Combinatorics
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Examples of basic counting

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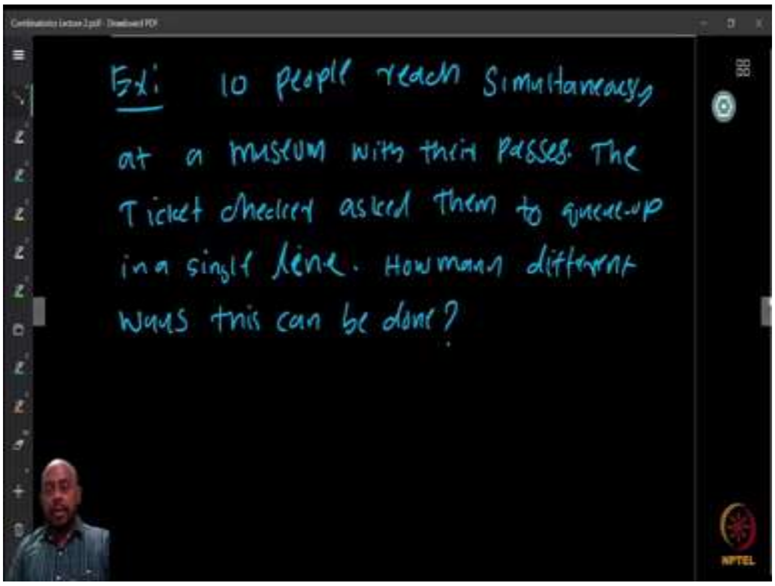

Now, another question, another example. So, I have 10 people who reach simultaneously at a museum and museum allows passes so, they all came with passes. So, you can use the pass to visit regularly maybe in once in a month you want to go a couple of times, you can use the pass to go. So, one day at the morning like, when the museum opens, 10 people stand in the front of the museum, and then they all reached at the same time.

So, basically when they wanted to enter, so the ticket checker said that, Okay, I mean, I cannot put everybody together, I want to check passes. So, please form a queue, so stand in a line and then come inside. Now since, all of them came together, they can queue in anywhere they want, if some person reaches first then we can say that, okay, because he reach first, he stand in the front.

But, now here everybody came together so in any order they are going to stand, it is not going to make difference. So, the question is that, how many different ways this queuing can be done? So, how will you count this?

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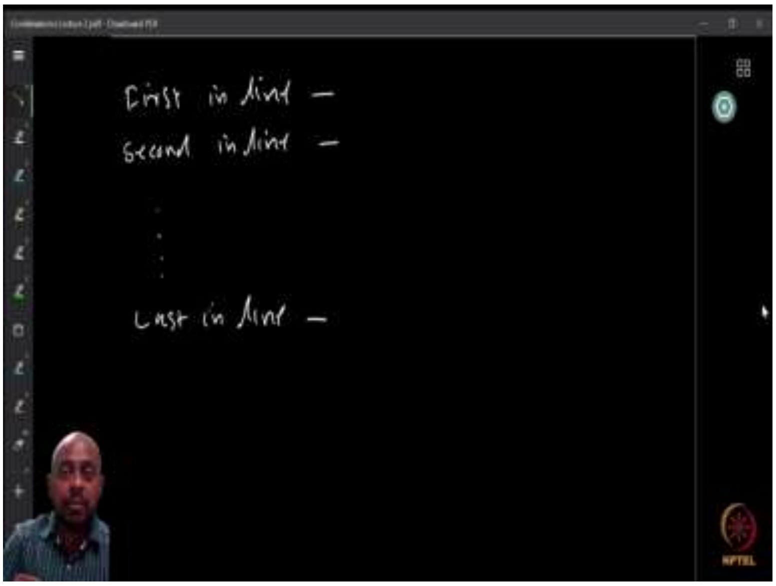

Let us look at the first person, who is going to be in the line. The first person going to be in the line, can be any of the 10 persons. So, I can choose the first person who is going to stand in any of the 10 possible ways. But, now when I look at the second person who is going to stand in the line that is going to depend on who was in the first? The choice of the person who is going to stand second is going to be dependent on the one who was standing in the first position?

Because, if, A, B, C, D etcetera are the people, if A standing in the first position, then A cannot stand in the second position. But, if B was standing in the first position, out of these 10, then, B can not stand in the second position. So, it can be only any of the other guys. So, the choices themselves are not independent, but the number of choices on the other hand, one can show that it is independent.

