Combinatorics
Professor Doctor Narayanan N
Department of Mathematics
Indian Institute of Technology Madras
Binomial Theorem and Bijective Counting

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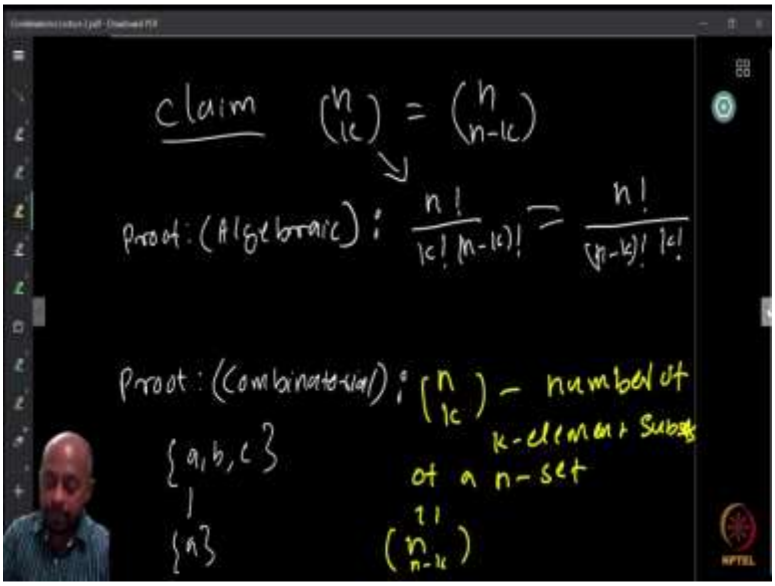

We already saw $\binom{n}{k}$ and its definition. It is $\frac{n!}{k!(n-k)!}$ in one way or we said that k-element subsets of an n-element set. Now, the claim is that there is some kind of symmetry with the binomial coefficient, $\binom{n}{k} = \binom{n}{n-k}$, for every k, k can be 0, 1 up to n.

Now, can you prove this? So, there is the standard proof you can use the algebraic proof very easly. How do you do that? $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. What is $\binom{n}{n-k}$? It is $\frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!}$. Therefore, $\binom{n}{k} = \binom{n}{n-k}$. So, that is a very easy proof.

But we do not want that. We do not want anything easy. The reason we do not want this is that this tells us nothing, nothing new that we do not know so far. It does not explain why they must be the same, it tells from the algebraic requirement; it must be the same by the definition. But intuitively, why this happens? It does not tell us. So, therefore, we look at combinatorial proofs.

So, in combinatorics, we look at combinatorial proofs, because that in many times, will give you other intuitions of why things must be the way it is. So, for that, we try to look at combinatorial proof. So, when I ask questions, sometimes I will say that give a combinatorial proof, you might have a direct algebraic proof, but we do not want that we will look at some other combinatorial arguments. Why the set that we are talking about, here that we are going to count, and the set that we are going to look at there, which we are also going to count must all have the same number of elements?

We are talking about two distinct sets, maybe, but they must have the same number of elements and what is the reason? This is what, we want to explain by using combinatorial argument. And that also tells us why they must have the same number of elements. So, here is a combinatorial argument for the same question.

Why $\binom{n}{k} = \binom{n}{n-k}$? Well, what is $\binom{n}{k}$? By definition $\binom{n}{k}$ is the number of k-element subsets of an n-element set, when I write n-set it is n-element set. Now given a set, let us say $\{a, b, c\}$ Suppose I want to form a subset $\{a\}$. The one thing I can do to form a subset is by taking this $\{a, b, c\}$ and then selecting the element a and form the subset $\{a\}$.

On the other hand, I can form the subset $\{a\}$ by choosing the other elements of the set, let us say b and c and removing that from the set $\{a, b, c\}$. So, if I want to choose a k-element subset, I can decide which k elements I want to choose. Or I can say that, I do not select these k elements, I select the $n - k$ elements that I do not want. How many ways I can choose? $\binom{n}{n-k}$ ways and throw them out.

So, I choose the $n - k$ elements I want and then throw them out that will give me a k- element set. And for any k- element set, I can choose $n - k$ -element sets, which I do not want, throw away those guys, then I get a k-element set. So, choosing the $n - k$ -element subset, is precisely choosing the k-element subset, its complement.

