Combinatorics
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Species: Substitution and Derivative

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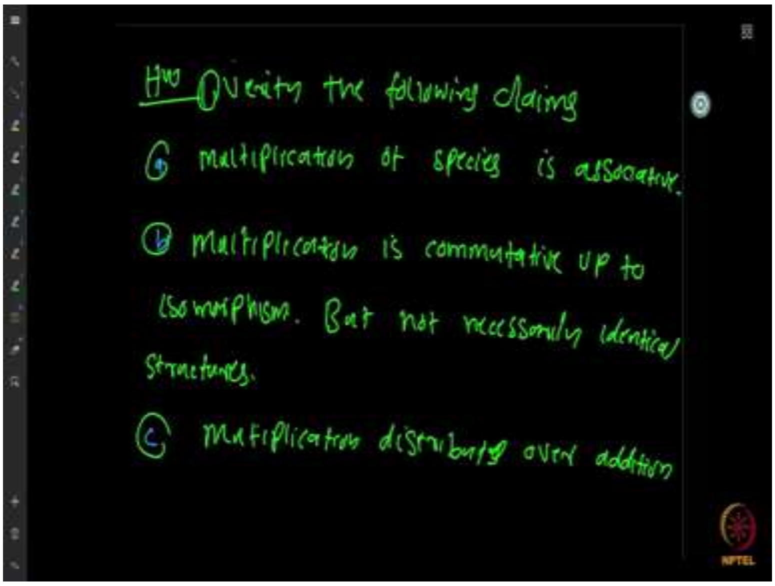

So, we continue the topic on species of combinatorial structures and start with the homework questions. The first question is to verify that the multiplication of species is associative. So if, A, B and C are species, then $A.(B.C) = (A.B).C$. So prove this or verify this then, and the second is that multiplication is commutative up to isomorphism.

In the sense that if I take two structures F and G then F.G may not be, identical to G.F. But, they are isomorphic. Then multiplication distributes over addition. So this is another homework that you want to do like, let us say (C . A) + B = (C. A) + (C . B).

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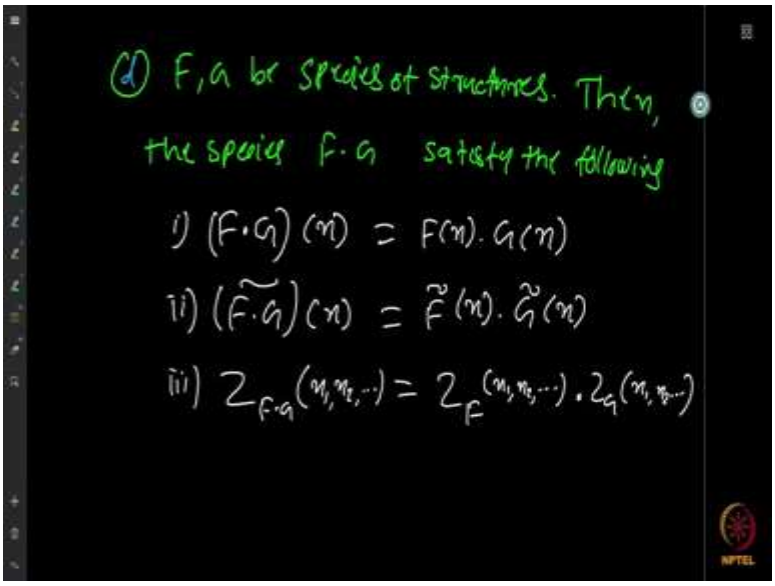

And the fourth part is to show the following. Suppose F and G are species of structures then the species F.G satisfy the following, that is the exponential generating function (F.G)(x), is the product of the exponential generating functions of F and G. That is (F.G)(x) = F(x).G(x) . Similarly for the ordinary generating function for the isomorphism type structures $(\widetilde{F.G})(x) = \tilde{F}(x).\tilde{G}(x)$. And the cycle index series $Z_{F.G}(x_1, x_2, \dots)$ is the product of the corresponding cycle index series for F and G. That is $Z_{F.G}(x_1, x_2, \dots) = Z_F(x_1, x_2, \dots).Z_G(x_1, x_2, \dots)$ So, these are the questions.

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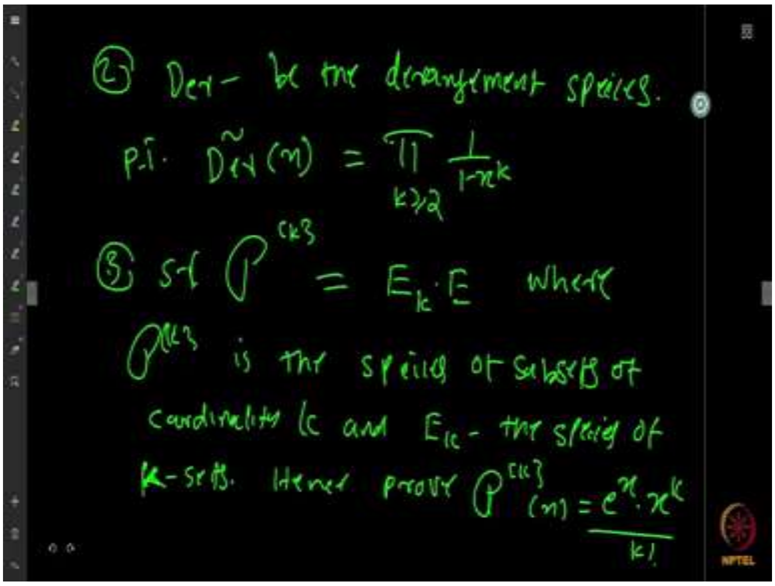

