Approximate Reasoning using Fuzzy Set Theory

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Lecture - 13
Triangular Norms: Algebraic Aspects

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Hello and welcome to the 3rd of the lectures in this week under the course titled Approximate Reasoning using Fuzzy Set Theory, a course offered over the NPTEL platform.

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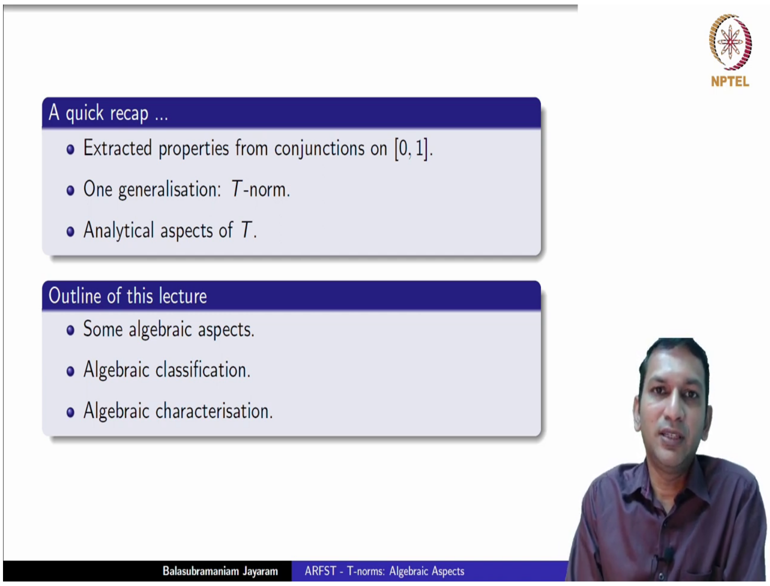

In the previous two lectures, we had looked at coming up with an axiomatic definition of one particular generalization of conjunctions to the setting of fuzzy sets that of T-norms, T-norms in brief, triangular norms. We have also seen some analytical aspects of triangular norms especially different types of continuity.

In this lecture, we will look at some algebraic aspects of T-norms both at the elemental level and also at the level of functions. And, with that we will give some simple classifications of T-norms themselves. And, what is interesting is we will also see that based on the order theoretic and analytic and algebraic properties that they enjoy, the confluence of these three sets of properties actually will give rise to some interesting characterization of T-norms on the unit interval [0,1].

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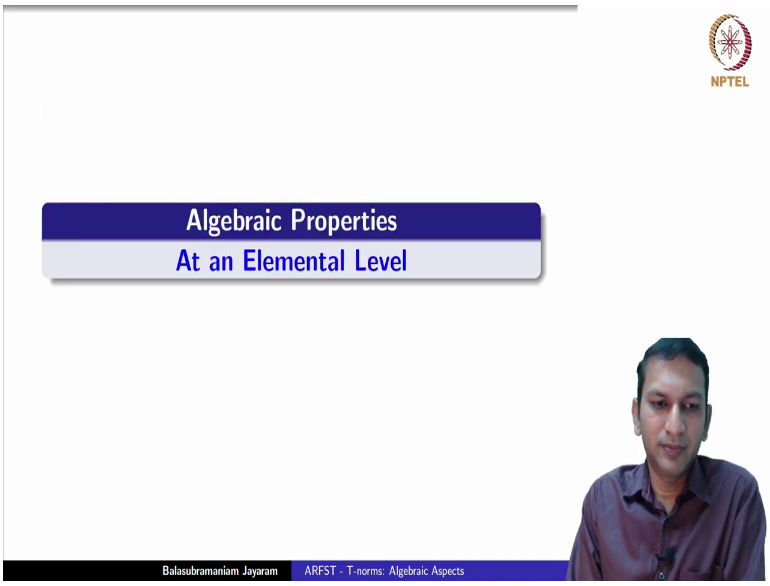

Let us look at some algebraic properties at the level of an element.

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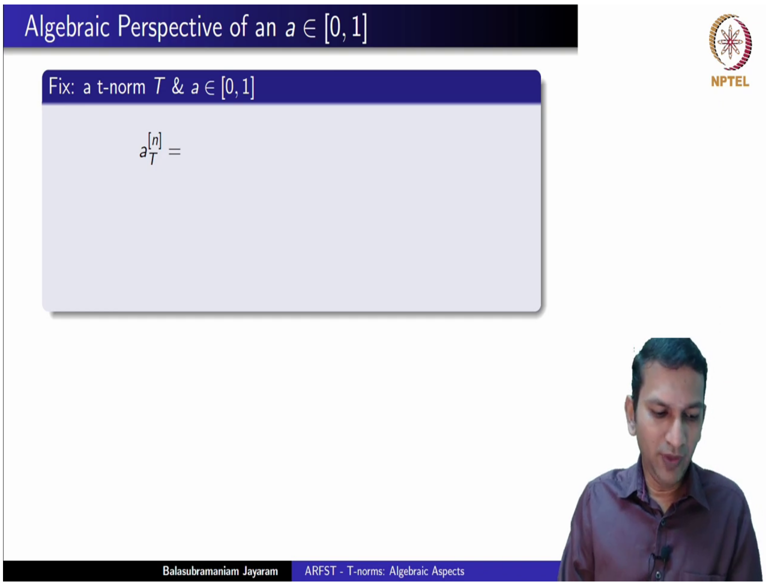

So, to begin with we fix at t-norm T and take an element a from the unit interval [0,1]. Let us introduce some notation. So, a is an arbitrary element, T is a fixed T-norm and what do we understand by this symbol?

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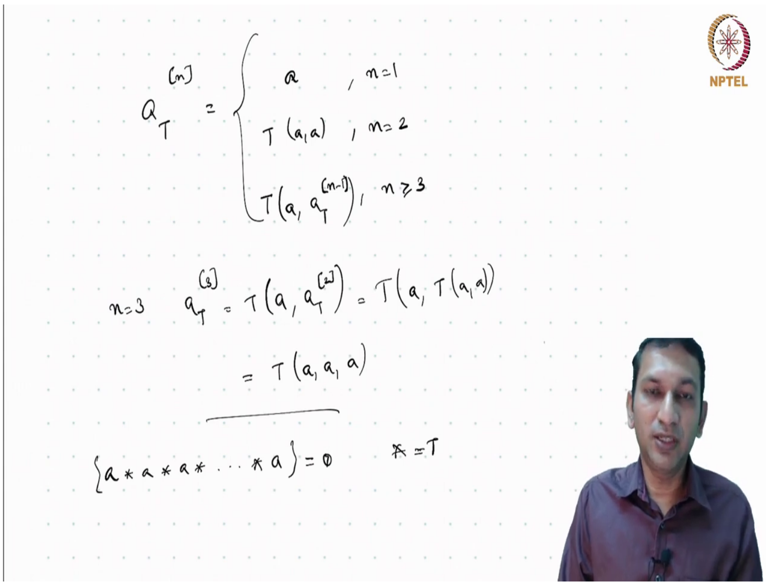

So, if n is 1 then this is simply a, if n is 2 then this is simply T of a, a and for all n greater than or equal to 3, we can write the T of a, a T of n minus 1. So, recursively we define this, it is clear that if n is equal to 3 then a T of 3 is T of a, a T of n minus 1 2 which can be written as T of a, T of a, a. Now, by associativity we can remove this and talk about this expression. So, by associativity, this binary operation can be extended to n-ary operation.

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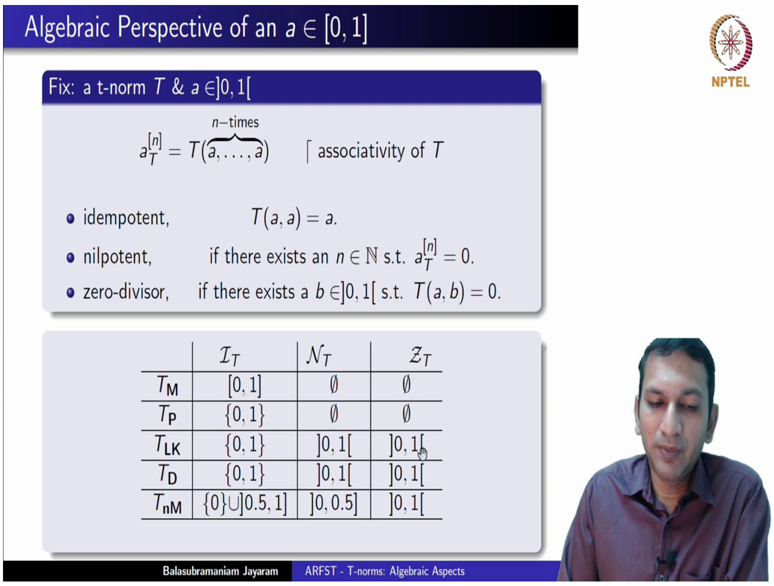

So, if you look at this, we could just think of it as operating a on itself n times. Once again the associativity of T allows us to look at this particular operation like this. Now, given an element a, when do we say it is idempotent? This is familiar to us, this with respect to the T; if T of a, a is equal to a. So, we are going to talk about properties of an element with respect to fixed T. So, idempotent is very well known to us. What is nilpotent? When do we say an element is nilpotent; if there exists an n a natural number N such that a T of n is 0.