So, therefore $\binom{n}{k} = \binom{n}{n-k}$, because choosing a subset is equal to choosing its complement that is the reason why they must be the same. Here, we are precisely doing that.

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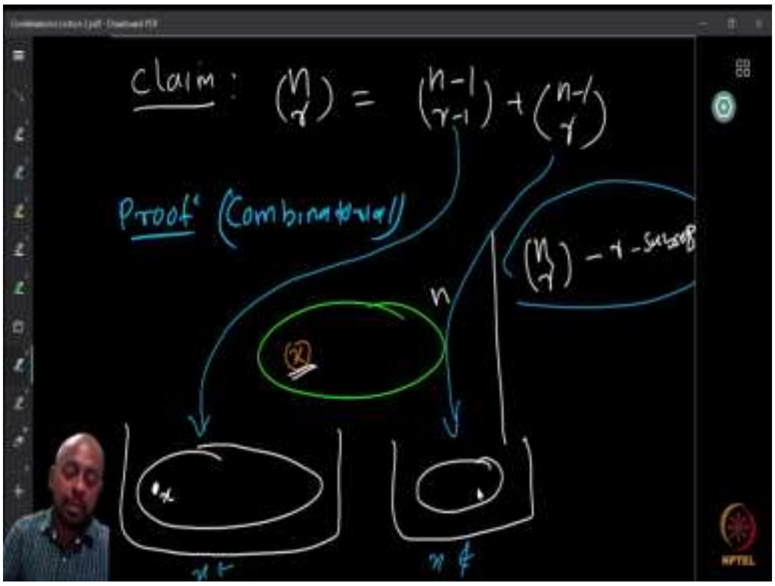

Now, here is another claim: let r is some positive integer less than n, then, $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$. As we said, we can always find an algebraic proof here. Again, try to write down these two things in the RHS, then cancel out the terms and then put them together nicely, you should be able to reduce it to $\binom{n}{r}$, but again, you know that does not tell us anything new.

But now, let us look at a combinatorial proof for the same thing. Why $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$? Again, if you can find out your own argument for this you should do that first. So, stop and think for some time, try to find out a combinatorial proof. Why this quantity $\binom{n}{r}$ which counts the r-element subsets of an n-element set must be equal to the $(r-1)$-element subsets of an $(n-1)$-element set plus r-elements subsets of some $(n-1)$-element set? Why this must be equal?

So, here is our combinatorial proof, I choose the n-element set that I have. So, I have this big n-element set and then out of which, I look at some particular element of this set, let us say x, so x is some element of this set that we are looking at in. Now, on the left-hand side, what we see is that we are looking at $\binom{n}{r}$, which is r- element subsets of the n-element set.

Now, when I look at the r-element subsets, I look at this subset and see whether this subset contains the element x or not. So, the special element that I just marked it down that element x, I look at whether the given r-element subset contains the element x or not.

Now, some of the subsets might all contain the element x. So, all the subsets that we are going to look at in this basket are all the r-element subsets which contain the element x. Then, of course there could be several r-element subsets which does not contain x.

So, these are the guys which does not contain x. So, this x is belonging to the set. So, here we have the r-element subsets where x is not a member and here we have the r-element subsets where x is a member.

Now, what I do is that I remove x from this set. Then, what do I get? I get $(r-1)$-element subsets, but all these $(r-1)$-element subsets are the subsets of the sets that we started with without the element x, which is an $(n-1)$-element set. So, therefore, there is precisely $\binom{n-1}{r-1}$ such subsets because I take the x away from the set. I get an $(n-1)$-element set I form the $(r-1)$-element subsets any of them I take, since x is not there, I can add x, so I get an r-element subset of the original set that we started with. So, these guys are all going to be precisely $\binom{n-1}{r-1}$.

On the other hand, here we are looking at r-element subsets, but these subsets are all without the element x. The starting set minus x. So, again it has only $(n-1)$ elements. So, all the r-element subsets without x are going to be present $\binom{n-1}{r}$.