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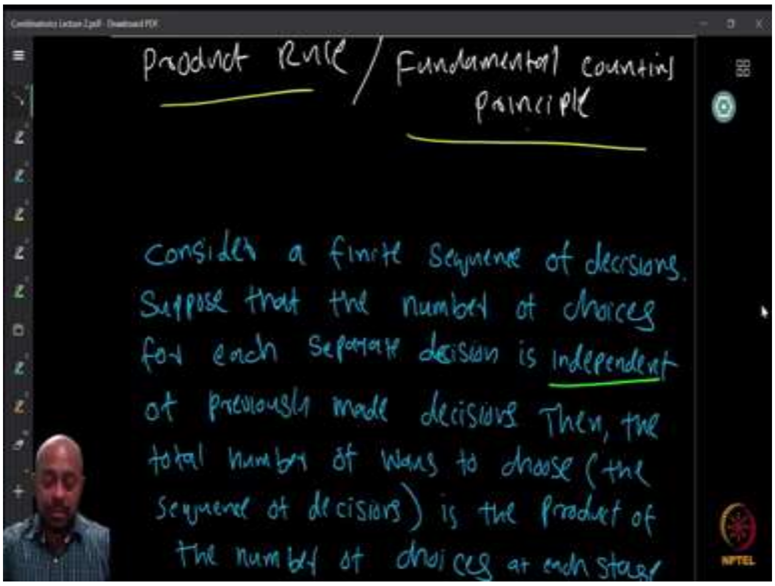

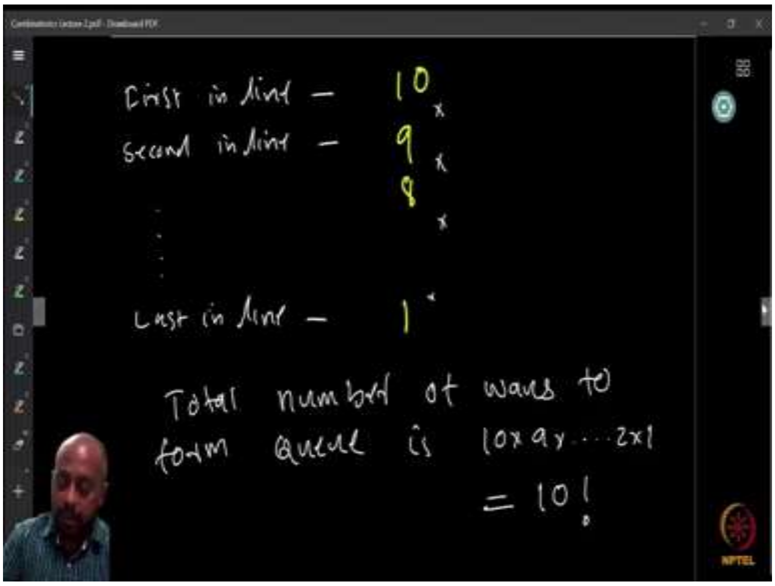

So, the framing of the rule, the product rule, is careful. So, it says that if the number of choices of each separates decision is independent of previously made decisions. So, the choices themselves need not be independent, that is important. So, the first in line, we have 10 possible choices for that guy. So, who are arrange there, one of these 10 persons must stand in the queue.

But, once a person is decided, the second position can be filled with any of the 9 possible remaining ways. Even though who is going to stand there is dependent on the first choice, the number of persons who is going to stand there does not dependent on this choice. So, therefore I can clearly say there are 9 possible other guys, standing for second position. Then I can choose the third guy in 8 possible ways etcetera, the last guy is going to be only one guy, he is going to stand in the 10th position.

So, we already said that, even though the choices are not independent, the numbers of choices at each stage were independent. So, therefore the total number of ways to form queue is $10 \times 9 \times 8 \times \ldots . \times 1$. This particular number has so much applications and it comes in so many places in any of the counting questions.

That it has a special name, it is called factorial, you might already have seen this. And it is denoted by a 10! in this case or $n!$ if you are looking at n in general. So, this is how we denote factorial.

