Combinatorics

Professor Doctor Narayanan N
Department of Mathematics
Indian Institute of Technology, Madras

Examples: Product and Division Rules

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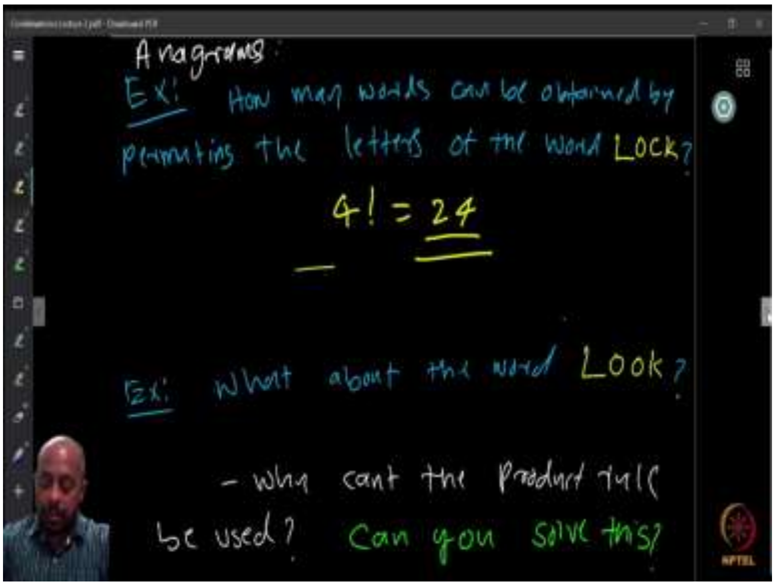

So, we continue from the previous class. So, where we were looking at the Anagrams of the word LOOK. So, did you think about the way how to count the anagrams of LOOK and why is it different from that of LOCK? And if you have solved it, very good, otherwise we have to wait a little more before we will look at this counting. So, we will look at another example now.

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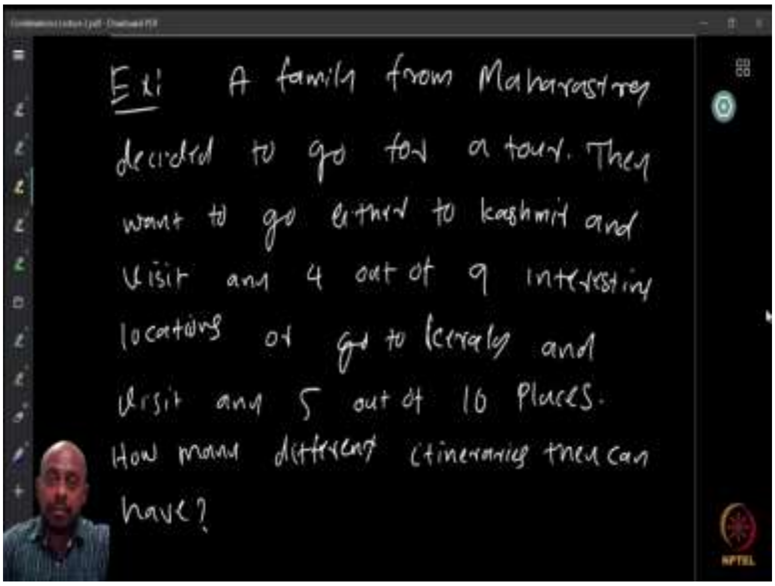

So, a family from let us say Maharashtra decided to go for a tour for their vacation. So, when they were thinking about this they decided, okay, either we can go to Kashmir or we can go to Kerala. So, either the South most or the North most part of India maybe and both are very beautiful, beautiful states and very nice places to see.

And then they said, okay, in Kashmir we can, we know we have 9 possible interesting places we found and then out of these 9 places maybe we can visit 4, with our budget we can visit 4 of these places or we can go to Kerala where we found 10 interesting places and maybe we can go to 5 out of these places. These are our choices.

So, we can either go to 4 places out of these 9 interesting locations that we have shortlisted or any of the 5 locations in Kerala out of the 10 places we have shortlisted. Now, the question is that how many different itineraries we can, they can come up with. So, finally, they have not yet decided, but suppose they are going to decide. It can be any of this, because they do not may be have a preference.

But they are depending on their preference, they choose one of these, but which one is it? So, what are the possible options that we have for their itinerary? So, itinerary means that there is the order is important. The way that they are going to visit like for example, if you are visiting, let us say 3 cities, going A, then to B, and then going to C, is different from going to A, then to C, and then to B, or from B, A and C because the travel tickets and bookings of the hotels, etcetera, are going to be different.