Therefore, this is precisely the number of ways we can produce these, but they basically exhaustively count all the r-element subsets of the set that we started with. That is how we arrived at these two baskets. So, therefore, they must be the same. So, this shows why $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

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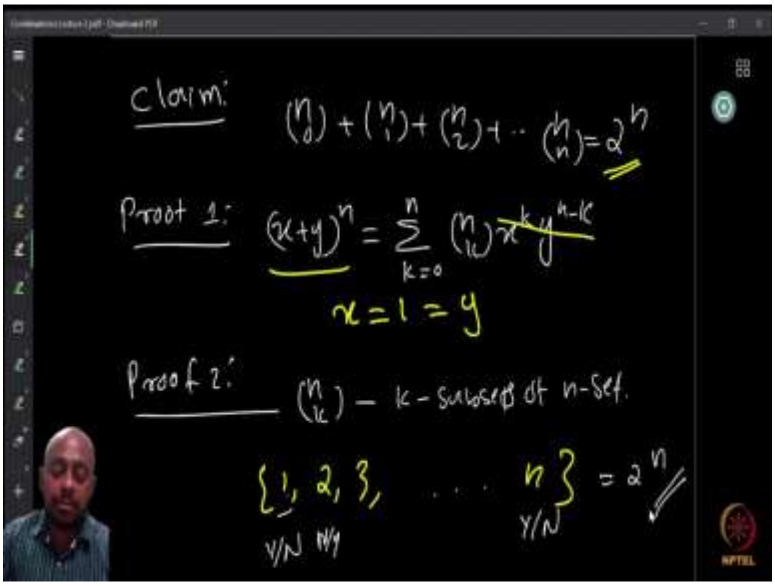

Now, here is a claim $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$. So, think about this and try to find your own proof, there are many ways to find proof. So, think about this question, try to come up with your own proof how many proofs you can come up with, you let me know.

So, here are two proofs I am going to give, first proof is by using binomial theorem that we just studied. So, we have the binomial theorem, which is$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$. So, how do you prove this equality using binomial theorem? Well, I just substitute $x=1$ and $y=1$.

If x and y are equal to 1, then on the left-hand side I get $(1+1)^n$, which is 2^n, which is the right-hand side here or what is on the right-hand side, well, x has become 1. So, therefore, $x^k=1$. Also y has become 1, so $y^{n-k}=1$. Therefore, these two becomes 1. So, I get $\sum_{k=0}^{n}\binom{n}{k}$ which is $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}$. So therefore, this proves the identity.

Now, what is the proof 2? Proof 2, well, $\binom{n}{k}$ is the number of k-element subsets of an n-element set. So that is what it counts. $\binom{n}{k}$ counts the number of k-subsets of n-set. Now what is on the left-hand side of the identity. So, it has $\binom{n}{0}$, which count this 0-element subsets of an n-element set which is the empty set, namely is going to be 1 always and $\binom{n}{1}$ which is the 1-element subsets, the 2-element subsets, etc., the n-element subsets. So, we are going to count the subsets of an n- element set looking at this cardinality.

So, by looking at the cardinality if it is 0 or 1 or 2 or etc. up to n there are only this $n+1$ possibilities. So, we count them separately looking at the cardinality I get $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+$

$\dots + \binom{n}{n}$. But what are the number of subsets of an n-element set? Well, that is 2^n.How? How do we form a subset of a set? You have 1, 2, 3, etc. up to n elements.

How do you form a subset? Well to form a subset. Now, I just decide for each element whether it is going to be there in this subset or not. So, I will say that this, this guy, element 1 is going to be there or not. So, it is yes or no. Element 2 again, yes or no two choices, element 3 two choices, element n two choices, yes or no. For each of the n guys I have exactly two choices to form my subsets whether to put it in a subset or, not throw it out.

So, because of this I have $2 \times 2 \times \dots \times 2$, n terms, which is 2^n choices that also count the number of subsets. This counts them all together, this one counts by them by separately by the cardinality. So, both must be the same. Therefore, the identity is clear. So, think of other proofs of this.