What does that mean? It means that if you take if an element were to be nilpotent, if you operate on it countably finite number of times it should become 0, where I am using the infix notation. The star is a t-norm T. So, if an element possesses this property with respect to the chosen t-norm, we say that element is a nilpotent element with respect to the t-norm T. When the same element is a zero-divisor, this concept should be familiar to you again. If there exists a b in the open interval (0, 1) such that T of a b is 0.

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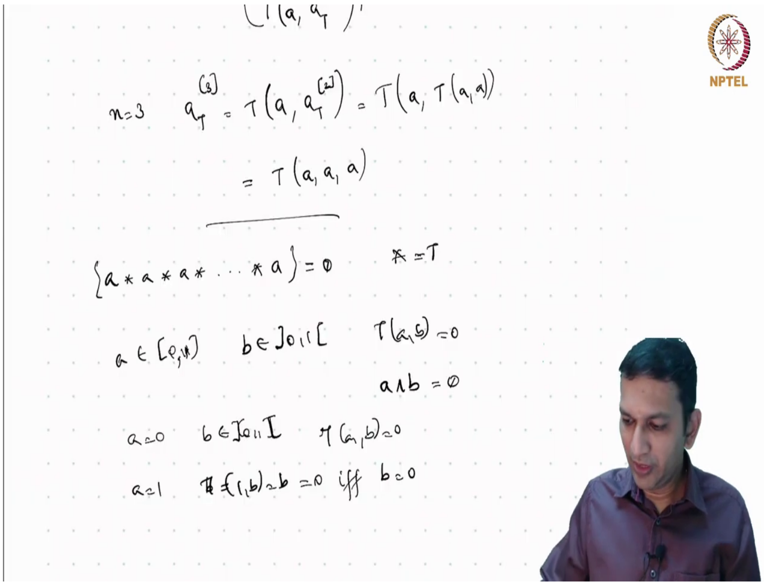

Now, this is much like you take an element of [0, 1] and find the b in (0,1) such that T of a, b is 0. Perhaps, you recognize this if you were to write it as mean, this a and b is 0. This is what we have seen as the complement of an element in the setting of lattices. Here, we are looking at it as a 0 (Refer Time: 05:02). Now, that when a is 0, then for any b practical in the open interval T of a, b is 0 for any T.

And, if a is equal to 1 then b cannot come from an open interval (0, 1) because T of 1, b is actually b and b is, this will be 0 if and only if b is 0. So, essentially when we talk about nilpotents and zero divisors, we will consider the a to come from open interval 0 ok. Let us look at these three algebraic properties that we can define with respect to an element and a given fixed T. How do our familiar t-norms behave or what are the corresponding sets of idempotent, nilpotent and zero-divisors with respect to these five t-norms? Let us look at that.

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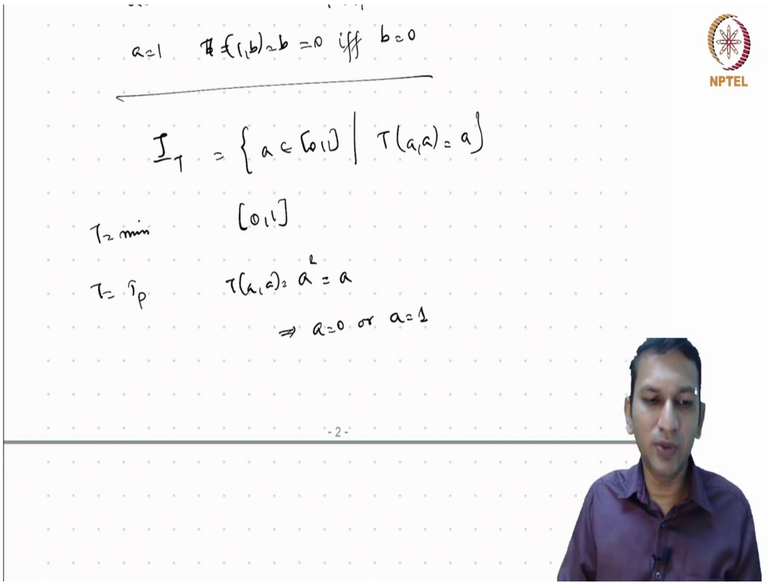

We know that; so, we are looking at idempotents that means; let us indicate by I_T set of all a in [0, 1] such that T(a, a) is equal to a. Now, if T is the minimum t-norm, then we know that this is the entire interval [0, 1]; because we have seen that minimum is the only idempotent t-norm. In fact, if we consider 0 a to be 0 or a to be 1, then for any t-norm these are idempotent elements. In fact, they are called the trivial idempotent elements. And, we also know that for the product t-norm T_P we do not have any idempotent elements.

Because, if you take this to be T_P, then T of a, a is a square; if you want this to be a then this can happen only if a is 0 or a is equal to 1. So, it has only the trivial a idempotent elements. We have seen that the Lukasiewicz conjunction is not idempotent. The drastic t-norm almost everywhere it is 0, it is not idempotent. But, what about the nilpotent minimum itself, does it have any idempotent elements? Well, let us look at the definition of nilpotent element.

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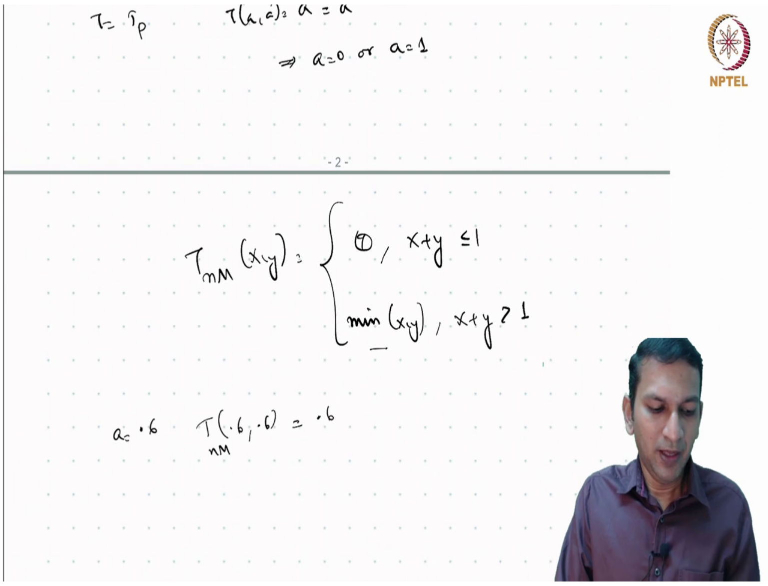

We preserve this phenomenon, it says it is 0; if x plus y is less than or equal to 1 and minimum of x, y if x plus y is greater than 1. Now, you see that we have the minimum operation coming into picture here. If you take 0.6 as a, look at T of 0.6, 0.6 with respect to minimum; 0.6 plus 0.6 is greater than 1. So, it is minimum of these two which means it is points; that means, nilpotent minimum does have idempotent elements, non-trivial idempotent elements. But what exactly are they? How do we find them?

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Let us look at their graphs. So, on the left you have the minimum t-norm, on the right you have the nilpotent minimum t-norm. And, it is easy to see immediately that we have obtained this nilpotent minimum from the minimum by reducing some part of the graph to the 0 reigon.

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Now, consider the diagonal of the unit cube, not diagonal of the square diagonal of the unit cube. We know that if it has to be idempotent that element the value that it should take at that point on the diagonal T of a, it should be actually; that means the graph of the corresponding t-norm should be touching this slide. You see that for the minimum it touches everywhere.

For the nilpotent minimum it touches only from open 0.5 to 1 whereas, on this line it is 0. So, if you see from this point of view, everybody here is actually the idempotent. So, it is easy to see, it can also be worked out from the formula that if you look at it the minimum t-norm, the idempotent set is the entire [0, 1] interval.

Whereas, for the product Lukasiewicz and drastic they have only the trivial idempotent elements. And, in the case of nilpotent minimum of course, other than the trivial idempotent elements, the entire open interval 0.5, 1 every element in this interval is an idempotent element. Now, let us look at what is nilpotents and how to find the nilpotent set for a given t-norm T.

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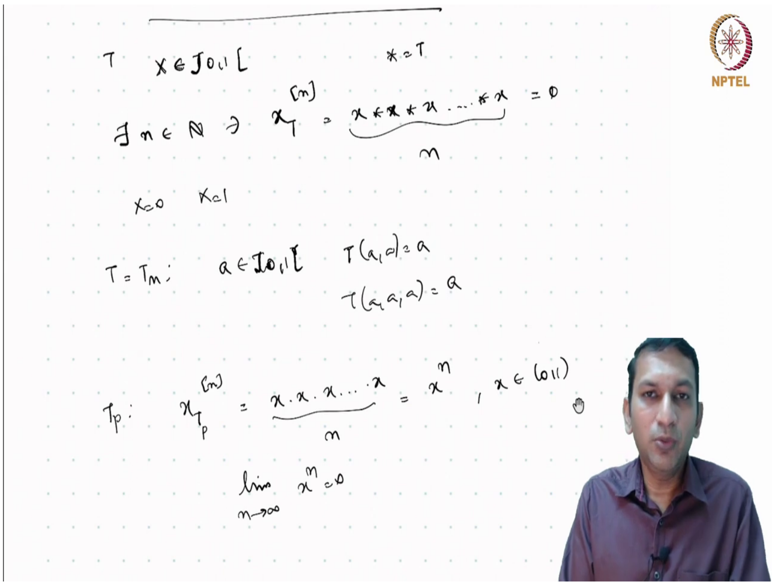

So, what is nilpotent? You fix a T, you take an X element of 0, 1 open 0, 1. Note that once again here that if X is 0, then it does not go anywhere. So, what we want here is that they should exist and n such that x T of n which is nothing but x composed with itself n times should become 0, where the n fix notation I am denoting the t-norm T. So, this is what should happen. If X is 0, then clearly 0 star 0 is 0 and if X is 1, we know that how many of times you operate 1 with itself, it is not going to go to 0.