So, the itineraries are going to be different depending on the order in which they are going to visit. Now, how do you find out the number. So, here we are going to use two different principles. We have both the addition principle and the product rule. So, these two principles we can use. Because, for example, if you take the first part alone, going to Kashmir and visit 4 out of the 9 places. We know how to find it. So, out of the 9 places, I want to select the order in which I am going to visit the 4. So, how do I do this.

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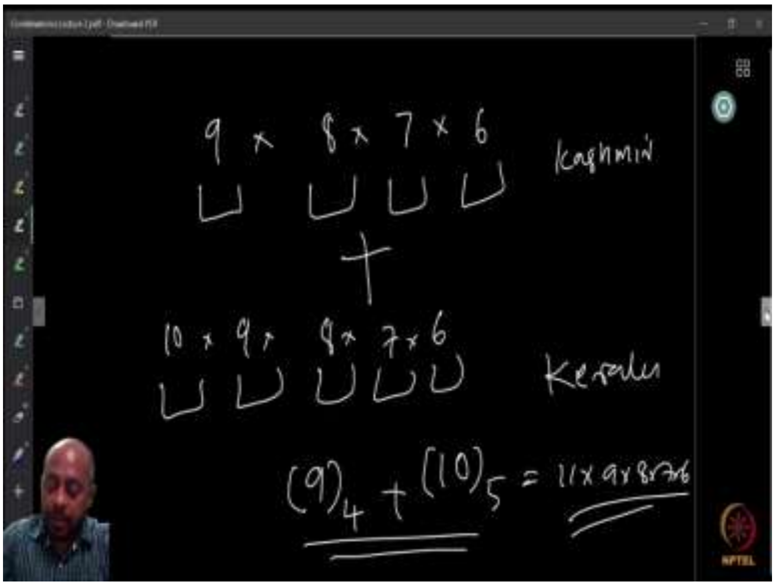

So, out of the 9, I can choose any of the 4 places. So, the first place I am going to visit can be any of the 9 places. The second place I am going to visit can be any of the remaining 8 places. The third place is going to be any of the 7 places and the last place I am going to visit in Kashmir can be any of the 6 places. But they may not decide to go to Kashmir at the end maybe they decide.

We can either go to Kerala or the Kashmir. So, maybe they instead of Kashmir, they have all these possibilities and then this can be multiplied because the choices are independent, or in the other case, they decide to go to Kerala, they have 5 possible destination they can choose from the 10 possible available places.

So, 10 choices here, then 9 choices here, 8 choices here, 7 choices here, and 6 choices here. So, multiply this and you get the possible ways to visit 5 locations in Kerala. But since it could be either this or that, we can, and they are disjoint choices, we can add using the addition principle.

So, therefore, the total number of itineraries is $(9)_4 + (10)_5 = (9 \times 8 \times 7 \times 6) + (10 \times 9 \times 8 \times 7 \times 6) = 11 \times 9 \times 8 \times 7 \times 6$. So, this is the total number of itineraries they can come up with. That is a huge number of itineraries.

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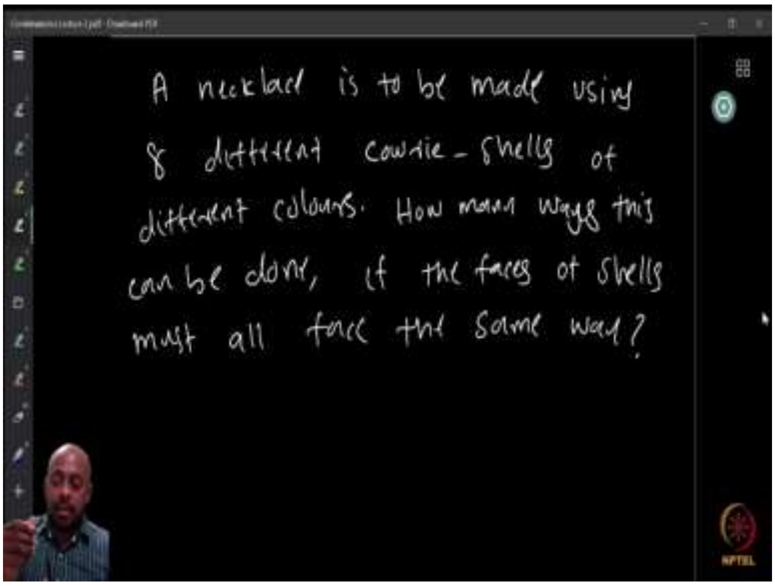

Now, here is another question. So, I have 8 different cowrie-shells, so cowrie-shells, you might have seen this kind of very beautiful, very nicely shaped small shells that you get from the sea. And they have a distinctive face it comes, I do not know how to draw, but I am sure that you can figure out. There are these or you can search in Google and find out what are cowrie-shells.

