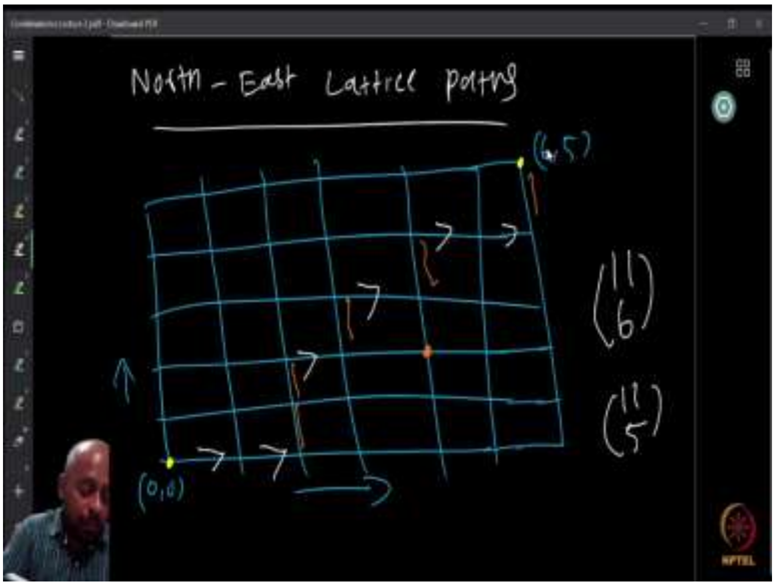
Combinatorics
Professor Doctor Narayanan N
Department of Mathematics
Indian Institute of Technology Madras
Counting Lattice Paths

(Refer Slide Time: 00:14)



So again, we continue the basic counting principles, more examples. So, today, we are going to look at a very interesting question to start with, it is called the North-East Lattice Path. This is going to be a very useful idea. And it could help you come up with many different proofs in nice ways, many combinatorial proofs for several of the things that we are going to look, so this is as follows.

So, the one question that we want to look at is as follows. You have this grid. On this grid, you are allowed to move from the origin, let us say (0, 0) is called the origin, you are only allowed to move either to the right or upwards from wherever you are standing.

Now, you are standing, let us say here. Here you are not allowed to go down or you are not allowed to go left, these two are not allowed. So, at any position, you can go either to your right or upwards. So, this is all North-East lattice path because we consider the upward movement as north movements and right movement as east movement, in a map usually that is the convention that we follow. So, this is called North-East lattice path. Now, the question is that, suppose I start from (0,0). And then I want to go, let us say 1, 2, 3, 4, 5, 6, and then I want to go 1, 2, 3, 4, 5 up, so I want to go to that (6,5) grid point.

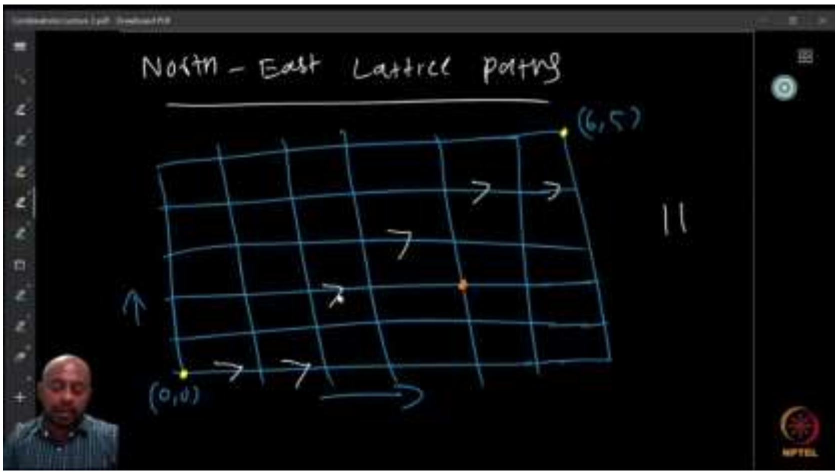
So, you can assume it is an infinite grid, but we are only going from (0,0) to (6,5). Now, how many different ways you can go to (6,5) starting from (0,0)? That is the question. Can you solve this? Can you count the number of ways you can go from (0,0) to (6,5) using only north or east movements from the grid? So, think about it for a few minutes and then continue with the lecture.