So, typically when we talk about nilpotent elements in zero divisors, we exclude the endpoints both 0 and 1. So, now, let x come from the open interval 0. Now, if you look at T to be T m, take any x any a in 0, 1 open 0, 1 and what we see is T of a is a, T of a, a, a is again the associated with a. Because, every element is idempotent element, it cannot be an idempotent element.

For the minimum we do not have any idempotent elements. What about the product? Well, if you take T_P then x T_P of n means x dot x dot x n times there is nothing, but x power n. Remember, x comes from the open interval 0, 1. Now, we know that a limiting case n tends to infinity x power n is 0. However, for no finite or countably finite n it is going to take 0 which means node element is a nilpotent element here. What about the Lukasiewicz?

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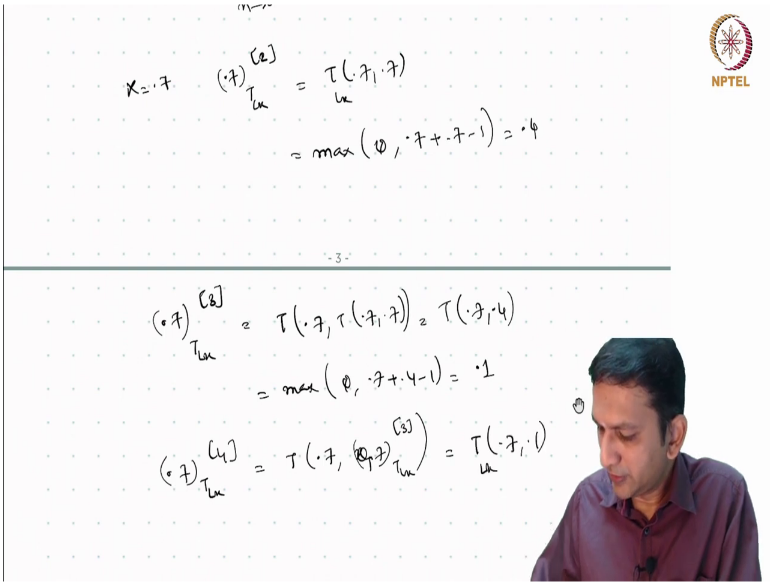

You know yes it can be seen that, if you consider this let us take X to be 0.7. Now, 0.7 with respect to Lukasiewicz, let us do 2 n is equal to 2; that means, this T of 0.7, 0.7. By definition minimum this maximum 0, x plus y minus 1. So, this will turn out to be 0.4. Now, if you write it once more 0.7 with respect to T and 3 and 3, what we get is T of 0.7, T of 0.7 T of 0.7 which is nothing but T of 0.7, 0.4. Now, this is max of 0 per 0.7 plus 0.4 minus 1, this is 0.1.

Now, you see that it is already decreasing, take 4 then it is T of 0.7, x T 0.7 T lk of 3 which you know is 1 0.1 is 0.1 with respect to you.

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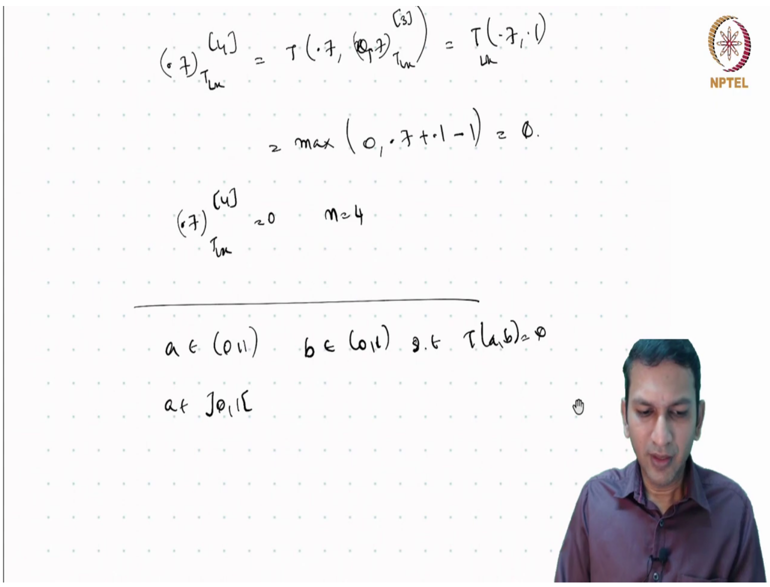

See here maximum of 0, 0.7 plus 1 point minus 1, this is 0. So, if you take 0.7 in practically 4 steps it is 0; that means, for n is equal to 4 0.7 being operated on itself with Lukasiewicz t-norm goes to 0. So, this is a nilpotent element of the Lukasiewicz t-norm. If you work out like this, you will see that the entire open interval (0, 1) are nilpotent elements for Lukasiewicz and also for the drastic t-norm.

However, for the nilpotent minimum t-norm what we have is only the values belonging to 0 to close to 0.5, they become the nilpotent elements. This can be easily worked out. Now, let us look at zero-divisor. So, what does zero-divisor say? If we have an element a, once again we are only considering the open interval (0, 1). As was already mentioned, we will interchangeably use either this notation or this notation to indicate open interval, openness of the intervals on either side.

Then, we need the b once again in open interval (0, 1) such that T(a, b) should be equal to 0. Now, instead of working out on the sheet, let us try to see whether we can look at it from a geometric perspective.

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These are the three t-norms that we have. On the top left is the Lukasiewicz, top right is the nilpotent minimum and on the bottom is the plastic minimum. Now, you will see here definitely all of them have zero divisors. Because you have essentially for zero divisor, what do we need? Two elements a and b neither of which is 0, but T(a, b) is 0. You see here everybody in this region almost satisfy, any pair you take here a, b, T(a, b) is 0. Similar is the case here and similar is the case here; that means, they have zero divisors.

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On the other hand, if you consider minimum product you see that they are actually becoming 0 only up at the boundaries when either X is 0 or Y is 0 as mandated by the t-norm, because of its monotonicity. Nowhere else they become 0; that means, these are two t-norms, that do not have zero divisors whereas, these three t-norms among the 5 that we are considering, they all have zero divisors.

If you work out, it will be interesting to see that minimum and product as we have seen the zero divisor set is empty. However, every element with open interval 0, 1 is a zero divisor for all the three t-norms of Lukasiewicz, drastic and the nilpotent minimum.

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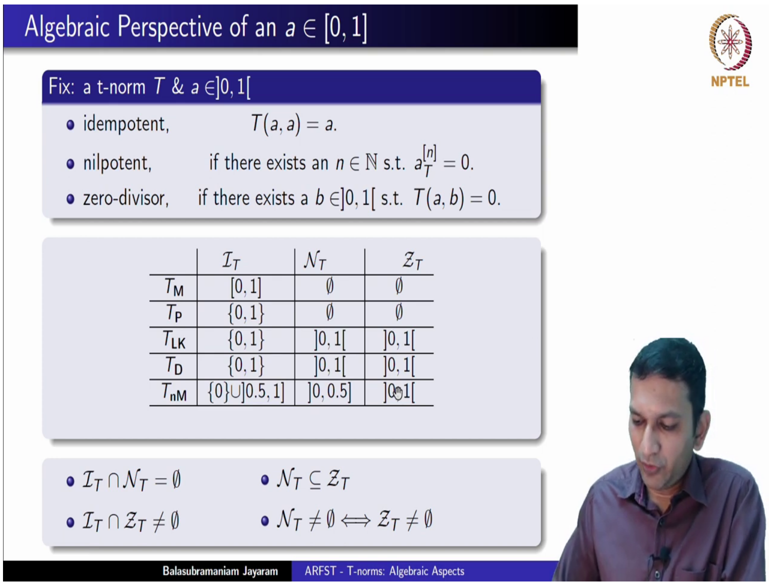

Now, some interesting observations can be made. If you look at the intersection between the set of all idempotent elements which you have denoted the I T and set of all nilpotent minimum nilpotent elements which we are denoting the N T. What about this intersection? Clearly, it is empty because if an element is idempotent, it cannot be nilpotent. If it is an idempotent, how many time times you iterate with itself it is not going to move on edge, it stays there. It cannot go to 0.

So, an idempotent element is never a nilpotent element. However, if you consider the intersection with the set of zero divisors of the T, then we see that it can be non-empty. For example, you see here this is for the nilpotent minimum we see that 0.6 is in fact, an idempotent element. However, 0.6 is also a zero divisor.