So, look at the cowrie-shells, you can see it has a distinct face, it has a nice rounded face and a flat face. Now, I want to make a necklace out of this cowrie-shells, it used to be in fashion in the pre-historic era's people or even not necessarily pre-historic even nowadays, people started using it, but it used to be very, the only things available before people have invented gold and things like that.

So, but they still you can make necklaces without using gold. So, now we want to use necklace, we want to make necklace using this eight different cowrie-shells. But they are of different colours, they come in different colours. Now, how many ways you can make this necklace? Because the order, when the way you keep, can be different, it looks different. But we assume that the faces of the shells must all face the same way.

And I do not want to put one shell this way and the other one this way, etcetera, like this or even one like this and one like this. That I do not want. So, everything must be faced in the same fashion. So, how many ways you can do this. Again, stop for a minute, think about this question. Can we use any of the earlier used principles or we want something new? Or maybe you need something that we already know and then something new or like we can use many of the earlier things used here or just one of them, so whichever way find it out.

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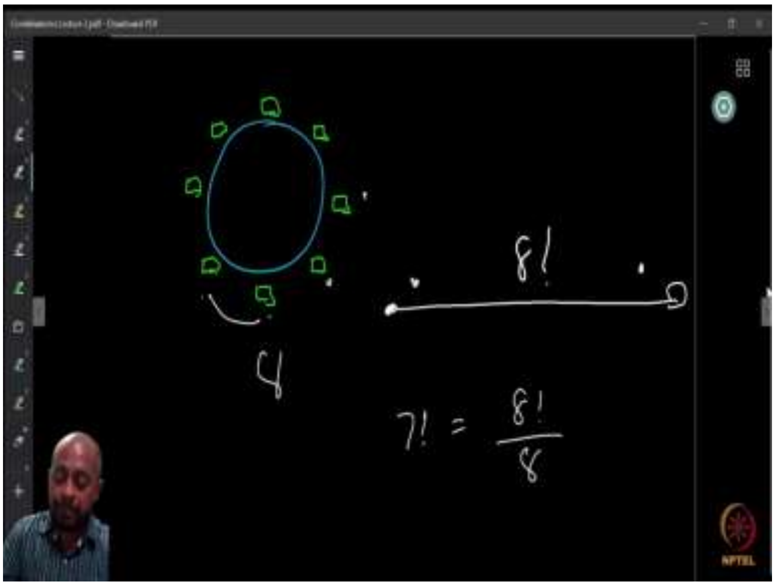

Suppose, we were using, so it is a, the structure of a necklace, let us say. So, you have this necklace and you can put 8 of the different cowrie-shells here, in a particular fashion. Now, they are of different colours, so the order in which you keep are going to be different, but since it is a necklace,

See, let us suppose initially like instead of this necklace, we had let us say that we just break the necklace by putting a connector here, let us say a connector here and the corresponding, a male connector here maybe and corresponding female connector. So, I can put this thing together. One inside the other, so that it locks. If I was doing this, then or in other ways, if I am looking at a linear arrangement of the cowrie-shells.

In a line, so I can just open this up, it forms a straight line. But here is a special hook maybe and here is another hook. So, the way I am going to put here are going to be different. So, there are all possible permutations or linear arrangements of these 8 shells. So, therefore, 8 factorial ways I can do this. So, the number of ways I can arrange in the line, is going to be 8 factorial.

But now, since I am going to put this in this circular fashion and without maybe, without using any of these connectors like this, then what I can say is that, see if I take this necklace and just rotate it, I just rotate this once. So, this shell come here, so this comes here, this comes here, this come here, etcetera.

So, I just rotate the necklace, it is the same, it does not matter, the rotating a little bit does not make any difference. So, if I rotate it, it is not going to make any difference. So, therefore, if I had this 8-factorial arrangements, I take one of this. I put into this form and then make this circular necklace. If I rotate, the thing, the cowrie-shell which was here, after the rotation will come here, after rotating it comes here.

So, I get a different permutation, but it is going to give you the same necklace. So, therefore, 8-factorial is not the correct answer, because I am over counting. Now, how many times I am over counting. So, what you can observe is that, since once I take any such permutation and make a necklace. If I just rotate it and put this guy to this position or put this guy to this position, put this guy to this position.

So, any of these will give the same necklace, or in the circular fashion, they are all going to give the same order, circular order is the same. Therefore, in the necklace, they are all going to be the same. But one necklace, one cowrie-shell, you can set in any of these 8 different positions, once you fix the order, and everything else will rotate together. And therefore, there are exactly 8 possible ways for this guy to be moved around in the circle.