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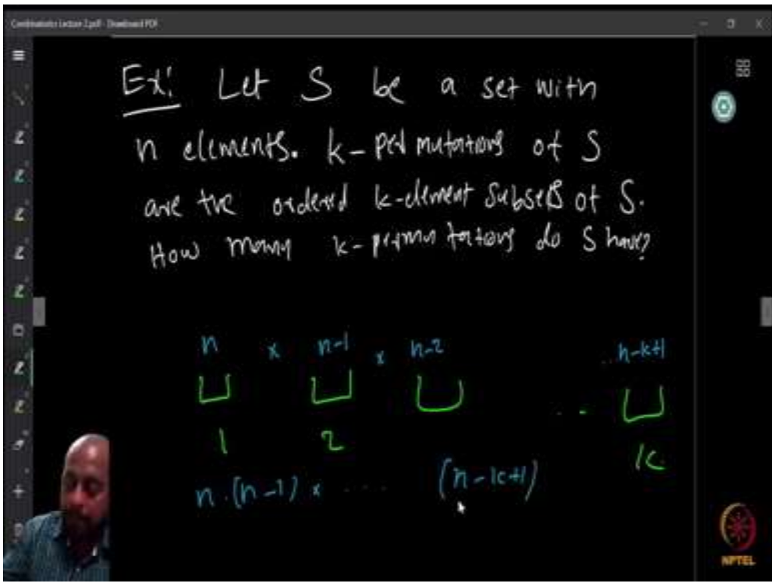

Now, another example, we are looking at too many examples, maybe, but, these examples are also telling some other question that comes, some parameters that comes very often. So, that might be also useful. So, let S be a set, which has n elements now, a k- permutation of the set S are the ordered k- element subsets of S. So, I can just take the k- element subsets of the set and then, order them. So, we start with a set, take some number of elements from them, put some in any order that I want.

Now, the question is that, how many k- permutations of S are there? Now, how do you count this? Well, from the previous question that we looked, we might have a good idea about it. So, I start with a set S which has exactly n elements. Now, I want to form k- permutation means that, I need to order these k elements in a particular manner. So, I fix let a say $'\,k\,'$ positions and then say that Okay, the first position I am going to choose so, 1, 2, 3, etcetera.

Now, first position, I want to fill up with some element so, I can choose one of the elements, there are how many choices? There is n possibilities for the first position because, any of the

n elements can be selected here. But, then the second position, the first person who were I chose cannot be there. So, therefore the remaining $n-1$, of them can be chosen here. Then $n-2$ and then up to $n-k+1$.

So, if there are k-element subsets that we are forming not subset like, permutation that we are forming, then we can choose n person for the first position, $n-1$ for the second etcetera, $n-k+1$. And each of these choices are independent so, therefore it is $n \times (n-1) \times (n-2) \times \ldots \times (n-k+1)$.

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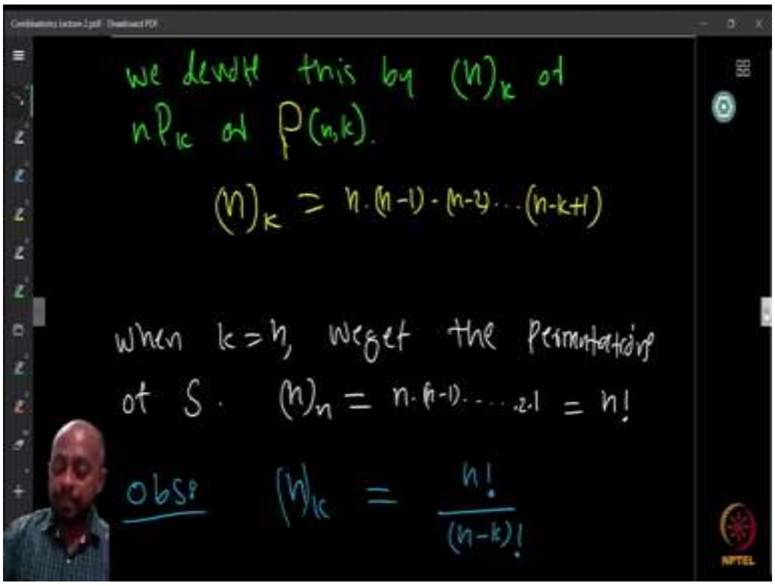

So, this particular thing has a special notation, this is a denoted by $(n)_k$. So, this is something, something a notation that will come again and it is also denoted sometimes of nP_k or $P(n,k)$. So, all these notations are used but, we will stick with one of them as far as we can but, sometimes many textbooks follow many different notations, sometimes we might just use one of these. Anyway, it is better to be familiar with these.