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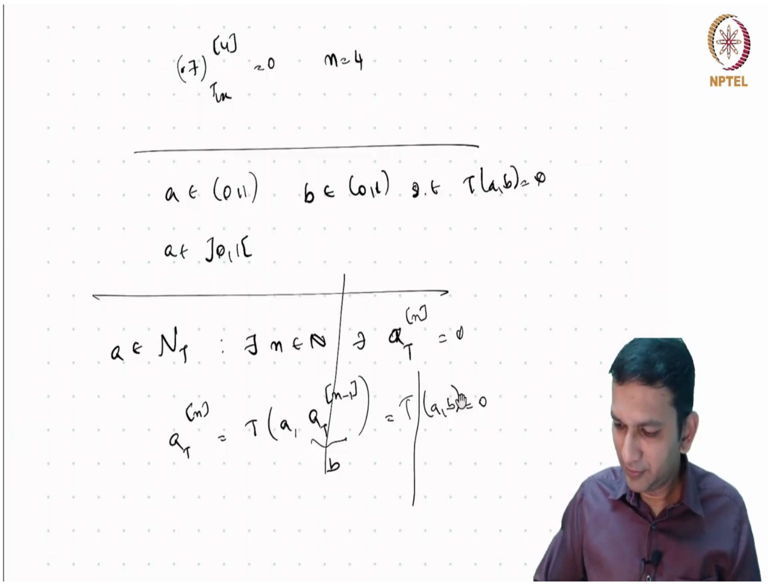

This can be seen like this. So, consider so, we have the definition of nilpotent minimum here. So, now, 0.6 we have taken, we know it is an idempotent minimum. But, it also can act as a zero divisor for whom? Say for 0.3. Look at what is T of 0.6, 0.3. This is 0.6 plus 0.3 is less than or equal to 1.

So, it is 0 so; that means, 0.3 is a zero divisor to 0.6 and so, is 0.6 divisor zero divisor 0.6. So, an idempotent element can also be a zero divisor. Now, what about these two sets right, the set of nilpotent elements of T and the zero divisors? It can be easily seen that every nilpotent element is also a zero divisor.

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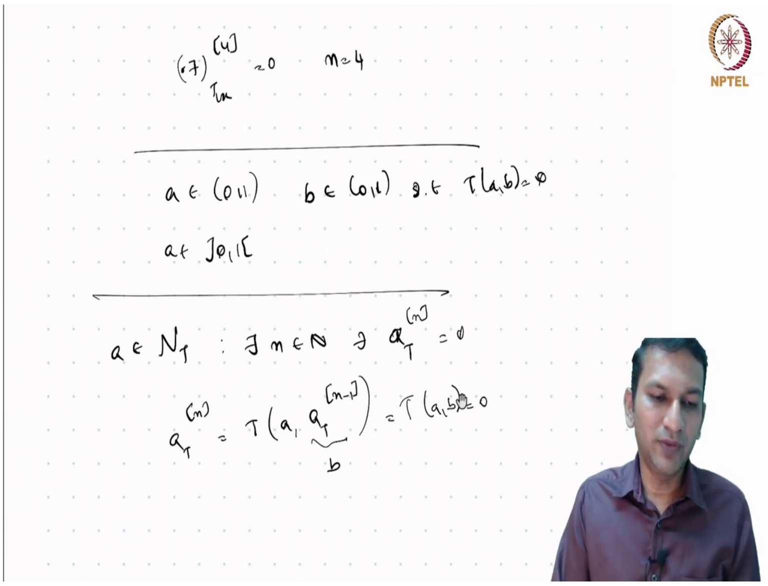

Now, how do you see this? So, if an a belongs to N T; that means, there exists a nilpotent element such that a T power n is equal to 0. Now, what is this a T power? You can write it as T of a, a T power n minus 1. Now, from this b; so, then what you have is T of a, b is 0. So, for this a there exists another b such that it is equal to 0. So, b becomes a zero divisor for a and so, thus a becomes the zero divisor for b.

So, every nilpotent element is also a zero divisor. Now, if you look at the final equivalent. So, if N T is not empty, in a set of nilpotent element is not empty; clearly Z T also cannot be empty. However, there refers the converse implication that is what is interesting. So, if the set of zero divisor is not empty, then it also showed that the set of nilpotent element is non-empty. How do we show this?

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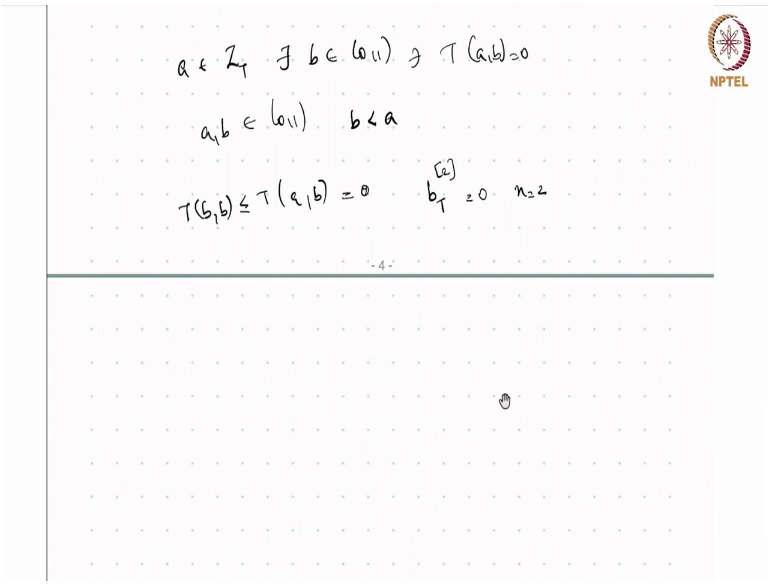

Once again let us look at this. So, now, let a belong to Z T. This means there exists a b element of open 0, 1, 1 such that T of a, b is equal to 0. Now, note that both a and b, they come from the 0, 1 interval, open 0, 1 interval; that means, either a is less than b or b is less than a. Let us assume for the moment that b is less than a. Now, what we know is that if you consider T of a, b; this is in fact, by monotonicity is greater than T of b, b.

But, this itself is 0. So, which means T of b, b is 0. But, what is T of b, b? b T of 2 is 0. So, there exists an n in this case actually it is 2 such that b T of 2 is 0 which means b becomes a nilpotent element. So, if but; however, note that not every zero divisor is also a nilpotent element. For example, in the case of nilpotent is minimum we saw that 0.6 is a zero divisor. It belongs to Z_T, but 0.6 also belongs to I T, idempotent element. And, it is an idempotent element, it cannot be a nilpotent element.

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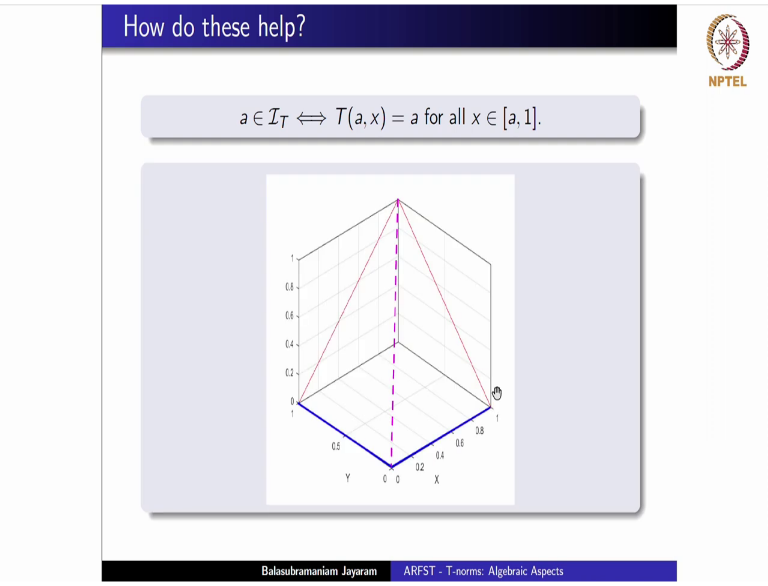

Well, we have introduced these three concepts of idempotents, nilpotentents and zero divisor; how do these help us? Let us look at this, if an element is an idempotent element of a T-norm; what do we know? We know that for every x greater than or equal to a and T of a, x is a.

