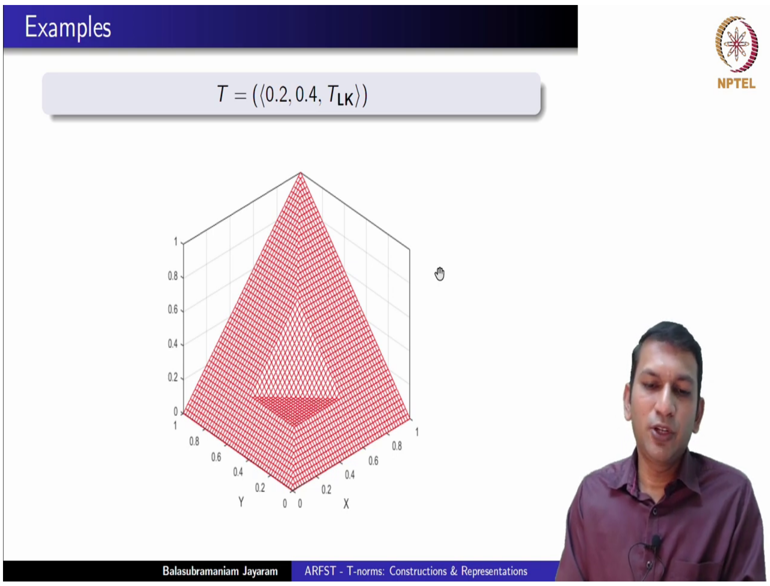

Let us look at some examples. So, each of the T-norms that we take in this ordinal sum, we call it an ordinal summand. So, in this case we have only one summand; that means from 0.2

to 0.4 we are using the Lukasiewicz T-norm to accordingly scale the domain and use the function T_LK to obtain the values for the T-norm.

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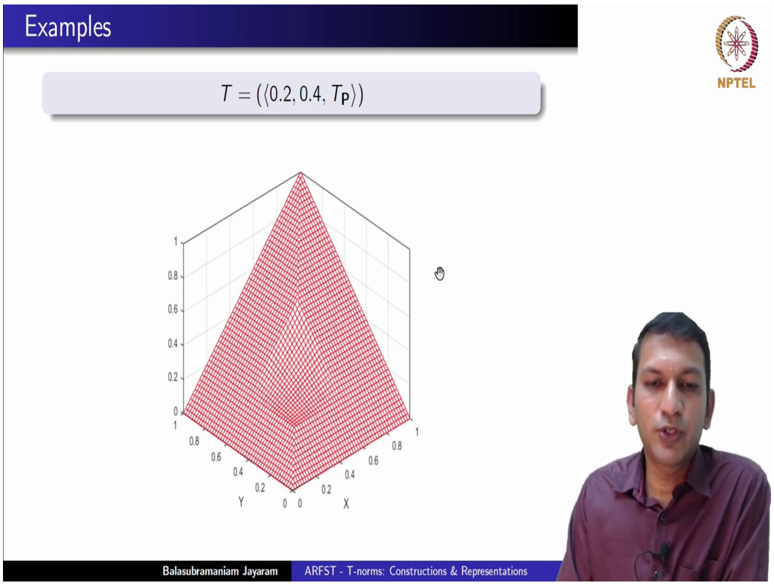

So, as you see here everywhere else it is minimum, only here it is getting the values has some transform values from Lukasiewicz T-norm. Instead if you use the product T-norm, you will see that essentially we are taking the Lukasiewicz T-norm and embedding it some scaled version of it; similarly a scaled version of T P is embedded over this domain 0.2, 0.4 square.