So, let me give you how to count this. So, how many steps we need to reach (6,5). So, we definitely need to take 11 steps to reach (6,5), because I definitely need to make 6 right movements before I can reach the column number 6, and I need to make 5 top movements before I can reach the y coordinate with 5, so the row number 5.

So, to reach (6,5), I definitely need to make exactly 6 right movements and 5 top movements or north movements. And I do not have any other, possibility because if I go one more left, there is no left movement allowed or one more up, I cannot come down because there is no down movement allowed. Therefore, I have to only move right or left and since I have to take exactly 6 right steps and 5 north steps. I know that I have exactly 11 movements, no matter which route I am going to take, I am going to make exactly 11 movements.

Now, I have 11 possible movements. Now out of these 11 possible movements, 6 of them must be east movements or right movements and 5 of them must be up movements or north movements. But suppose I decide which are the right movements, I am going to make. I will say that, here I am going to take a right movement, here I am going to take a right movement, then we will say that, I am going to make a right movement here and then a right movement here and the right movement here. Then maybe the right movement here.

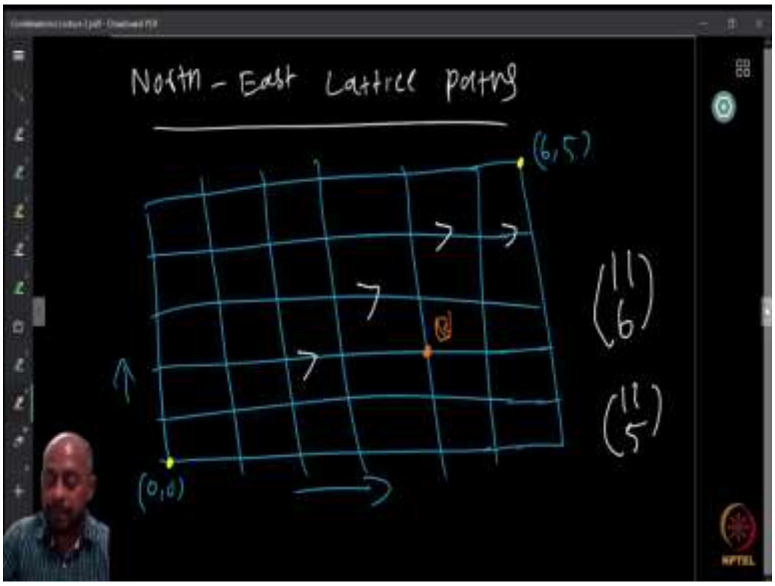
(Refer Slide Time: 5:44)



So, once I decide which are precisely the 6 right movements I am going to take, the other up movements are pre decided and there is no choice, because if I want to make the movement here, then I know that I must have moved up here and also here, there is no choice I also must have moved up here, up here and up here. These are without any choice. Therefore, the moment I decide the 6 right movements, my 5 north movements are decided.

So, therefore, I just need to decide the 6 choices out of the 11 possible movement options. So, $\binom{11}{6}$, possibilities are there. Or instead, I can choose the up movements, the 5 possible ways $\binom{11}{5}$, both of them will give the same number because we already know that $\binom{11}{5} = \binom{11}{6}$. Therefore, I have $\binom{11}{6}$, or $\binom{11}{5}$, possible north eastern lattice route, from $(0,0)$ to $(6,5)$, you can replace $(6,5)$ with any number and you will get the arbitrary (m, n), lattice path. This $\binom{m+n}{m}$ or $\binom{m+n}{n}$.

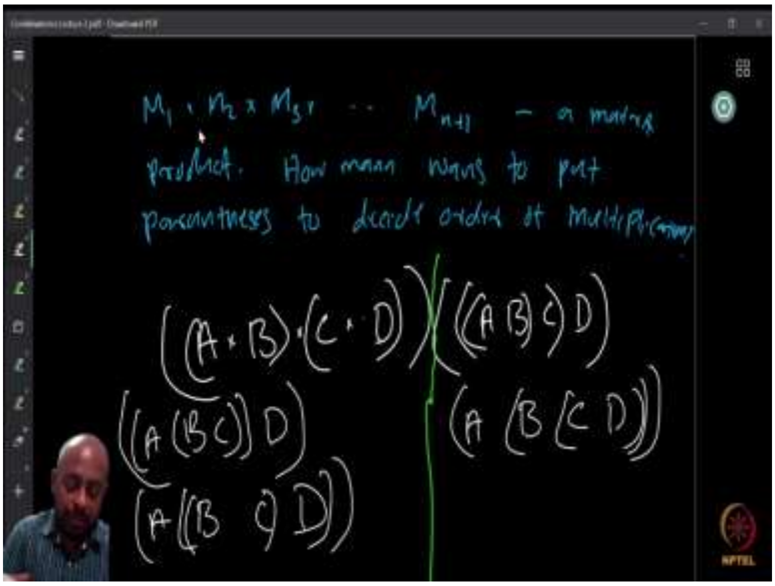
(Refer Slide Time: 7:12)