Therefore, out of this 8! arrangements, 8 of them are going to be giving the same circular order. But now, this is true for any given order, instead of these, I take some other arbitrary order, then I do this again the same thing, each one correspond to 8 cases in the linear order. So, therefore, for all of them, I have exactly 8 possibilities, which are giving this exactly the same necklace.

So, therefore, I can divide by 8 intuitively, so I divide 8! by 8, so I get $7! = \frac{8!}{8}$. So, this tells us that the number of possible arrangements of the necklace, I mean the cowrie-shells into a necklace is7!

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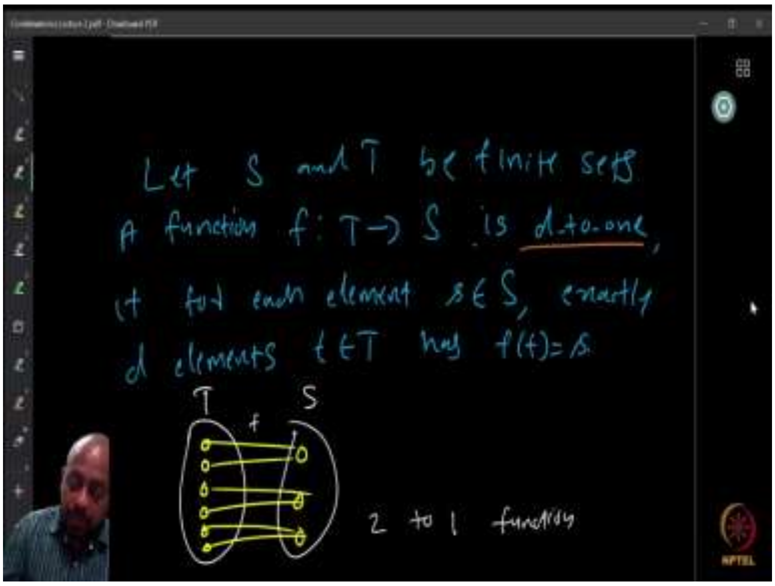

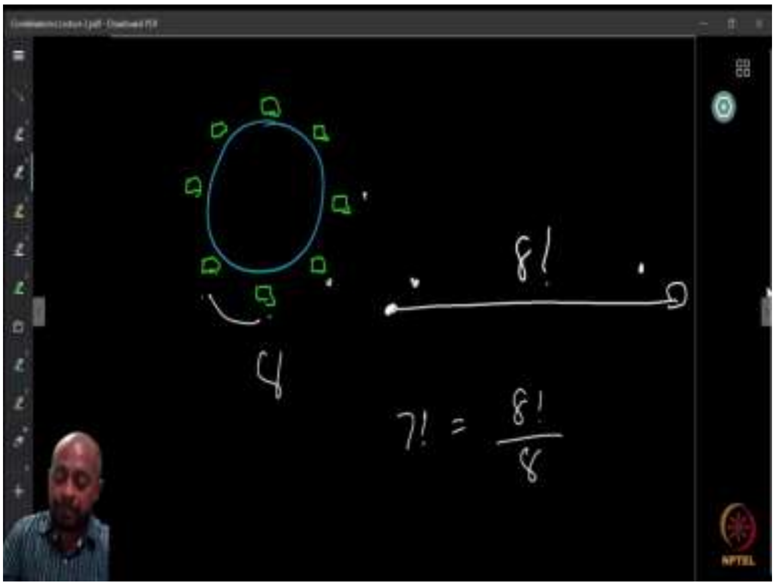

So, here we use two principles, we use the product rule and then we use something, some observation to be able to divide. Now, how did we divide, what allowed us to do the division. So, let us make it precise. So, let us consider two sets S and T both are finite sets and consider a function, let say $f: T \rightarrow S$.

We call a function f to be d-to-1 function, where d is a positive integer. If for every element $s \in S$, exactly d elements $t \in T$ has $f(t) = s$. That is, for exactly d elements of T, their image is going to be s. So, with respect to the function, each element in S has, exactly d inverse elements. It is called a d-to-1 function, here is an example of a 2-to-one function to make it more clear.

If you are not taken math courses, maybe you are not familiar with the terminology. So, here you have one element of S and correspondingly exactly two elements, according to f. Similarly, for this element, I have these two elements for this guy I have exactly these two. So, because the pre-image or the inverse elements of each of the element in S, are going to be exactly two, it is a 2 to 1 function. Similarly, you defined d-to-one function.