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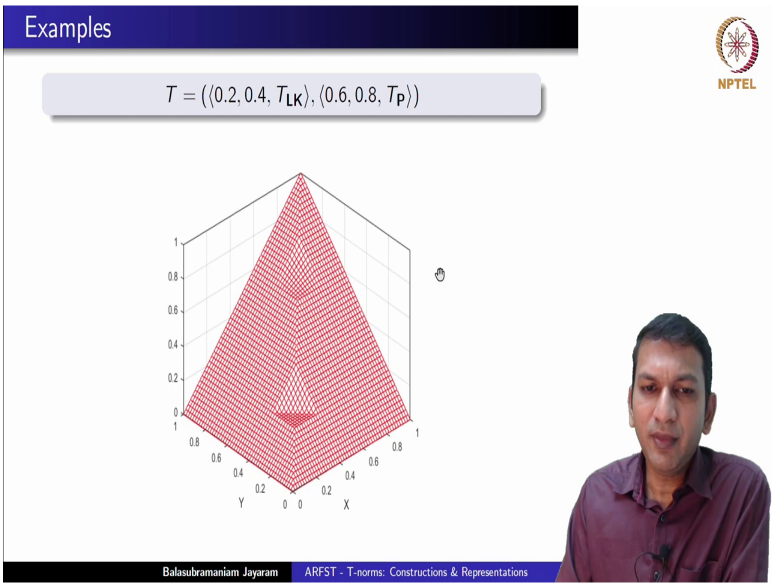

So, now you see here there are two summands between 0.2 and 0.4 it is Lukasiewicz, 0.6 and 0 0.8 that is product and you see this. Note that we are modifying the minimum with two continuous T-norms, Lukasiewicz and product and still it remains to be continuous.

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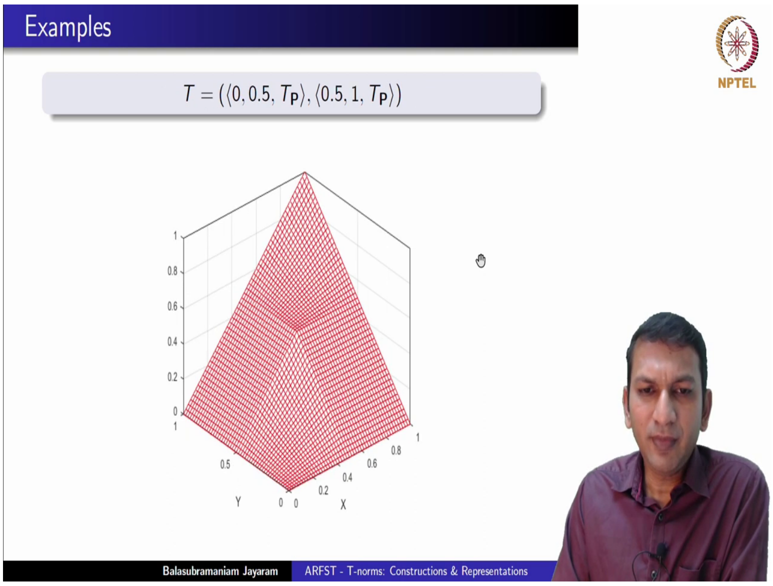

Now, this is two summands, it is not a mandatory that both these summands the T-norms that use are different; they can be same, this is one example of it.

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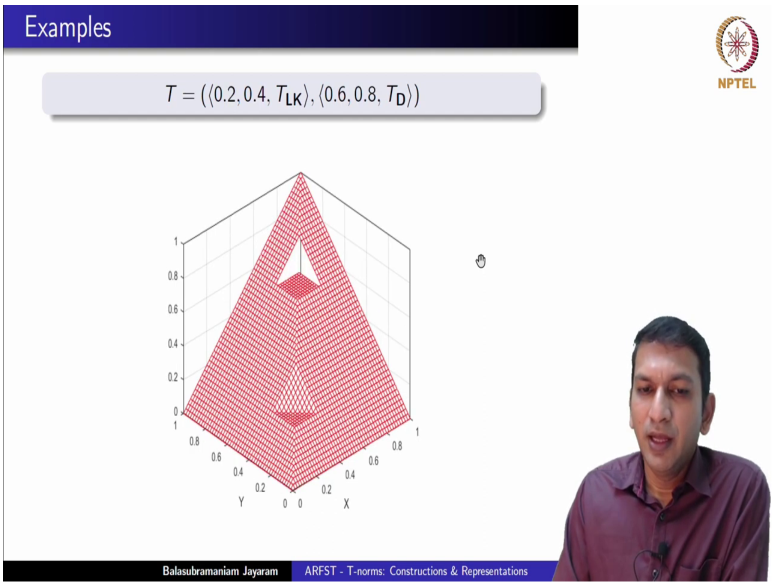