Now using this now, we can easily add some further questions. For example, I will say that, instead of these I am going to need the paths I can make starting from $(0,0)$ going to this particular point, so maybe these paths are representing some one-way streets in some weird city, where you can only move north and east movements. And, there is happened to be a very famous bakery at this position B, so I want to, I start from my office at $(0,0)$ and going back home, but then I am going to visit this bakery to buy some stuff from there. No, it is not healthy, maybe you do not want bakery, you want something else, some vegetable shop maybe we want to buy some fruits or vegetables from here and then I want to go home. So, then I want to say that, I want to stop from my office at this place.

Now, how many ways I can go from here to home, if I am guaranteed that I will stop at this particular shop? So, try to come up with an answer by using the previous methods that we know and the one that we just followed.

(Refer Slide Time: 08:43)



Now, here is a question that, algorithm people will be very interested in, for example, like you have a matrix product, let us say $n+1$ matrices are there M_1, M_2, M_3, …, M_{n+1} these matrices are to be multiplied as $M_1 \times M_2 \times \ldots \times M_{n+1}$. Now, this matrix multiplication, of course, it is not commutative. So, I cannot arbitrarily change the order of these things.

So, I have to keep the order of the matrices. But now, when I do the multiplication. I can only multiply two matrices at a time I cannot multiply all of them together. I have to multiply two of them, then I can multiply this multiplied number with something else in the next one in that order whatever or I can insert like I say that I multiply M_1 and M_2 first, then I multiply M_3 and M_4. Then I multiply $M_1 M_2$ with $M_3 M_4$.

That is possible or maybe, I can just multiply M_1 and M_2 first, find the answer then multiply with that with M_3, then find the answer, multiply the product of these three, now with M_4, so there are several ways I can do this product to find the same product. And some of them may be more efficient than the other depending on the type of matrices that we are looking at. But, without looking at that, we want to say how many different ways I can do the matrix multiplication. So, find the number of ways to parentheses this product, so that I can find different ways of multiplying.

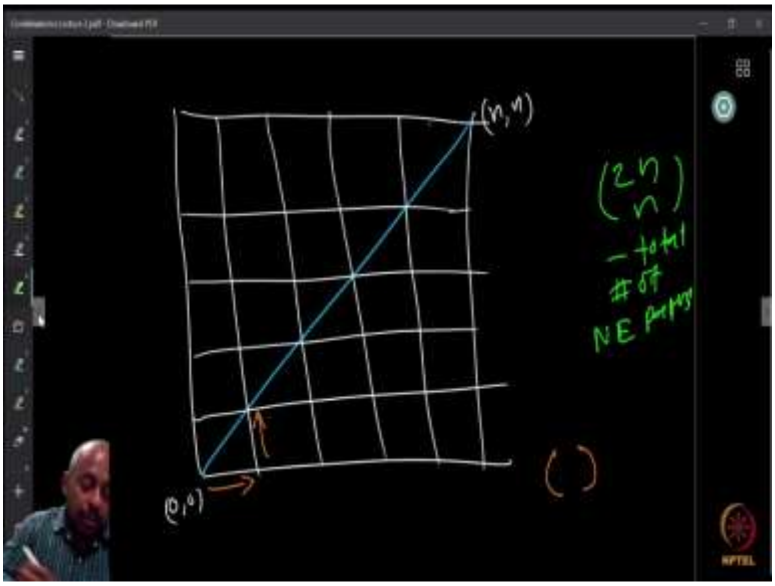
For example, let us say that you have four matrices A, B, C, and D. So, if I want to multiply A, B, C, and D, I can multiply A and B first, then I multiply C and D, then I multiply the product AB, and CD. That is ((AB)(CD)), this is how I give the bracket. Then I can also say that A, B, C, D, I take, I multiply A and B first, then I multiply with the C, then I multiply with D. That is (((AB)C)D). So, this is the second possibility, I want to just separate them out, so that does not look like the same thing, we have just extended here.