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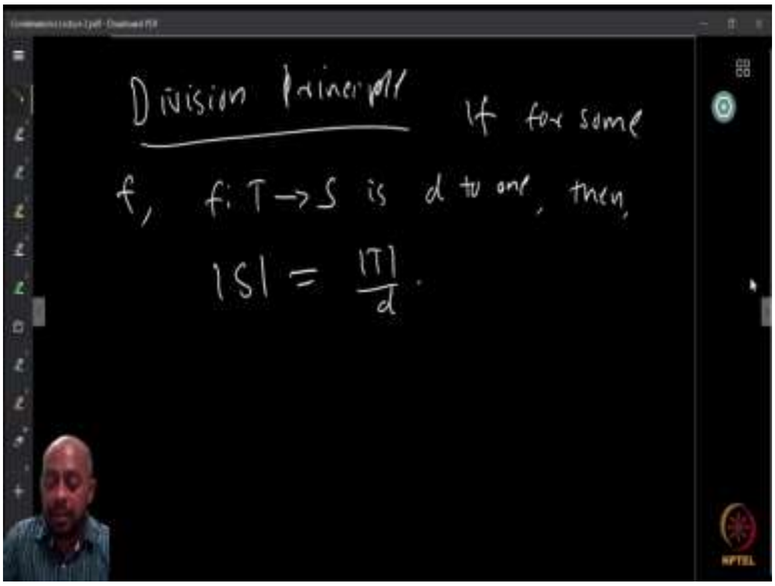

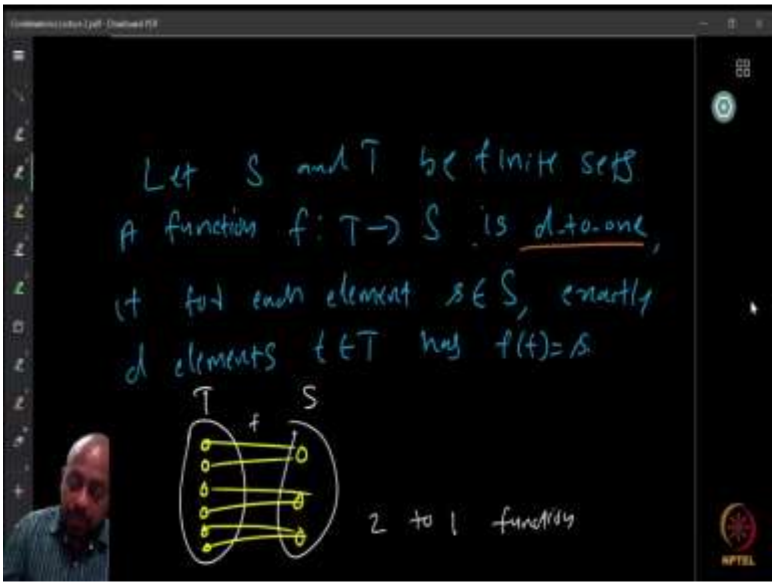

Now, the division principle says that if for some function f, $f: T \rightarrow S$ is d-to-one, then $|S| = \frac{|T|}{d}$, for finite sets S and T.

Once you see how this function behaves, it is immediately clear to us, intuitively, but we are going to comment as a division and say it as a division principle. And this principle can be used precisely when you can define such a d -to-one function, that is important.

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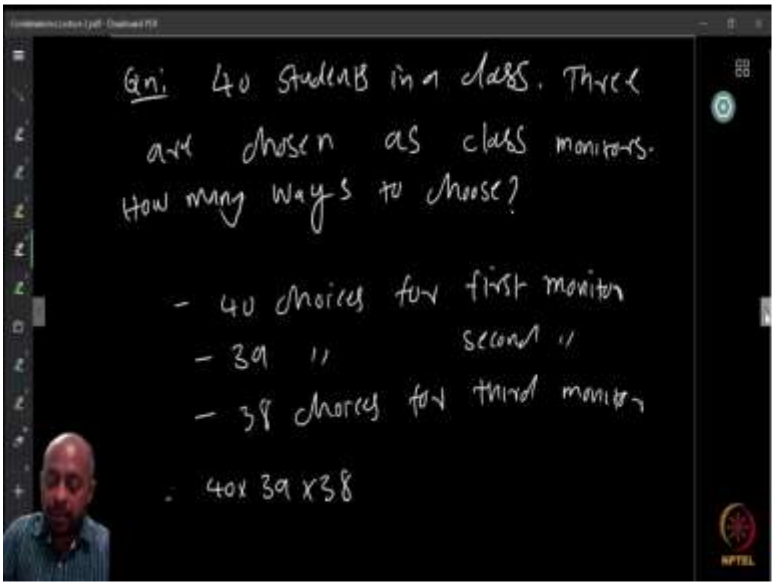

So, here is a question. 40 students are there in a class and 3 are going to be chosen as class monitors. Now, how many ways we can choose? From the previous counting that we have done, we can now see that, I want to choose 3 persons for the monitors. So, for the first monitor, I can choose any of the 40 students. So, there are 40 choices for the first monitor. Then for the other monitors, second guy. I do not, I am choosing three different people.