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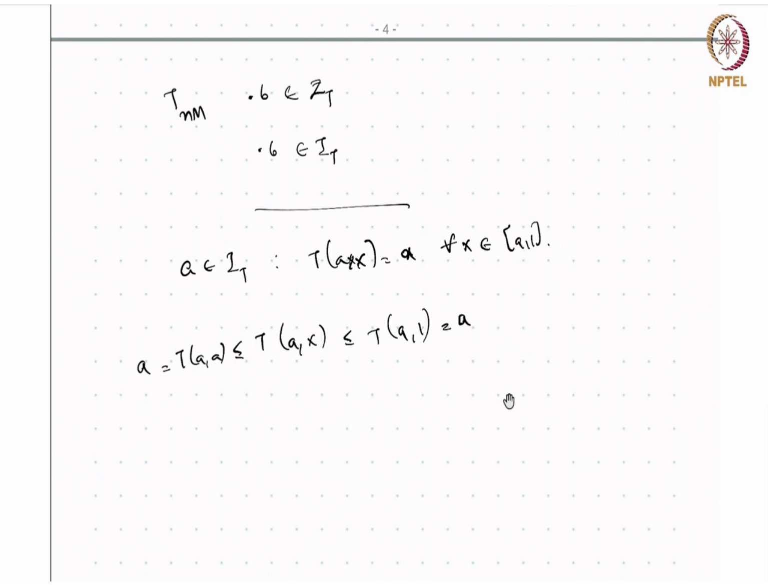

Now, once again this can be easily proven by monotonicity. What do we have? We have a element of I T and what we need to prove is T of a, x is equal to a for all x in (Refer Time: 21:01). So, if you take any x, x is greater than or equal to a. So that means, this is greater than

T of a, a, but we know the monotonocity this is smaller than T of a, 1. This is a because 1 is the identity and this is a because it is idempotent element; that means, T of a is a.

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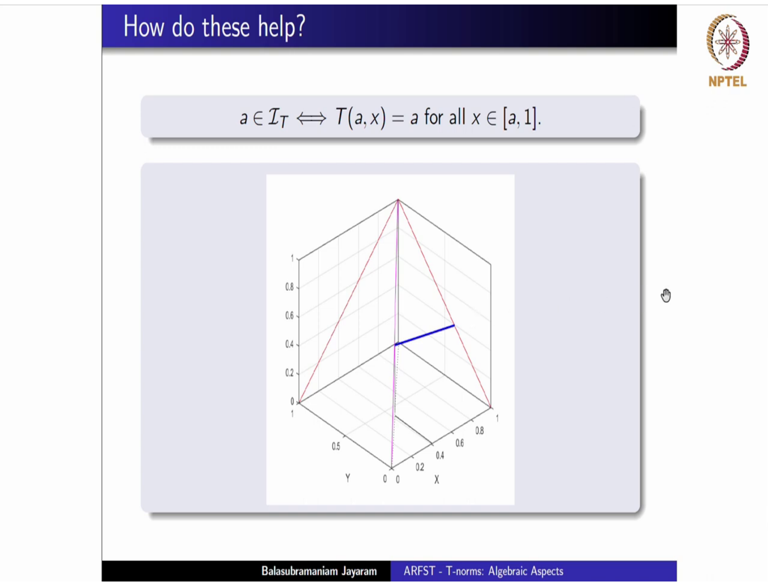

Now, other than the algebraic manipulation that we have done and shown this; what does it mean geometrically? Look at this, this is a skeletal framework of a T; that means, the line the dotted line magenta is actually the diagonal of the cube. We know that if a point is idempotent, then it actually falls on this diagonal of the cube. So, let us assume that 0.4 is one such a point, its idempotent for a particular T that we are considering. What it means is at T of 0.4, 0.4 it is 0.4.

Now, T of 1, 0.4 is also 0.4 and what we see that is because of monotonicity it has to be constant flow. Now, this allows us to imagine the graph of a t-norm at the point where at the point which is an idempotent element.

So, it allows you to visualize that t-norm quite effectively. And, if you want to sketch the graph of a particular t-norm making use of these things, making use of elements which are either idempotent or zero divisor you will have a good idea of how the graph would look like. Often this is very important when you are choosing t-norms in a particular context.

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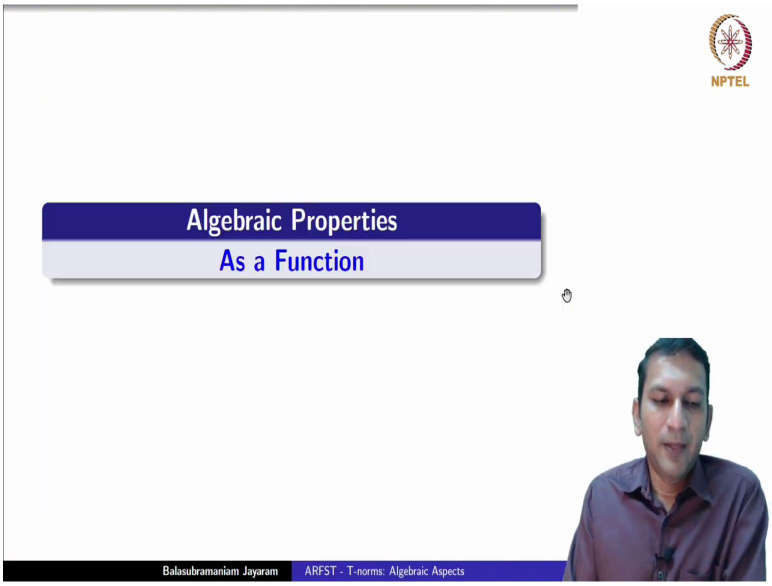

Well, so far we have seen algebraic properties of an element with respect to t-norm. Now, let us look at some of the algebraic properties that t-norms themselves enjoy or how to classify them based on the algebraic property they enjoy.

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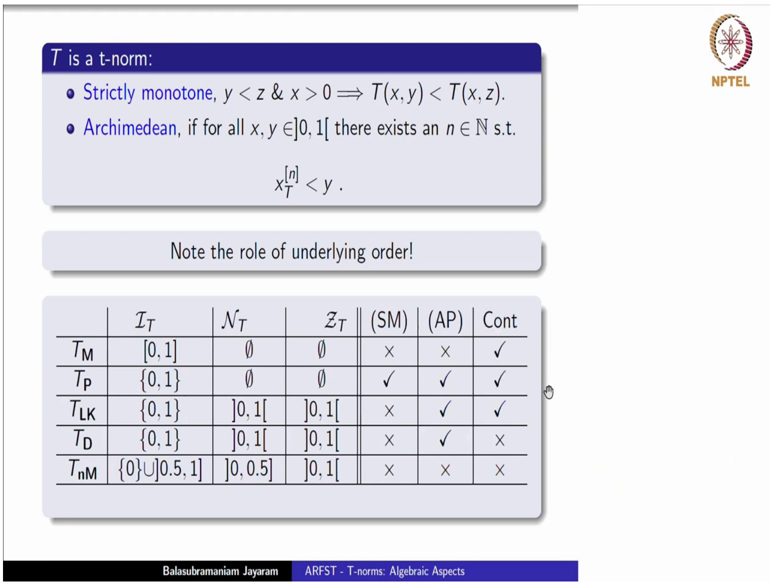

Given a t-norm, we say it is strictly monotone if we have an x which is greater than 0 and y less than z, then we should have that T of x, y is less than T of x, z. Essentially, it says that it should be strictly increasing other than on the boundary and x is 0, anyway it is a constant 0; everywhere else it should be strictly increasing. We can also talk about Archimedean elements. What it says is take any pair of elements x, y and open interval square 0, 1 square; there should exist an n such that after operating x with itself n number of times the value should become smaller than y.

We look at it presently. Once again we want to discuss this in the light of the five t-norms that we are considering. But, what we would like you to notice specifically is the role of the underlying order. For Archimedean, it is strict monotonicity. We see that the order of that is available on the 0, 1 interval is actually playing the role. Archimedean means we take the element and operate with respect to the t-norm iteratively. In the case of strict monotonicity which we are claiming that it should be compatible with the underlying order that is monotonicity. But, we also want it to be strictly monotone.

Now, let us consider these five t-norms in terms of strictly monotonicity. Once again we will go for some visual cue to help us identifying whether these t-norms are strictly monotonic or not. Look at these t-norms; the Lukasiewicz, nilpotent minimum and the drastic t-norm. Strict monotonicity we understand, means the graph cannot be constant anywhere other than on the 0, 1 that is where x is 0 or y is 0.

And, you see here while in this region it is strict, we see that there is a constant region here, similarly here and similarly here. So that means, these three t-norms are not strictly monotone. What about minimum on product? Well, they are 0 only on the boundary, but if you look at minimum because every element is idempotent, from that point onwards it actually becomes a constant. So, it increases say 0.4 as an idempotent element, it increases till 0.4.

But, as we have seen the few slides earlier from 0.4 to 1, T of 0.4, x when x comes from 0.4 to 1 is actually a constant at 0.4. So, this is also not strictly monotone. However, we know product we take two elements which are not which are distinct same, we always get some values different from both the operands; unless one of them is 0 or x is equal to 1. So, among the five t-norms what we see is only the product t-norm is actually strictly monotone. What about the Archimedean property, how do we understand this?