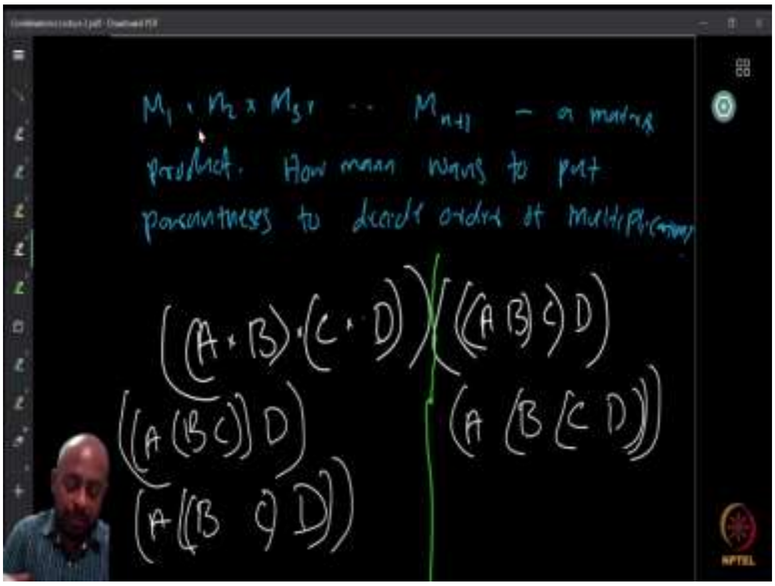
So, then what are the other possibilities? Well, A, B, C, D, I have, then I multiply let us say B and C first, and then I multiply with A with B and C, then I multiply this product with D. That is ((A(BC))D). This is another possibility. Or I can multiply C and D first, then I multiply it with B, the I multiply with A. That is (A(B(CD))). This is another possibility. Any other possibility? I multiply, let us say A and, what is the other way. There is another way, maybe I multiply with, so I multiply B and C first, then I multiply it with the D, then I multiply with A. (A((BC)D). So, this gives a different order.

So, here, I multiply A and B, C and D separately, then multiply them together here I multiply A and B multiplied with C product and then multiply that product with D, here I do B,C first then A with BC and then that product with the D, here I do B, C first then BC with D first and then A with the product, and here I do C, D first, then B with C, D and then A with B, C, D.

Now, I have found out five different ways. Can you find more? Maybe there are more, I do not know. So, you figure out if there is any more or if it is the exhaustive list. Now, but I wanted to find out for general $n + 1$ matrices or whatever number, I just put $n + 1$ for some reason, but that is not really important.

(Refer Slide Time: 13:48)





Now, the interesting thing is that to find out this, we can use the idea of the lattice path that we are looking at the North-Eastern lattice path. So, if you have $n + 1$ matrices, then you are going to put n pairs of brackets that we saw. So, like we have four things, which is $3 + 1$. So, I put 3 pairs of brackets for the product. So, I put two together. So, therefore, we need one less.

Because there is going to be exactly whatever the number of terms minus 1 products happening. Number of products is exactly one less so therefore, that is why, you have one less bracketing. So, we are going to put parentheses now, n parentheses to define this product, but that is precisely saying that how many different ways you can put parentheses, so that you always have the number of parentheses on the left more than the number of parentheses on the right because if otherwise we are going to be in trouble, because when you want to balance it out we want to go from the left.

So, we want to find the balanced parentheses and count how many of them are possible if there are going to be n parentheses. So, we want to find the total number of balanced parentheses, where there are n pairs of them. Now, finding this is going to be reducible to counting a type of path from origin to (n, n), you want to find a n pairs of brackets and find the number of balances that is possible with this it is basically counting the number of paths from $(0,0)$ to (n, n) with some property.

Now, can you think of what is the property that we are looking at? We want to look at the path the North-Eastern lattice path starting from (0, 0) to (n, n). But we want to make sure that this will give the total number of balanced parentheses. So again, stop and think about it for a few minutes before you look at the answer. How to do this? So, here is how it is going to be. So, I want to start from the $(0,0)$ and I want to produce path which reaches (n, n).