So, I cannot choose the first guy. So, that I can choose any of the 39 remaining students. Now, once I chose the 39, the second person out of the 39 then the third person can be chosen other than the first two. So, there are 38 choices. And these choices are independent. So, therefore 40 into 39 into 38 ways I can choose these 3 guys. Is it correct? So, if you do not think it is correct, how do you improve the answer? Or if you think it is correct, see, think about it again and see whether there is something that we are missing.

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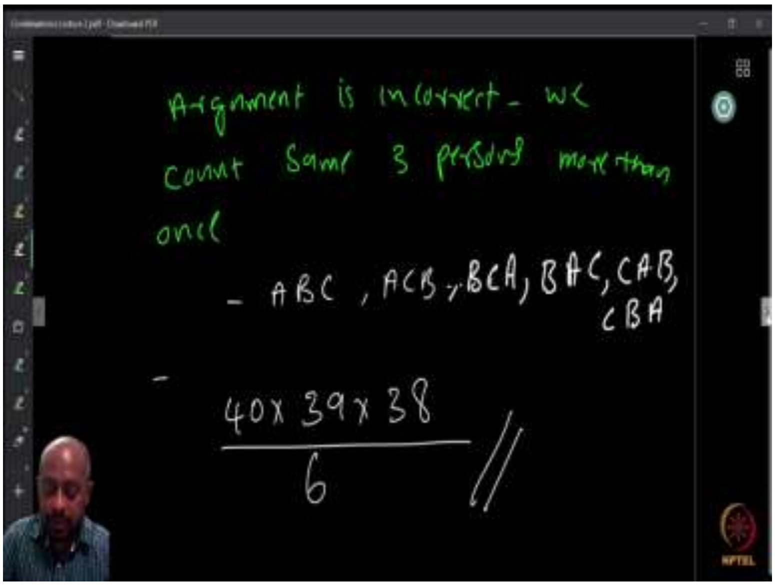

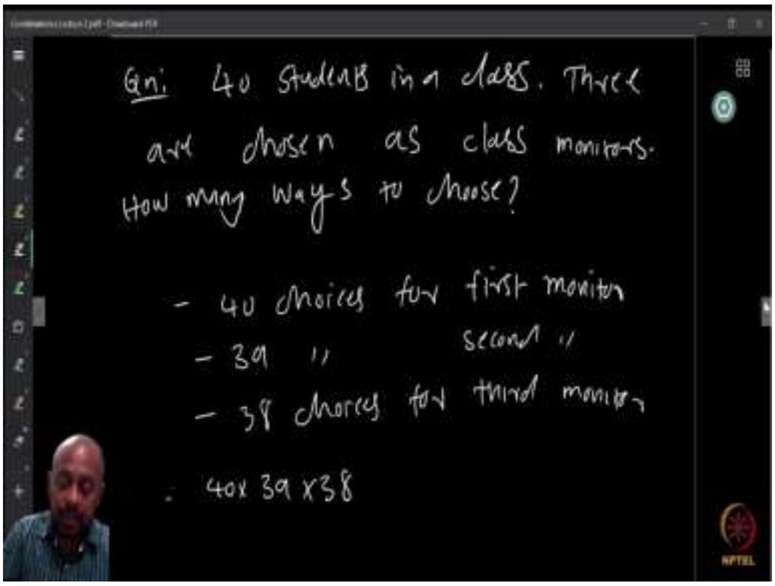

So, now the argument is incorrect, because we count the same three persons more than once. For example, when you look at, let us say, persons, A, B, and C. So, the first person to be chosen as the first monitor was let us say A, the second person chosen was B, and the third person chosen was C. In another choice, we could have chosen let us again, A as the first person, C as the second person, and B as a third person, and we will still get the same three A, B, C as the monitors.

Similarly, I could have B as the first choice, C as the second, and A as the third choice, we will still get the three guys as the monitors. Because we did not distinguish between the positions of the monitors, we just said there are 3, exactly 3 class monitors we are going to use, did not say that one guy is superior to the other. One guy has higher rank than the other and nothing like that.