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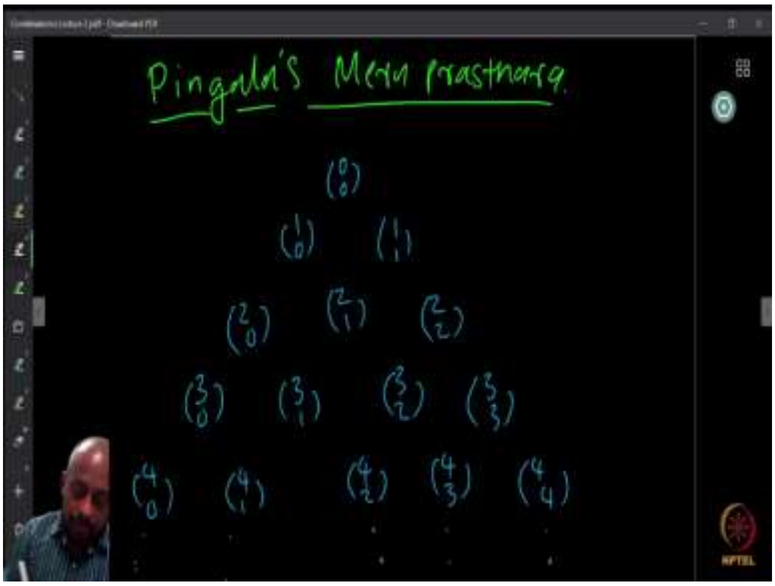

Now, I want to introduce you to something that many of you may have seen. So, this is Pingala's Meru Prasthara. So, Pingala's Meru Prasthara is known to be discovered by Pingala who was a very famous person, who studied the meters, basically language expert from ancient India in Sanskrit, he has written some book called Chandah Sutra. So, it basically talks about the meters, the poetry. Regarding the poetry, so there are many vrithas somethings like that. So, those vrithas are basically the meters. And he wrote a, it is called Chandah in Sanskrit, so, its called Chandah Sutra. So, there you do lot of mathematics.

To find out how many possible meters are there over a particular length, what are the things that you allow. So, all these things you can count there is a lot of mathematics there in this

Chandah Sutra, this is known to be somewhere in the BC 600 or earlier, just maybe, after Panini or something. So, his lifetime is around that time, so many 1000s of years before, he has described, what is called Meru Prasthara.

So, the, Meru means it is the name of a mountain in that, the olden days. So, a particular mountain, very famous mountain. And so, there is like mountain like structure, so that is the reason it is called Meru Prastara. So, you put $\binom{0}{0}$, which is 1 as the, the top, then you put $\binom{1}{0}$ and $\binom{1}{1}$, then $\binom{2}{0}$, $\binom{2}{1}$, $\binom{2}{2}$, etc. For each integer, you put this, so this continuous. So, this is the mountain that he is talking about, it has a nice structure.

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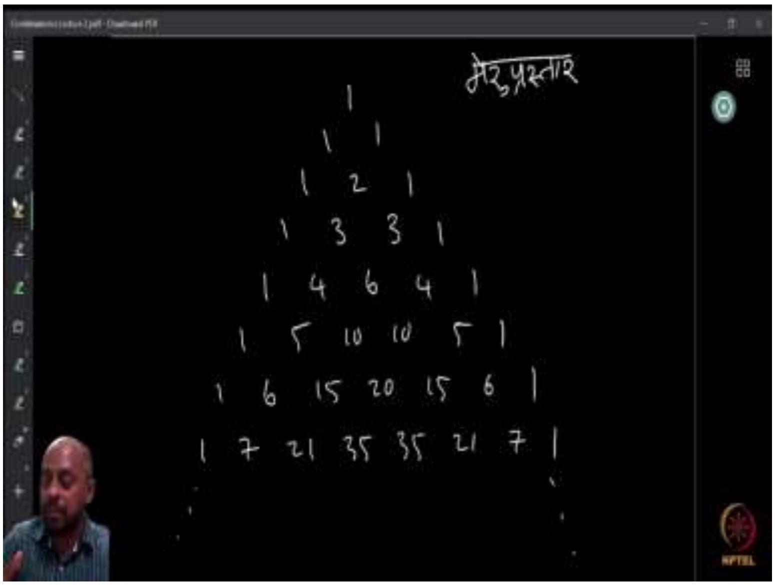

So, these numbers you can put, and then you can see this is going to be 1, 1, 1, 1, 2, 1, 1, 3, 3, 1, 1, 4, 6, 4, 1, etc. which are the binomial coefficients, as we saw earlier, which is appearing, the k th binomial coefficient appears at the $(k+1)$th line and we have this nice triangular way, which is called Meru Prasthara, or, later Blaise Pascal discovered is called Pascal's triangle also, maybe that is the way you have heard, but the name should be Meru Prasthara because more than 2000 years before Pascal, this was discovered.