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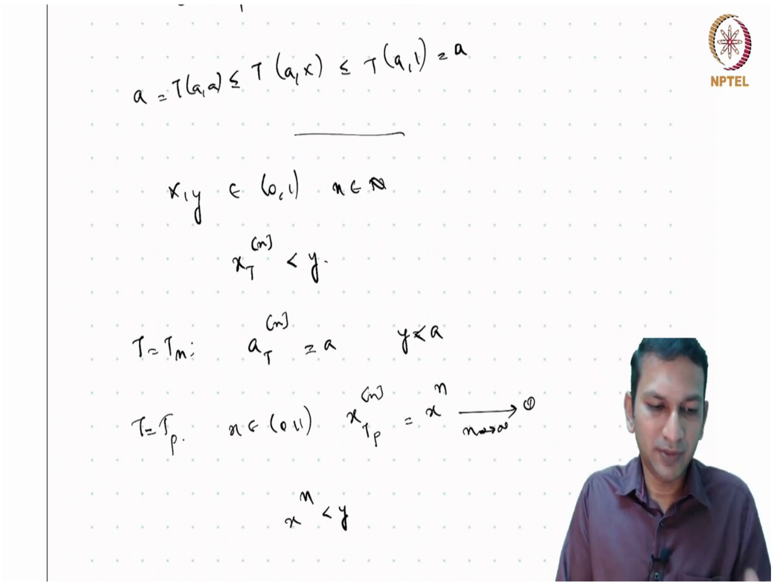

So, now, we are getting two elements x and y in the open interval 0, 1. What we want to do is if you want to say that the t-norm enjoys Archimedean property for any arbitrary x, y. We should be able to find an n such that x when operated with itself n times should becomes smaller than y. Now, let us look at the different t-norms that we have. In the case of minimum, we see that this is not going to be possible because, any a if you take a T of n will actually be a.

So, if you take your y to be smaller than a, then this is not going to happen. So, the minimum t-norm does not satisfy or enjoy the Archimedean property. What about the product t-norm? Now, if you take n x in open interval, then we know that x T of n with respect to product is nothing, but x for n. We see that in the limiting case as n tends to infinity, it does go to 0. So, now, no matter which y you give, you will be able to find some n such that x power n is smaller than y.

So, this is basically coming from convergence of this sequence. Now, let us see whether we can take some visual cue for this. These are the minimum nilpotent minimum t-norms. You see that they have an idempotent element. The moment they have an idempotent element, immediately we see that it cannot enjoy the Archimedean property. Because, iterating the idempotent element with itself does not lead you to some other value.

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So, these two t-norms are not are not do not enjoy the Archimedean property. However, we can show in the case of drastic t-norm, any two points you take T of x is equal to 0. So, clearly you see that given x you are going to fall to 0. So, is the case with product and we have seen that every element in the Lukasiewicz for Lukasiewicz minimum is a nilpotent element.

That means after finitely minimum or countably finite minimum for a particular name that we can choose fixed. So, in that sense we could say it is also finite x T power n actually is 0 which means again for any y that you will get, we can find an answer that x T power n is less than 1. So, based on this we say that among these five t-norms, other than minimum nilpotent minimum other three do enjoy the Archimedean property.

Now, this perhaps the first time that we are seeing more blue than red. Let us also add continuity into this t-norm and you will see that the first three are continuously norms, the last two are not. We have seen that many times before. Now, it is interesting to note that if it is put strict an Archimedean, we seem to see that it is also continuous. But, is it really true, can we generalize this from just one example?

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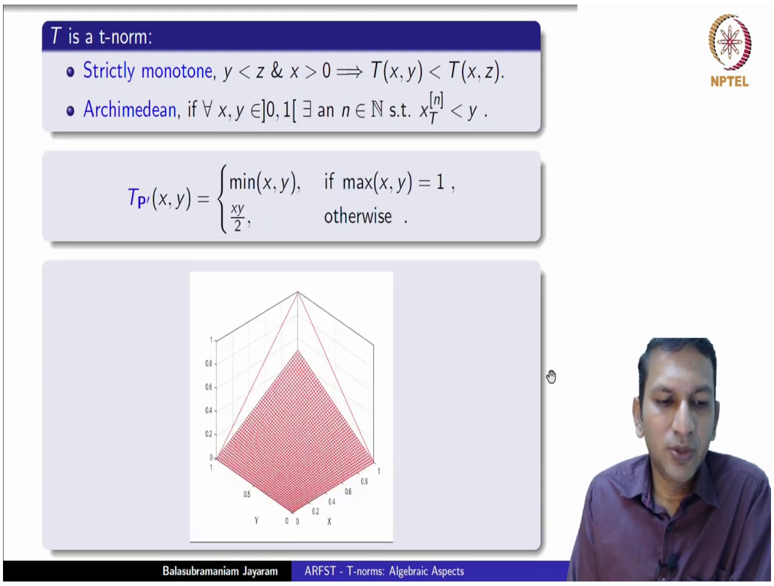

Let us look at another example. This essentially product t-norm half by half. So, this is how the graph looks like. It is clear now that this is strictly monotone, no where it is constant. However, it is not continuous and also we see that it will enjoy the Archimedean property because, it essentially goes to 0.

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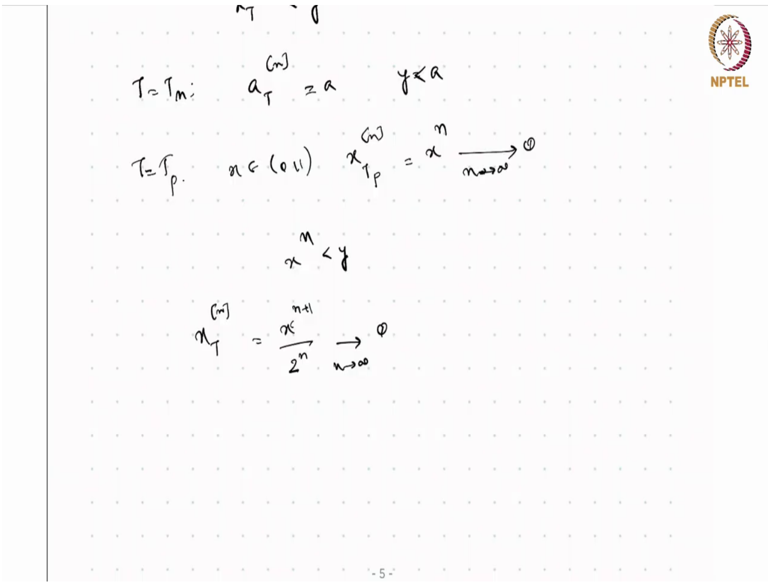

For instance, if you iterate for this t-norm, take an x and iterate x T power n will be essentially x n plus 1 by 2 power n, it keeps on 1. So, clearly this goes to 0 as n tends to infinity which means that even anyway we will always be able to find the name such that x T power n is smaller than y. So, this is a t-norm which is both strictly monotone and satisfies the Archimedean property, but it is not continuous. Once again you will see that it does not have any non-trivial idempotent elements and once again there are no nilpotent elements and, there are no zero divisor also.

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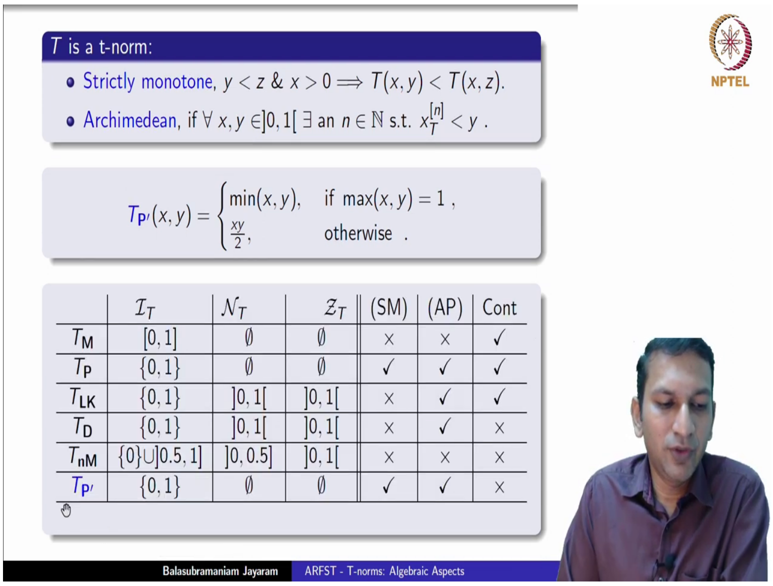

So, if you look at introduce this t-norm also into the table, this is how it look like. Well, why these two concepts of strictly monotone and Archimedean's?

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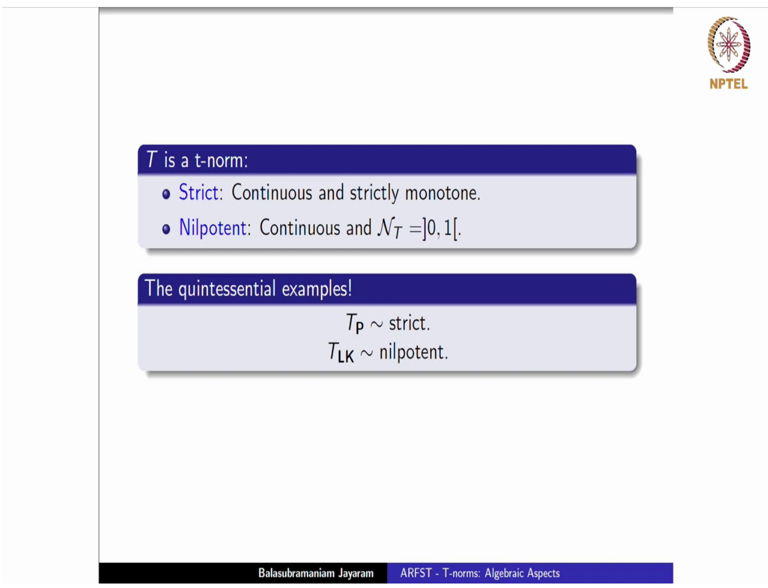

It will become clear soon enough when we look into the applications also soon. But, for now let us look at two particular classes of t-norms. We call a t-norm to be strict, if it is both strictly monotone and continuous. We understand when a t-norms continuous, a usual definition of continuity is here. And, we see a t-norm is nilpotent if it is both continuous and the set of nilpotent elements is entire 0, 1 interval; that means, every element is nilpotent.