Now, let us say that my right movement is going to denote a left parenthesis, the opening bracket and an upward movement denotes a right parenthesis, the closing bracket.

Now, when I start from (0,0), I have to make a movement, but now whenever I put a parenthesis I have to start with an opening parenthesis before I can put a closing parenthesis. So, only after I move a right movement, I can go upwards, now, can I go again upwards from here, if I go upwards from here, what happens is that I am going to put another right parenthesis but then this is no more balanced.

It cannot be balanced because if you are going just to the right, you can never add things to the right. You cannot know if you make a movement here and then here, what you are going to do is to do this and this, this is not going to make the parentheses balanced. So, therefore, I always need to make sure that the number of left parentheses is never less than the number of right parentheses that has appeared so far.

Or in other words, when we are making the movements in the lattice, movements in the lattice we cannot cross the main diagonal. From (0, 0) to (n, n), there is a diagonal because if you are going to cross that diagonal that means that we have made more upward movements than the number of rightward movements.

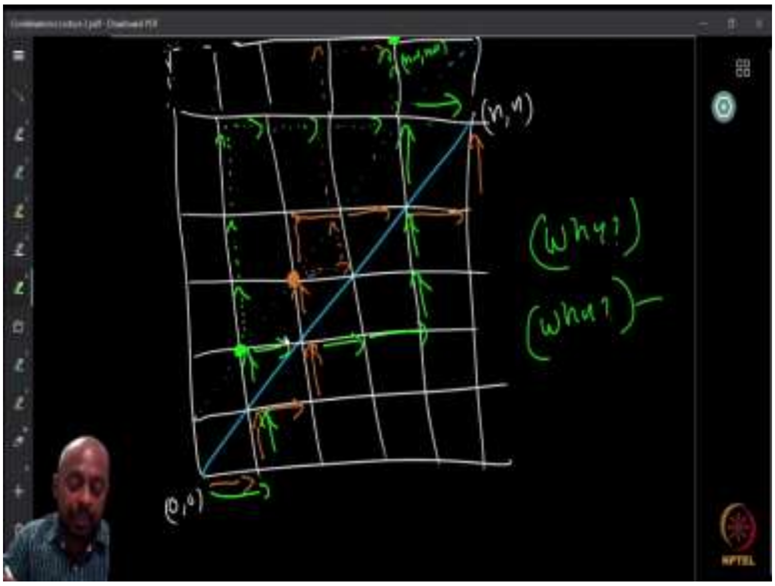
So, therefore, that will say that the number of right parentheses is more than the number of left parentheses, that is used so far, which means that the parentheses is not going to be balanced. So, therefore, I want to find out roots from $(0,0)$ to (n, n) that does not cross the origin but only makes right and upward movements.

So, I want to count the number of such paths. But the question is can you count the number of paths from (0, 0) to (n, n)? So, we know that all such paths, we counted that is n plus n, which is $\binom{2n}{n}$. So, we know that total number of lattice paths without looking at the diagonal is $\binom{2n}{n}$. Total number of North-East paths.

But now we put the condition that it cannot cross the blue diagonal, the main diagram that we came up with, it must always lie below it. Not that it should not cross, it should be below it. Because the left parentheses must be more than the right parentheses before we come in, not to claim or is that at least as many as the right parenthesis.

So, how do you count now, the guys who does not go up? So, we want to find the guys who does not go above the diagonal. Can you count this? So, think about this for some time. In fact, I would even recommend that you think about it for even a half a day or spend some time thinking about this before you look at the solution. So, let me tell you what is the solution I can come up with. There are actually other ways to solve this.

(Refer Slide Time: 21:28)



And what I am going to do is to use our well-known subtraction principle which means that I am going to find the bad guys and remove them, I think that is easier to count. I am going to count the bad guys. What are the bad guys? The bad guys are the paths which crosses the diagonal. It goes above the diagonal. So, I want to count the North-Eastern lattice path which goes above the diagonal at its some point. So, suppose some path goes above the diagonal, then it must cross the diagonal at least at some point.