Even before I do not know maybe people have discovered but what we know the earliest discovery by Pingala, so it is Meru Prasthara. And, this triangular formation has very, very nice properties. Here, we can find, so some of these properties you should try to figure out yourself, so play with it, look at, try to make this triangle, look at what all possible properties you can come up with.

Whatever you find yourself, also let me know, you can just send me some of these nice property that you find and write it down and send to me, I will see. Maybe we can, look at this properties, and try to come up with some, some more questions regarding this.

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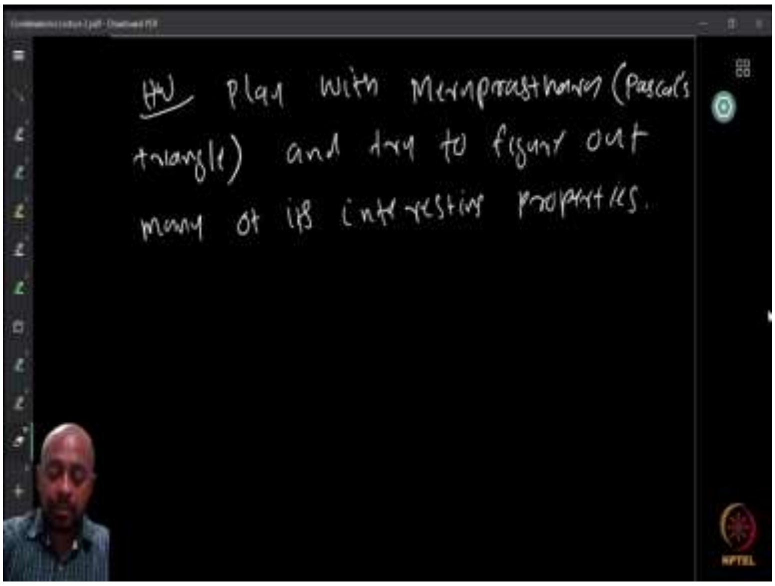

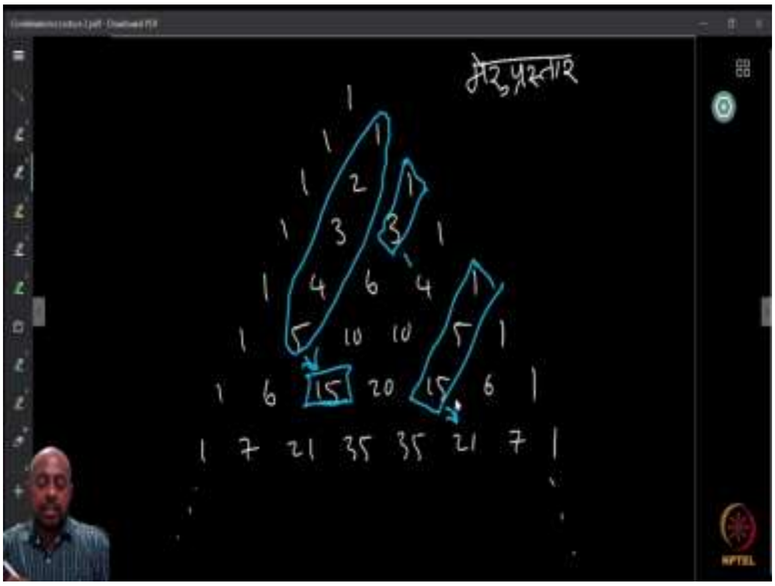

So, this is the home work I want to give you, just play with Meru Prasthara or Pascal's triangle try to figure out many of its interesting properties. So, let me give you one property just to know, just as an example. So, let me take this kind of a diagonal, and I start from the top I go 1, 2, 3, 4, let us say whatever 5 which ever, how many times I want I can go.

Now, I just add up these elements, add up this 1 + 2 + 3 + 4 + 5, I will get 15 that is precisely the element that is going to appear here or anything, I want plus 3. This one is going to be 4, or 1 + 5 + 15 is going to be 21. So, this way, I can find out that no matter how much I go I can see that this property is true.

So, similarly many properties are there like you can draw in many different diagonals, many different things, properties you can observe from this. Try to come up with such nice properties, we will give an example based around this property I showed you, in fact to prove this property using the binomial coefficients. Or, maybe you can try to formulate this property on your own. So, I will not write the exercise now, you think about this and by the next class, try to come up with a formulation and maybe a proof. So, we will stop for today and continue in the next class.