And, among the t-norms that we have seen, we see we saw that the product t-norm is both continuous and strictly monotone. Similarly, the nilpotent t-norm an example of that is the Lukasiewicz t-norm which is both continuous and every element is an nilpotent element. In fact, what is interesting is these are essentially the prototypical examples for this class or these classes of t-norms; that means, it can be shown that every strict t-norm is somehow related to product and every nilpotent t-norm is related to Lukasiewicz t-norm. We will see this perhaps in one of the next following lectures.

Now, this we have so far what we have seen is algebraic properties of elements and its properties and the function of the itself; algebraic properties and the function of (Refer Time: 31:57).

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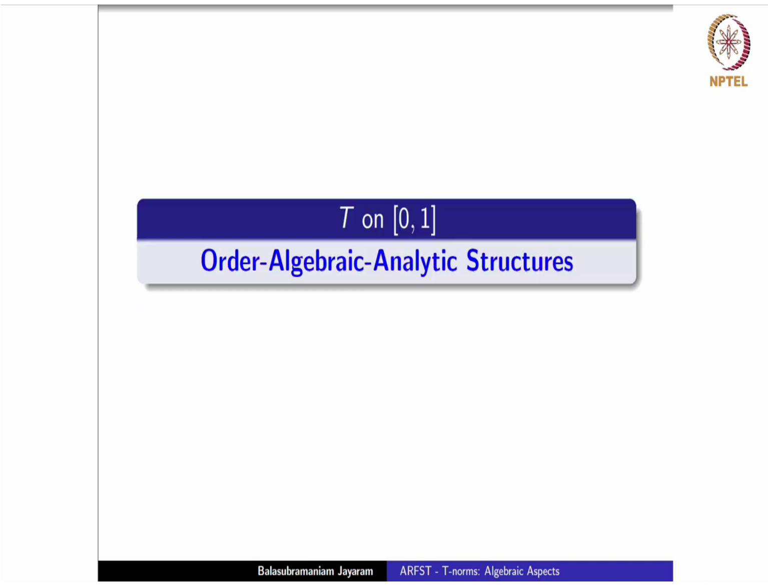

Now, let us do something more. Let us take T which is a binary operation of 0, 1 and look at it from order algebraic and analytic perspectives. Because it has continuity, it has some order underneath and it has some nice algebraic properties also.

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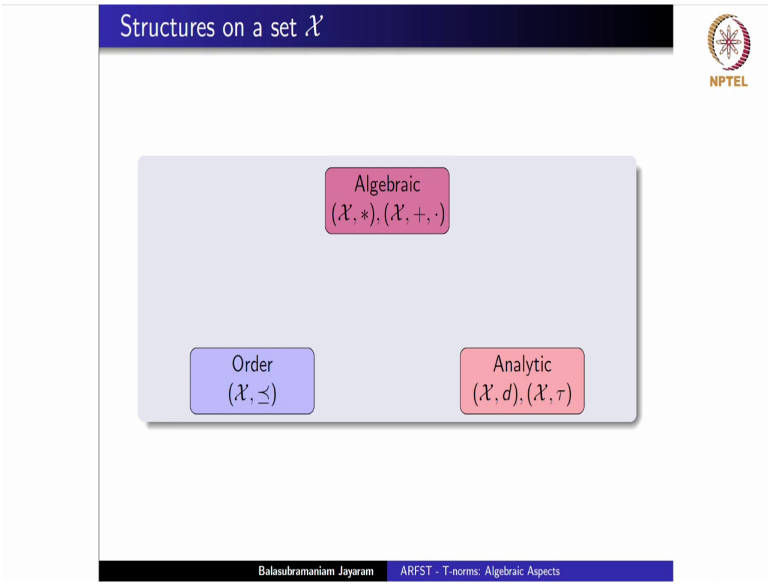

Now, if you look at the kind of structure that you can have on the syntax, you could have an algebraic structure like groups, vector spaces, you could have an order theoretic structure. We have seen four sets, lattices, chains, different special types of lattices and on the same set X you could also have some analytical structures like the or metric spaces or topological spaces.

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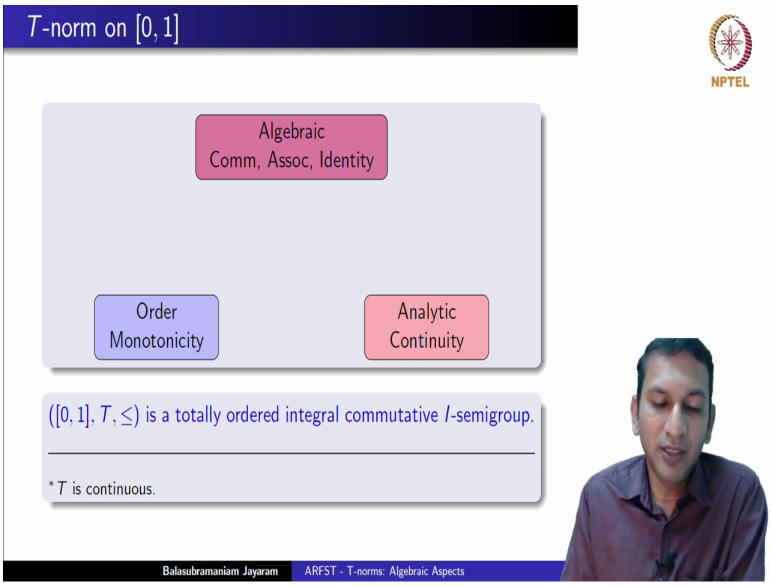

Now, what we see here is for a t-norm if you look at it in terms of the properties that we can enjoy, there are some algebraic properties like commutativity, associativity and identity. There are other properties which is essentially making use of the underlying order on [0, 1] and monotonously come from there and it also can be made continuous. So, you can also consider continuous demand. So, now, let us look at how these three properties come together and show what a t-norm can do to the unit interval, what kind of structures it can impose on the unit interval [0, 1].

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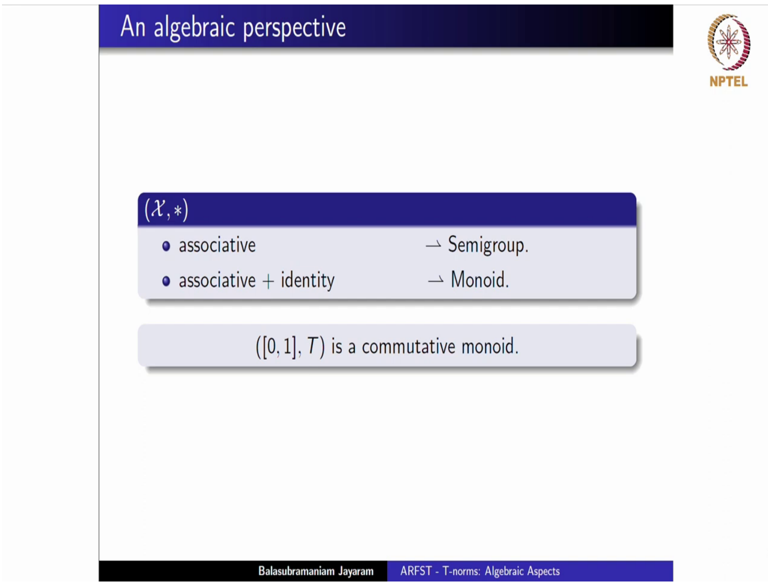

Given a set X and a binary operation which is closed on it; that means, star is to next cross X to X, if it is also associative we call it the semigroup. If it also has an identity we call it a moniod. And, what we know is if you take the set 0, 1 and T consider a t-norm as just a binary operation; we see that based on its properties it is actually a commutative monoid, associative commutative and an identity of the cases.

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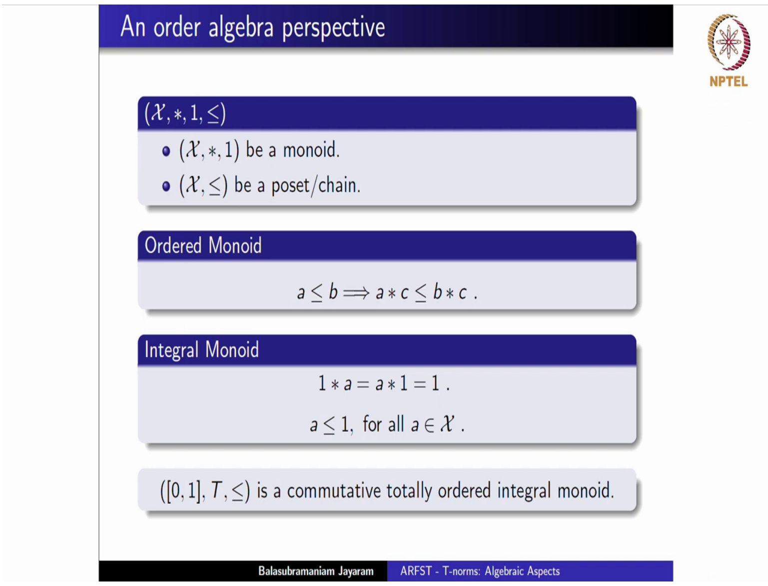

Note, that if you have a set with a binary operation, closed binary operation on and a special element 1 and also an ordering on the underlying order, such that this X star 1 is a monoid and with respect to this ordering it poset, poset or a chain would happen. Now, we can define what is an ordered monoid. It is nothing, but the operation the binary operation star respects the underlying order. It is compatible with the existing order on the set X. So, a less than or equal to b implies a star c is less than or equal to b star c.