So, this is the number of permutations, k-permutations of a set. Now, if $k=n$, so we are looking at $n, n-1, n-2, \ldots, n-k+1$. So, when $k=n$, you can see that this immediately reduces to $n!$. This is what we get. So, it also makes sense because, we are talking about k-permutations. Now, what we looked at earlier was basically an n-permutation of the n set. Because, we are looking at all possible arrangements, this is precisely what we looked at in the previous question, where we were looking at the number of ways to queue up those 10 people?

So, the 10! came because, they are all the permutations of the people. We are arranging them in the linear order, so the linear arrangements of these objects can also be called as permutations of them. So, our observation is that $(n)_k = \frac{n!}{(n-k)!}$, because by definition whatever $(n-k)! = 1 \times 2 \times ... \times (n-k)$, and $n! = 1 \times 2 \times ... \times n$. But, in the $(n)_k$, we have only $n \times (n-1) \times (n-2) \times ... \times (n-k+1)$. So, the 1 to $(n-k)$ is missing so that I can just divide from the whole product. So, therefore $(n)_k = \frac{n!}{(n-k)!}$.

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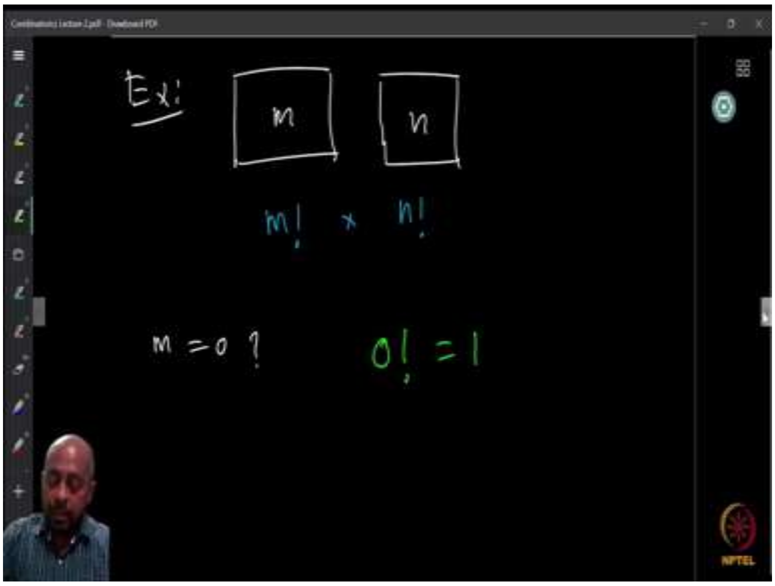

Yet another example, I think we are looking that many examples. So, what we have here? So, I have 2 rooms so, 1 room has m people standing inside that room, and the next room has n person standing there. Now, I ask the question, how many different ways are there for me to ask the people in the first room to queue up and also at same time in the second room to queue up. So, I want the persons in the first room to queue up, and at the same time I want the persons in the second room also to queue up.

How many ways we can do it? So, we already saw that if you have m persons and you want to queue them up, then I can do this in $m!$ possible ways. In the second room, there are n people so, I can permute them in any of the $n!$ ways to queue them up, but these two I can do independently because, the persons in the first room are queued up and the persons in the second room are queued up separately. So, therefore there is $m! \times n!$ ways.

So, the total number of ways to queue up them separately is $m! \times n!$. Now, comes the question, what happens if $m = 0$, if $m = 0$, I can still arrange the persons in the second room to queue up in $n!$ ways. So, when I say that people in the first room to be queued up in whatever ways they can and the second room to be queued up in whatever ways they can, I still want the answer that, the $n!$ guys who are going to be permuted in the second room or all possible different arrangements for that.

So, I want $m! \times n!$ to be the answer for this, but then I want to get $n!$ as the answer because, that is a number of ways to arrange the people in the second room. So, therefore $m!$ in the first case when $m = 0$, must be 1. Because, otherwise, $0 \times n!$, will not be $n!$. I want $n!$ times something to be $n!$ itself. So, therefore $m!$ in that case must be also 1.