It is also not mandatory that we should only use continuous T-norms to construct this. So, Lukasiewicz is continuous and as you can see the drastic T-norm is not.

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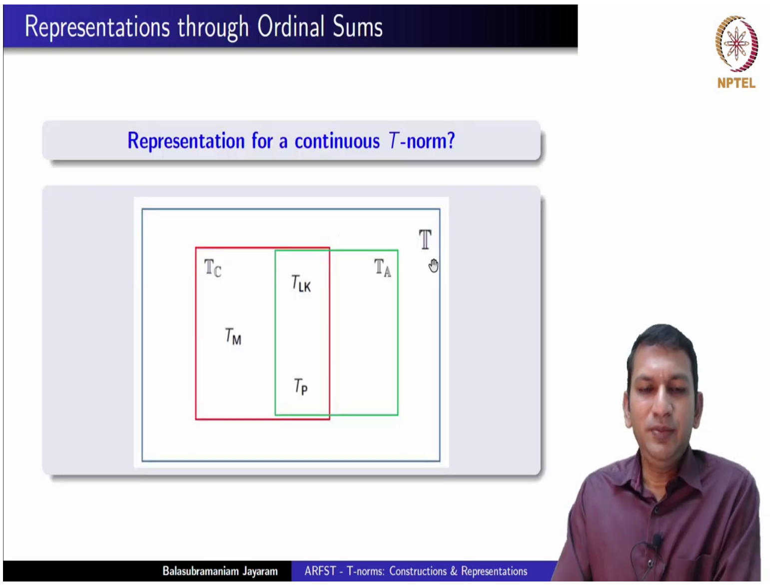

Now, we ask the question, can we have a representation for any continuous T-norm? So, we know that we have for this part we wanted for the entire red square.

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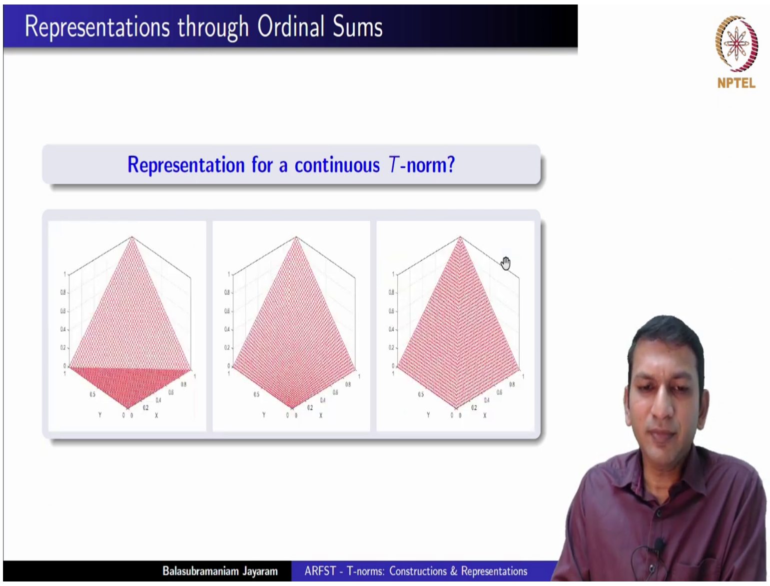

Now, what are the continuous T-norm that we know this prototypical examples? Lukasiewicz - continuous and nilpotent, product - continuous and strictly monotone, minimum which is continuous, but not Archimedean.