So, therefore all these ABC or ACB, or CBA, each of these choices are going to give us exactly the same three guys. So, which means that we have counted them more. So, we have already read some over counting. So, now can we do something to undo the over counting. So, we observed that in each of the cases, we do the over counting also exactly the same number of times.

Because if A, B and C are going to be the monitors, it could be either of these ABC, ACB, or then what are the other choices BCA, BAC, or CAB, or CBA. And exactly these possibilities only. Any other choice of order related to the people will not give exactly these 3 guys. So, therefore, to choose the 3 guys we have counted them in how many times 1, 2, 3, 4, 5, and 6 times.

So, six times we have over counted the same triplet of people. And one can think about a little bit and see that for every triplet this is true. Because ABC were arbitrary. So, therefore, we have a d- to-one function, you want to define formally the sets and all you can do it, I will skip that part.

But think about it, and try to define it, it would be a nice way to see. So, we have this function and therefore we can use the division principle. And say that since we counted this $40 \times 39 \times 38$, and every triplet, we counted exactly 6 times I can divide by 6 now. So, this is going to be the number of ways to choose the three class monitors.

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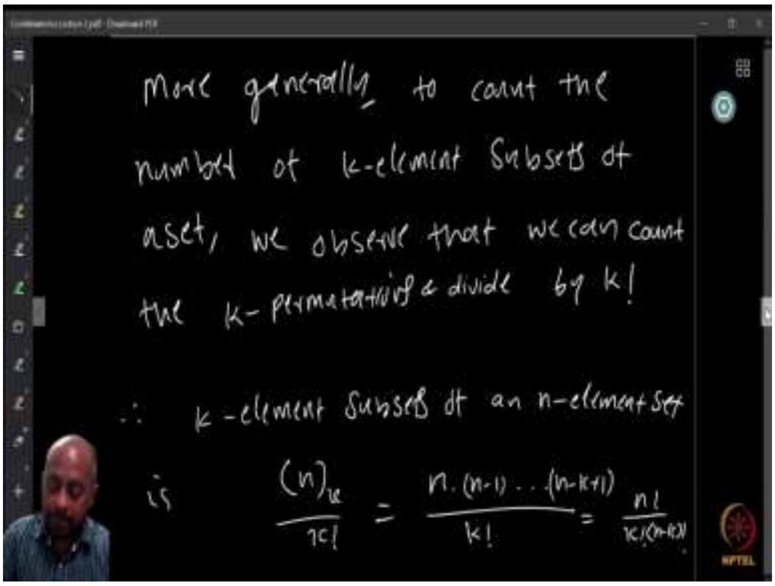

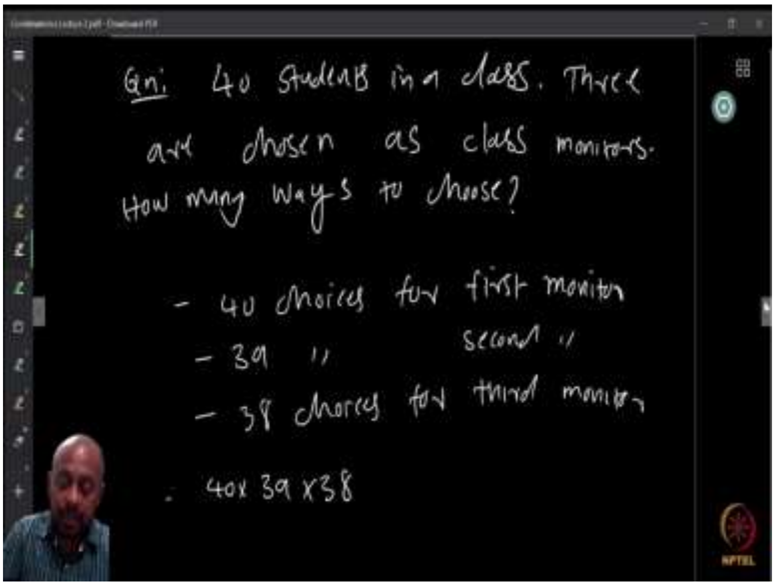

Now, more generally, this same principle can be used to count the number of k- element subsets of a set. So, what we did precisely was here was to find a three-element subset of this 40-element set. Because we wanted to just choose three persons. So, that was a three-element subset, but what we did was to basically count the three permutations, and then we divided by 6. But in general, what we can do is that we can count the k permutations.