So, therefore we justify the reason we use, $0! = 1$. In fact, there will be several other occasions, we will see that why it is required, to define it this way. But, now we have at least some reason to see why.

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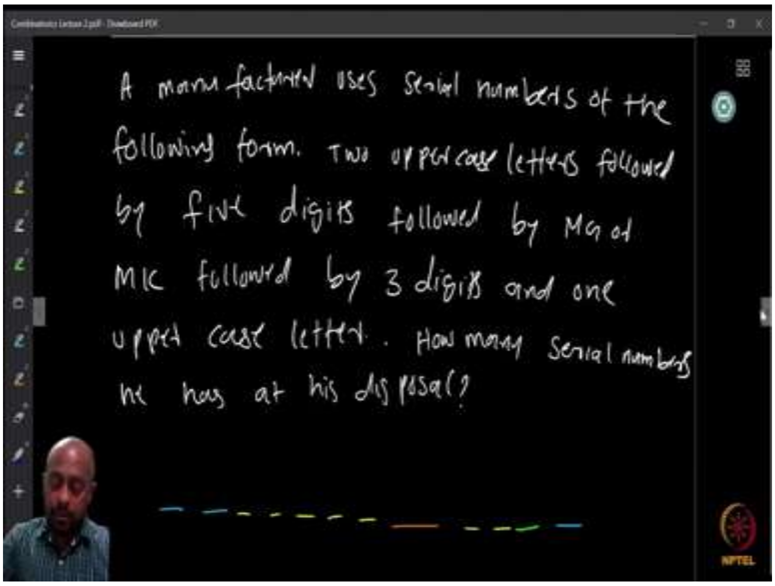

Now, here is a question, a manufacturer uses let us say serial numbers of the following form. So, every product that he makes, he put some serial number, may be using for warranty purposes etcetera. Now, two uppercase letters followed by 5 digits and then, they are followed by the letters capital M and G or capital M and K. Then, this can be followed by MG and MK followed by again 3 digits and one uppercase letter. So, the serial number has 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13; 13 letters or digits appearing in the form.

Now, how many possible serial numbers he has at his disposal? So, he makes these products everyday may be 1000 of them, now question is that, totally how many serial numbers he has at his hand. Because, after sometime we know it is a finite number so, after sometime it might repeat. So, till then, when we can do this, produce many, many numbers of this form. So, this is very easy to compute, I want you to think about this and work it out to yourself, may be that is better way.

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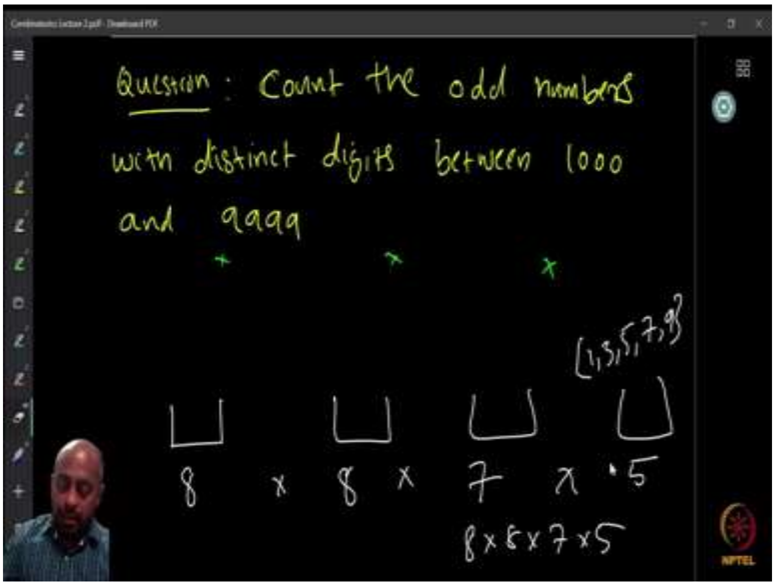