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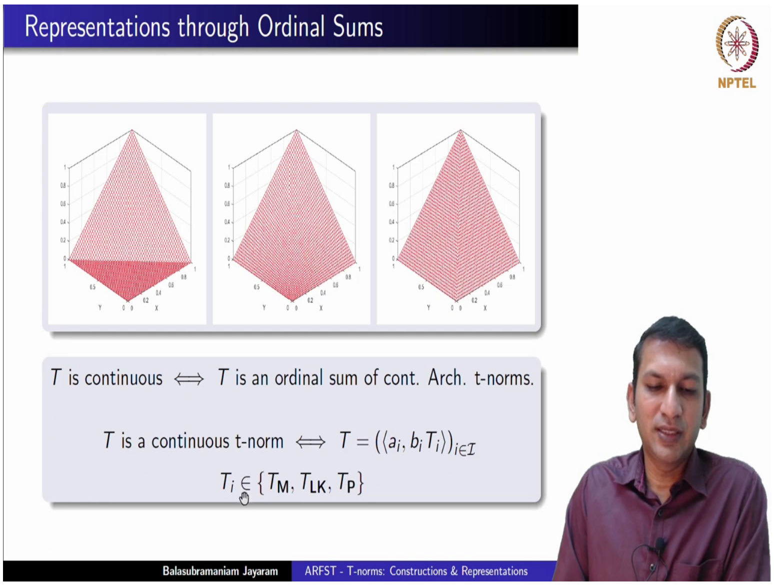

What is interesting is it can be shown that any continuous t-norm is in fact an ordinal sum of continuous Archimedean t-norms. Now, remember continuous Archimedean t-norms; the continuous Archimedean t-norms themselves are represent t-norms; that means they can be obtained from continuous additive generators and they can either be strict or nilpotent, which means either they are isomorphic or phi conjugate of T P or T L K and outside of that you have only minimum. So, which means these are the three T-norms that are playing a role in constructing continuous t-norms.

In fact, it can be written like this, T is a continuous t-norms if and only if T is an ordinal sum of some t-norms and each T_i looks like this, each T_i is either minimum, Lukasiewicz or product.

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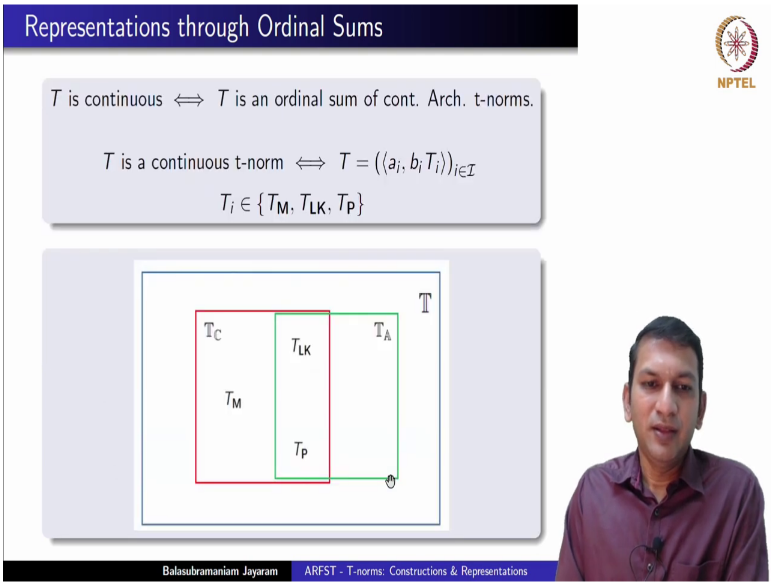

So, with this we have found for a large class large subclass of t-norms, a representation for them either in terms of five conjugacy or in terms of continuous additive generator or in terms of the ordinal sum representation.