And once you do the k permutations, we observed that the k permutations of a fixed set are going to be exactly $k!$. And once you have any of these $k!$, they will all give the same set. So, therefore, I can divide by the $k!$ to get the total number of k- element subsets. So, to count the k- element subsets, I can just use this. So, first count the thing that we already know how to count, the k permutations, and then divide by $k!$. So, we have the k permutations here, which is $(n)_k$, and then I divide by $k!$. So, I get $\frac{n!}{k!(n-k)!}$. So, this is our number of k -element subsets of an n-element set. So, we can count the subset by counting the k permutations first or listings, k-element subsets that we can list.

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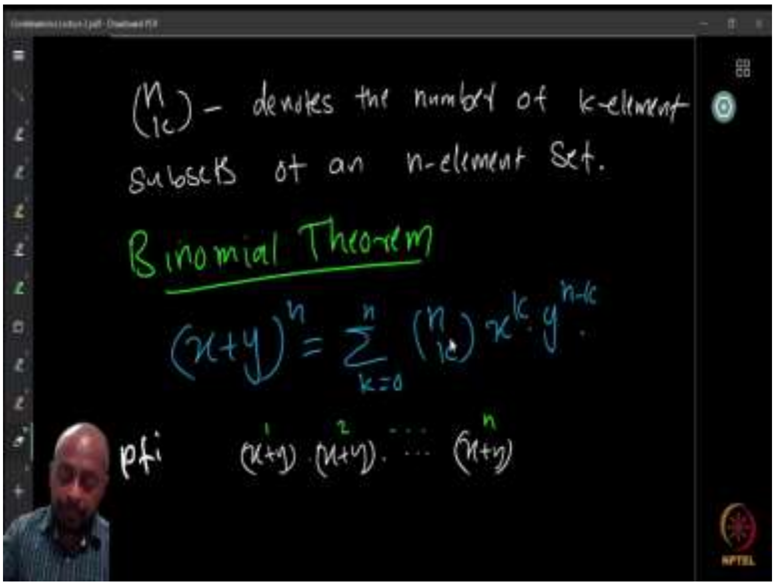

So, we denote this number $\binom{n}{k}$, read as n choose k we have seen this many times I am sure, maths students definitely know. So, this is the standard notation to say that we are going to choose k -element subsets of an n- element set, that is a number of ways to do this. The number of k -element subsets of n -element set, $\binom{n}{k}$. Now, this number is called Binomial coefficient, k th Binomial coefficient of n. And the reason is because, this appears in the well-known theorem called Binomial theorem.

So, what is the Binomial theorem? Binomial is a polynomial with exactly 2 terms. So, you have these 2 terms and that is called a Binomial. So, $x + y$ is a Binomial. Now, I take the binomial and take the n th power of it. So, it is basically product of n copies of $x + y$. So, $x + y$ is its factor, there are n copies of this factor. So, how do you expand this and what are they going to be the terms.

So, Binomial theorem says that $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. So, what we know is that when you take the product you can only have the factors coming from x and y. Every factor must contribute one of the factors x or y. So, it will be some power of x into some power of y. These are going to be the terms.

All the terms are going to be this way. So, it can be $x^0 y^n$ or $x^2 y^{n-2}$ or any these things. Now, how many of these guys are there? So, basically, if you want to write there is going to be like you know many, many terms coming here. So, we do not want that. So, we want to write them together all the similar terms I can put together.

But how many of them are there? So binomial theorem say that precisely $\binom{n}{k}$ terms of the form $x^k y^{n-k}$ are going to be there. Why is this? But this is immediate actually, if you look at the product $(x+y)^n$, it is $(x+y) \times (x+y) \times \ldots . \times (x+y)$.

Now, how do I form the term $x^k\, y^{n-k}$. To form this I should be able to choose exactly k copies of x and the remaining all must be copies of y. Because I do not have a choice but to choose from every factor at least once. Because the product has all this n terms, they are multiplying. So, from each of this I have to choose one.

So, how many ways I can choose x^k, the remaining must be all y. So, there is no choice. So, if I choose k of the x's, then I am done. So, how many ways I can choose k x's? Well, so we have exactly n right factors $x+y$. So, out of these n copies, I decided which of the k copies I want to select the extra. So, how many ways I can do this?

Exactly $\binom{n}{k}$ ways, so the $\binom{n}{k}$ ways I choose x^k. I do not choose the remaining terms. Whatever is except for this set, that is the remaining $n-k$ sets are going to give exactly y. Because otherwise, the power of x will increase. So, therefore, by definition of $\binom{n}{k}$, we can see immediately that this is a coefficient of $x^k\, y^{n-k}$ is exactly $\binom{n}{k}$.

Now, how many terms can appear? Well, k can be either 0 or 1 or 2 or up to n. So, therefore, $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ that is the expansion of $(x+y)^n$. So, that is the binomial theorem.