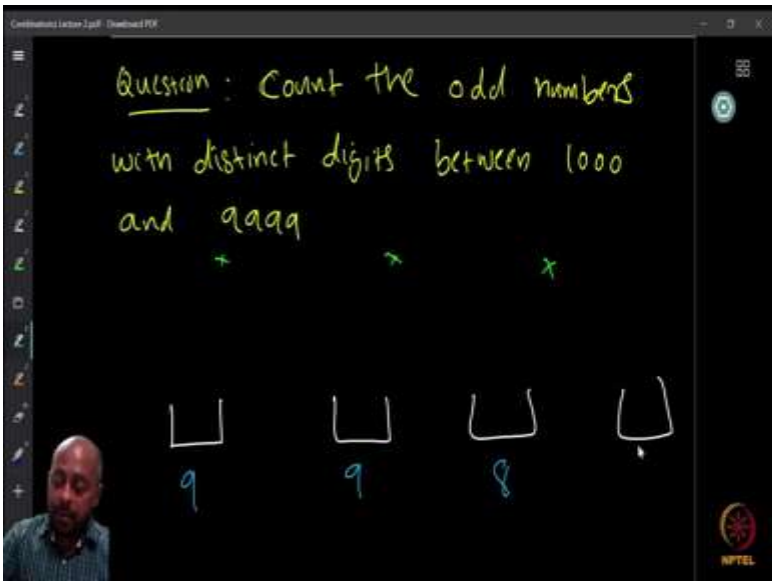

Now, another question. Count the odd numbers with distinct digits between 1000 and 9999. This is a question I want to discuss, because of some interesting facts. But, before I discuss, please stop the video and look at this question, think about it, and try to come up with your own answer. And see whether it matches with our computation. So, how are you going to count this? So, here is how I am going to do it. So, I know that I am looking at odd numbers with distinct digits between 1000 and 9999.

So, I know that all these numbers has exactly 4 digits. Now since, I know that the number is odd, I know that the last digit is going to be either 1, 3, 5 or 7 or 9 of course, yeah. So, the last digit has the possibilities 1, 3, 5, 7 or 9. So, there are 5 possible choices for the last digit 1, 3, 5, 7, 9. Now, the first digit what is the choice I have for the first digit? The first digit cannot be 0.

Now, it cannot be the digit I chose for the last digit because, it needs distinct digits. So, therefore I have 8 possibilities here. So, other than the first one I chose, I can choose any of the remaining 8. So, I have 1 to 9 out of this 9, I cannot choose the one I chose for the position here, so therefore I have remaining 8 choices. Now, to select the second digit, how many ways I can? Second digit can be any of the 0 to 9, the 10 digits but, I cannot have the first digit as well as the last digit.

So, therefore 8 choices are there for this, and then for the third digit I cannot use the first digit the second digit and the last digit. So, therefore out of the 10 I have 7 choices here. So, now I say that "Okay" there is $8 \times 8 \times 7 \times 5$, possible numbers with this property. Now, you might have noticed that I counted it in a very special order, like I counted the numbers for the last digit first, then I counted the numbers for the first digit, then I counted for the second and third digits.

Now, question is that can we do it in another order and of course you may be even to do it in another order. But, again you have to be very, very careful. Suppose we did in the other order, suppose we started with the order that as follows, so I start with let saying that, Okay, I am going to look at 4 digit numbers, I have these 4 possible functions, the first position can be any of the numbers 1 to 9 because, it cannot be 0. So, I have 9 choices here.

Then, the second position it can be 0, but it cannot be the number that I chose the first so, there is again 9 choices for this, and for the third position I have to avoid these two guys, so therefore I have 8 choices here. Now, when it comes to last position, what you will say? So, you have to say that, Okay, I have the possible numbers like 1 to 5; 1, 3, 5 etcetera, 9, and out of this I want to avoid the numbers that has appeared before but, the trouble is that, we do not know, whether how many odd numbers appeared here or how many even numbers appeared here?

I mean if we know the first two digits we have 1 and 7, then I cannot use one answer here. But, on the other hand, if it was like 2 or 4 and says that we are using, then I can use any of the 5 numbers. So, I do not know exactly how many possibilities I have for the last choice. This is because, the number of available decisions, is not independent of the previous decisions. So, we need to keep the independents and for that we might have to choose a particular order of computation.