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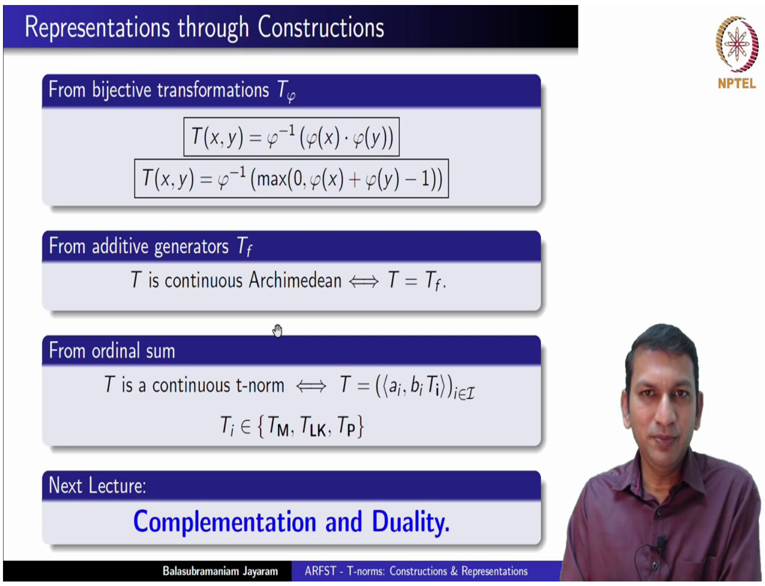

So, this so far we have seen three constructions, each one of them have led to a representation for particular subclass of T-norms. From bijective transformations we saw that any strict T-norm is phi conjugate of product; similarly any nilpotent T-norm is a phi conjugate of Lukasiewicz. From additive generators we found that every continuous Archimedean in

T-norm is generated by continuous additive generator. And from ordinal sum we saw that, any continuous T-norm is an ordinal sum of T-norms, each of which is either minimum, Lukasiewicz or a product T-norm.

Now, what next? In the next lecture, we will look at complementation and through complementation dual function of a T-norm. We have seen complementation from a lattice theory point of view; it is similar to that what we will take is the beginning approach, but of course later on we will diversify. This is what we will discuss in the last of the lectures this week.

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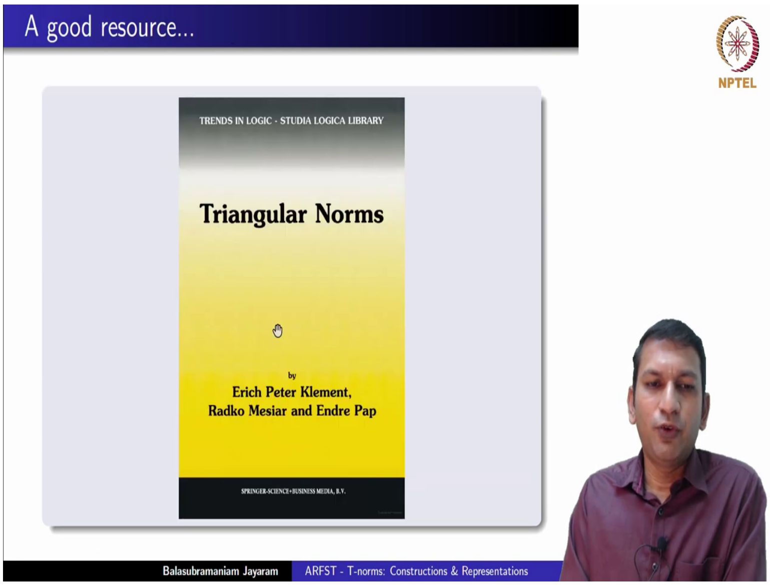

And for this lecture the topics that are covered in this lecture this book by Klement and Mesiar and Pap; titled Triangular Norms is a very good resource for you.

Thank you once again for joining me in this lecture and hope to see you soon in the next lecture.

Thank you.