Now, the first time it crosses the diagonal, I marked that point as special. So, you take whatever your path is, your path may do like this. Or it might even go further. Then it might go right, again right, again maybe right, then go up. Maybe this is one of your paths. Or maybe, you started in another way, you started maybe right itself, then went up, and then went up again here. Then you went right, right, maybe you went right. Then you went up again here, up again here, maybe up again here, then you went right. But this path also crossed the diagonal here. So, these are the bad guys, we do not want such paths.

Now, can you count these kinds of bad paths in a different way? You might ask like, what is the difference in counting the one which is below and then the ones which just crossed ones. Now, there is a difference we will see. So, what I am going to do is a trick. So, the trick is the following. So, every bad path, which means the path that crosses the diagonal crosses it for the first time, somewhere along this path.

So, the movement it crosses I marked that point as special. Now, what I am going to do is that, I am going to reroute this path by doing the following. So, after the point, till the point it cross, I will not do anything. After the point cross this, exactly once the first time it crosses. Any path, the remaining path, I will look at the remaining path and say that, anytime it goes to the right, I move instead, I move upwards.

So, I am going to continue this path as follows. So, instead of going right, I am going to go up. The next movement is again right. So, I again, move up, again the next movement is right, so I again move up. Now, then the next movement is well it is going upward. So, therefore, I move right, again up therefore right. I will use the dot, just to show it is the virtual path that I am creating.

So, again three upward movements, so, therefore, I am going to make three right movements, then I am going to right again here. So, therefore, I make an upward movement. I will extend my infinite grid to this. So, once I made this movement, I have reached this point somewhere, whatever. What was that point? It is a point $(n+1, n-1)$.

So, I did this so, after it cross once, I decided wherever it was going right I am going to go up and wherever it is going up I am going to go right. Now, I do this for every path. So, the red path that we are looking at for example, it cross here, now since it cross here it was going up and then next movement was up, so instead I will go right.

Then, the next movement was right, so instead I will go up again the next was right, so I will go up, again the next was right, so I will go up and then the next was up. Therefore, I will go right. So, I happened to reach $(n+1, n-1)$ again. But my claim is that every path after this switching of the things will always reach this point, particular point.

So, all the bad paths I can do this idea that right makes left and the right movement becomes the upward movement and the upward movement becomes the right movement. So, this is what we are going to do is precisely looking at this diagonal. And we are going to reflect the path which is above this with respect to this diagonal, I am looking at the reflection of the path. So, this path was like this, so, now it has with respect to this diagonal its reflection is going to be like this.

Similarly, this path, its reflection is going to be like this. Now the claim is that every such reflection will come to the point $(n+1, n-1)$. Can you think why? So, why does the reflection always reach $(n+1, n-1)$? You need to answer this because if I give all the answers, there is no fun.

But it is easy, so to think about this, all the reflected paths will finally reach this point $(n+1, n-1)$. So we only reflected the bad paths. We are going to count the bad path. So, every bad path after the reflection reaches this point. Now, every path from (0,0) to $(n+1, n-1)$ on this grid. This sub grid I mean not sub of the original one, but this grid that we are looking at, so every path from (0,0) to $(n+1, n-1)$ on the grid, corresponds to a bad path after the reflection the other way.

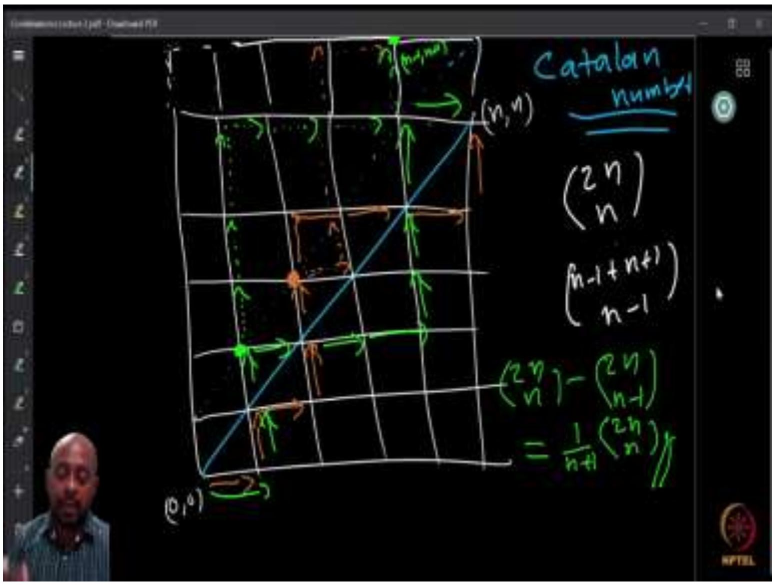
So, instead of doing this way reflection, I take the other way reflection, I mean the same rule but you take the reflection of the path going from (0,0) to $(n+1, n-1)$. They will all reach (n, n) and they will all cross the main diagonal at least once. This is something, you can observe and see and try to answer, why again.