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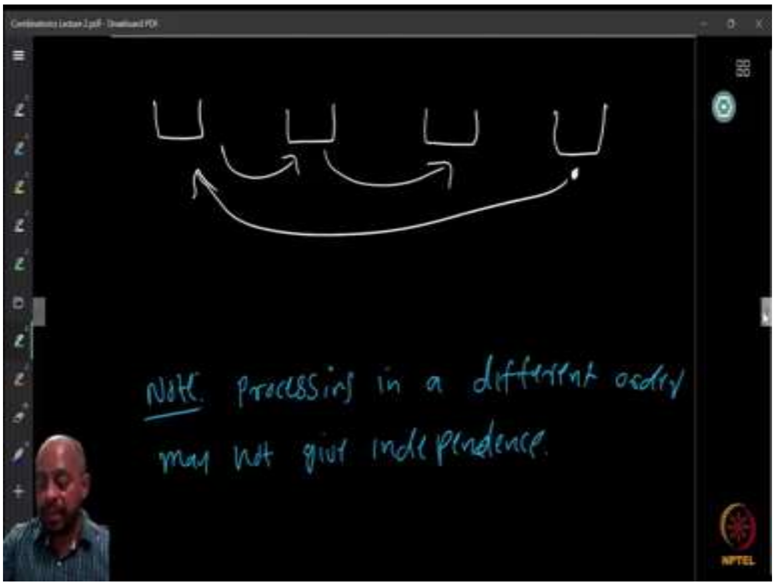

So, we computed in this order, we started with the last position, then we said that, Okay, I already chose the most restricted one, it must be odd and then I went to the first position, because, it cannot be 0, and it must be distinct from the last digit and then again I know that, all the things are possible here but except for the two which are used here.

So, I can freely choose so, processing in a different order may not give the independents always. So, for we have to be careful in deciding which order we are going to choose, so that the choices, at the time we make are independent with the previous, I mean the number of choices are independent of the previous choices.

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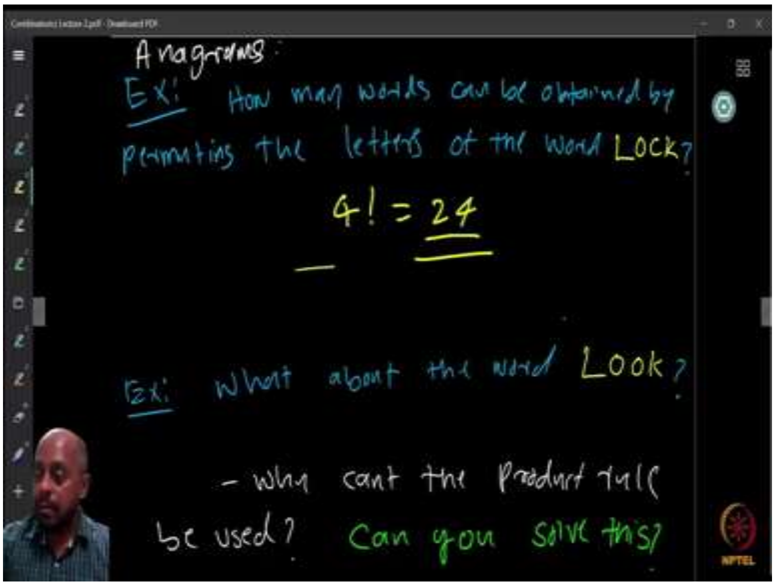

We are looking that at Anagrams. So, anagrams are basically permutations or basically words so, given a word you can permute the letters of the word in whichever way you want. Then, to find out different other words, they need not makes any sense but, they can be just words, not a word in the particular language but, they are words made from, basically strings made from permuting the letters.

So, how many words can be obtained by permuting the letters of the word LOCK. This is easy because, as we have already seen we have 4 letters here L O C and K and we are allowed to use only these 4 letters, the 4 letters can be ordered in any of the possible ways. So, we are looking permutations for anagrams, so there are 4 different letters and how many permutations are there? 4! .

So, we can immediately say that, Okay, there are 4! which is equal to 24 different ways we can make. So maybe has an exercise try to write down the 24 different permutations of LOCK and make sure that, there are exactly 24 of them. Now, I ask, okey, what about LOOK instead of LOCK? Can you use the same argument, or why cannot you use the, why we cannot use the product rule to find out the same thing? So, how do we use product rule here? Well, out of L O C K the first position can have all the 4 possible choices.

Second position once you chose the first one cannot be used in the second, therefore, I have 3 choices, then 2 choices and 1 choice. So, why it cannot be used to solve this in this case, L O O K? So, think about this, think about this and find out why we cannot use this and why we could use in the previous thing? So, we will stop this lecture at this time, then continue in the next lecture.