We know that immediately T satisfies this with respect to the usual order on 0. What is an integral monoid? Since, it is a monoid, it has an identity both left and right. If this identity is also the top element, then we call it an integral moment that is the identity should be the maximum element. Every other element should be smaller than that with respect to the order that you have on the surface.

Now, clearly you see that if you consider 0, 1 with the usual order on it and any t-norm, it is a commutative totally order plus 0, 1 with respect to usual order is totally ordered. This structure is a commutative totally ordered integral monoid.

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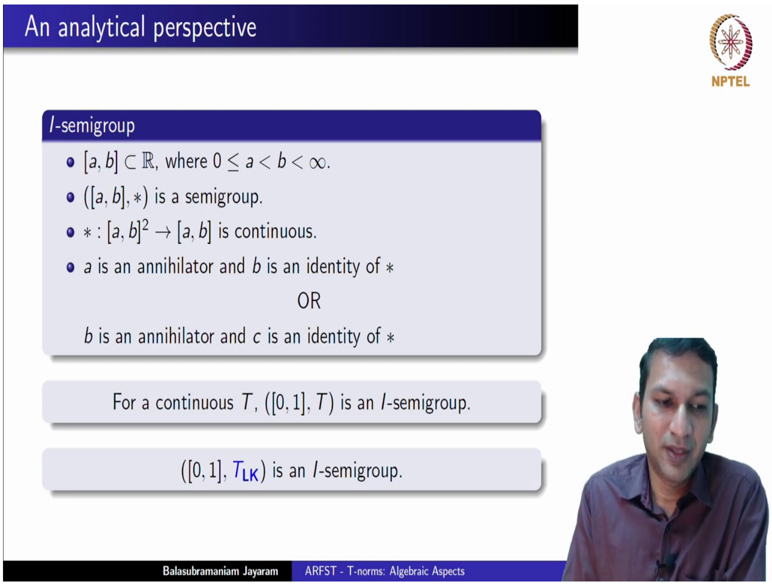

Now, let us throw in some analytical properties. Consider a closed sub interval of R finite and consider an operation on that which makes it a simple. Also, since on R we have the usual metric in topology, let this the star be continuous with respect to that topology or metric. Further, in the structure let the endpoints be such that one of the endpoints is an annihilator and the other is an identity. So, either a is an annihilator and b is an identity or b is an annihilator and a is an identity of star. What do we understand by annihilator?

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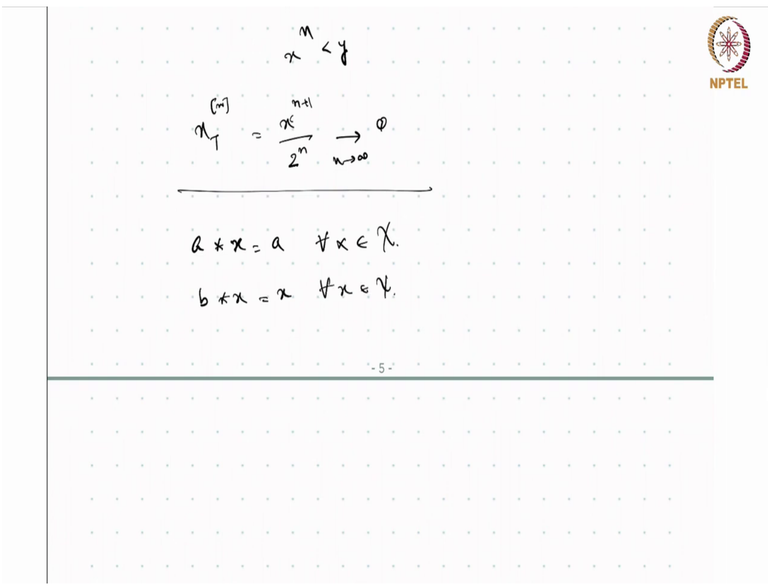

So, annihilator means if you have a, we say a is an annihilator if a star x is equal to a from all x coming from that set X; by identity we understand b is an identity, b star x is actually equal to x (Refer Time: 36:22). So, now, what do we want? If you consider a continuous T on the 0, 1 interval; we consider T as a binary operation of continuous T. We immediately see know or see it as a I semi group.

Now, comes in the familiar Lukasiewicz conjunction. So, since Lukasiewicz conjunction is continuous, what me see is on 0, 1 it forms an I semigroup. You may remember that Lukasiewicz conjunction in its dual disjunction, the pair came very close to giving us a lattice. But, now they are happily giving us an I semigroup. It is not only that, if you put all these properties together what we get is this very interesting result.

If you consider 0, 1 with the usual order and the continuous T, it becomes a totally ordered integral commutative I semigroup. There is a nice very rich algebraic structure that you could have on the unit interval through a continuous t-norm.

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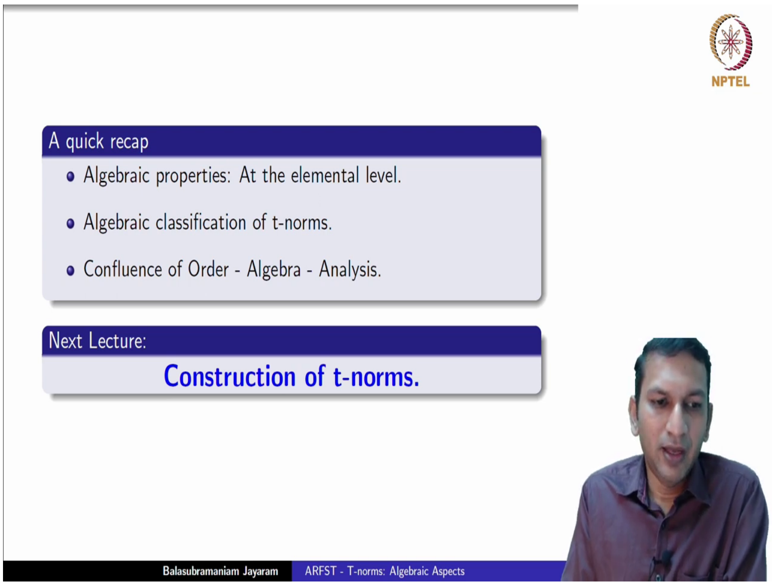

A quick recap of what we have seen today. We have looked at algebraic properties at elemental level and also at the functional level which allowed us to talk about at least two types of t-norms should be monotone and those which are Archimedean. We have seen how the confluence of order theoretic, algebraic analytic properties has given a nice characterization of t-norms on the set of; I think you would have done now.

What next? Using some of these results, they have come up with many constructions of t-norms. We will look at a few of them in the next lecture. They would definitely be useful for us when we deal with application of context. So, that would form the topic of our next lecture for this week and different types of constructions of t-norms.

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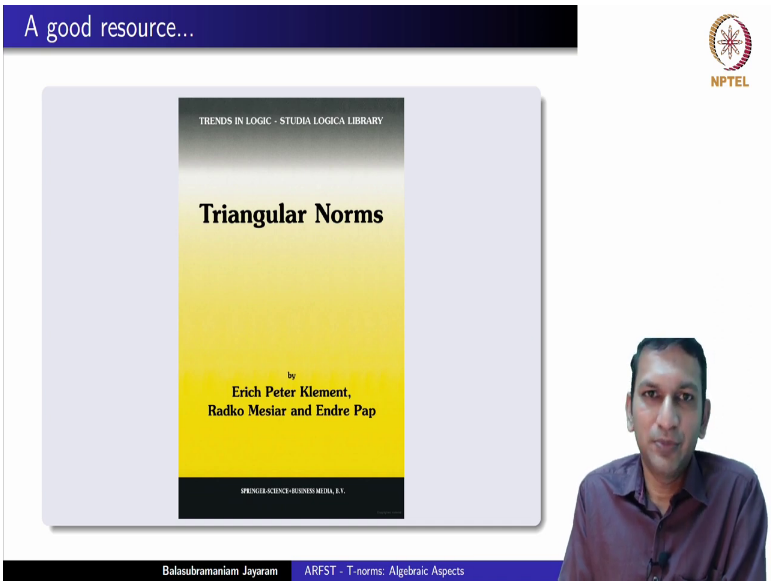

A good resource that you could look up to for the topics that are covered in this lecture is the book by Klement, Mesiar and Pap. So, thank you once again for joining me in this lecture and hope to see you again in the next lecture.

Thank you.