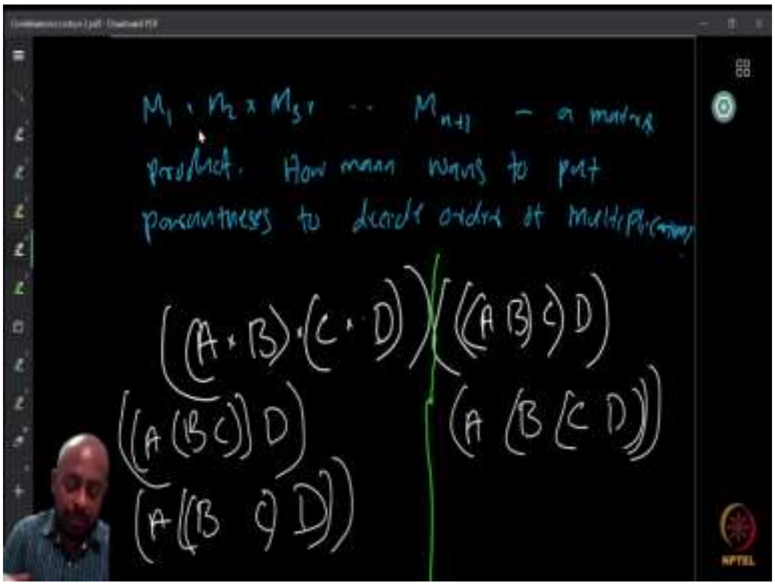
So, why the reflected path always reaches here and all the paths to $(n+1, n-1)$ reaches back to (n, n) and all such reflected paths have crossed the diagonal at least once. Now, once you have this observation, once you have this you can do the counting because, we are saying that, we look at all possible paths starting from (0,0) to (n, n).

And which are the paths which crosses the diagonal? They are the paths which after the reflection reaches $(n+1, n-1)$. And we said that there is a one-to-one correspondence between the paths that start from (0,0,) to $(n+1, n-1)$ and the bad paths from (0,0) to (n, n)

which crosses the diagonal. Because every such path gives a bad path here and every bad path always reaches here, so try to explain why this is true. Convince yourself, with this case and then we have the answer.

(Refer Slide Time: 31:36)





So, we have $\binom{2n}{n}$ possible total paths from (0,0) to (n, n) and then going from (0,0) to $(n+1, n-1)$, we have $\binom{n-1+n+1}{n-1}$. These are the bad paths. So, I can subtract, but this is going to be equal to $\binom{2n}{n} - \binom{2n}{n-1}$

And you can use either your algebra or some other nice arguments. Think about this whichever way you want to finally show that this quantity is going to be equal to again, $\frac{1}{n+1} \times \binom{2n}{n}$. So show this and we get the answer. And this number is very, very important. It has a special name, it comes as solution to so many combinatorial questions, there are even, books on this. So, it is called Catalan number.

So, if you look at Richard Stanley's book on enumerative combinatorics the updated edition has some four hundred or five hundred combinatorial questions to count things, whose answer is going to be the Catalan number, so many different areas, so many different things it comes from, now all of them finally goes down to the Catalan number.

So, it is called Catalan in the memory of mathematician called Catalan, Catalan number which says usually write with C capital, as it is a name. So, this is the number $\frac{1}{n+1} \times \binom{2n}{n}$. So, Catalan number we just found out is the number of ways to parentheses the $n+1$ matrix, product to do a multiplication.

However, it is a number of ways to do the balanced parentheses with n pairs of parentheses. And it is a number of paths starting from (0,0) to (n, n), which does not cross the main diagonal. It just stays below the diagonal. So, all these things are, of course, Catalan number already. Now we can find so many other things. So, think about some other situations where you can think of Catalan numbers.