And then, there is couple of more questions. The second homework is to do the following. Suppose, by Der we denote the derangement species, we discussed this earlier, then show that the isomorphism type is counted by $\widetilde{Der}(x) = \prod_{k \geq 2} \frac{1}{1-x^k}$.

Third question is to show that, if $\mathcal{P}^{\{k\}}$ denotes the species of subsets of cardinality exactly k, and E_k denotes the species of k-elements sets, then show that $\mathcal{P}^{\{k\}} = E_k.E$. And hence prove that the exponential generating function, $\mathcal{P}^{\{k\}}(x) = \frac{e^x x^k}{k!}$

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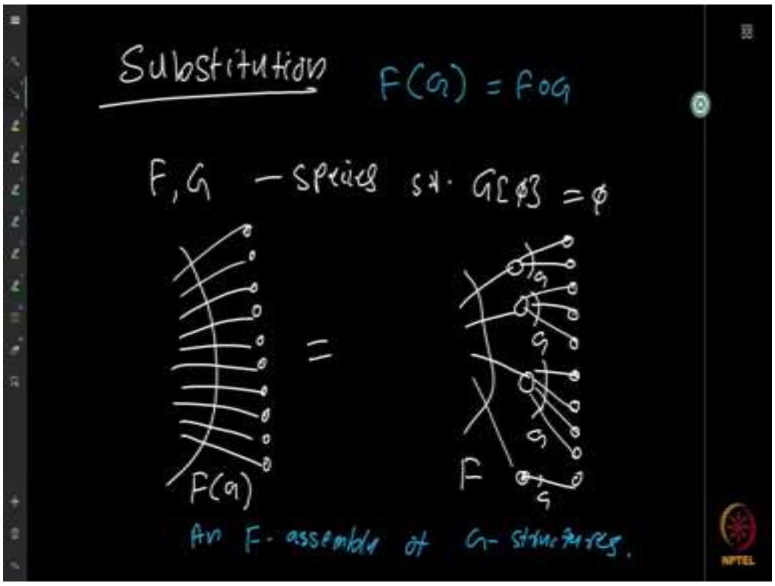

Now, we introduce a new operation called substitution. So suppose F and G are two species such that the species G has no structures on the empty set. That means, you cannot create structures of type G from the empty set. If this is true then, we can define $F(G)$ as $F \circ G$ as follows. So given a set U we define $F(G)$ on U as follows.

So, putting the structure $F(G)$ on the set U is actually taking the set U then partition the set U into non-empty parts, then for each part which is a small set, a subset you put a G-structure. So, you put G structures on each of the parts and then considering these G-structures as elements of a set. You put an F-structure on these elements, so on the G- structures. But basically it is an F assembly of G-structures. You are assembling G-structures into an F-structure. So, this is called the substitution.

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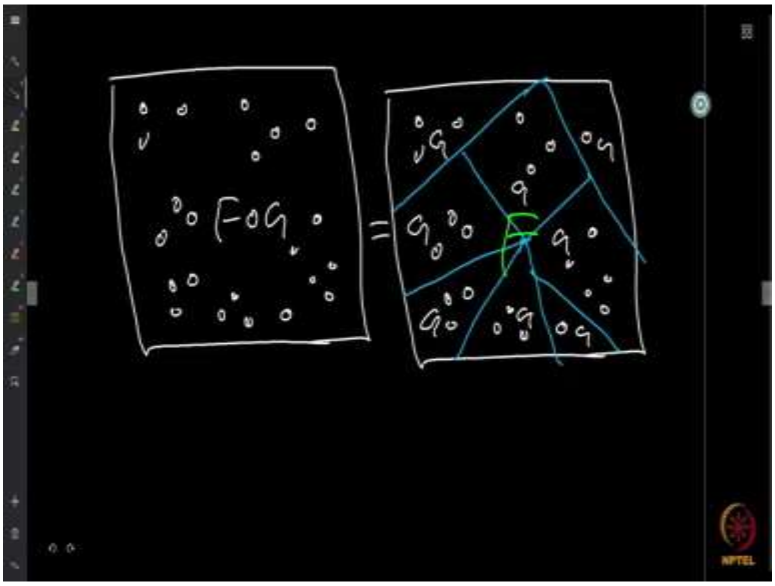

Now, another way to represent this is that like, you have this box representation of F∘ G, which is basically you partition the set into different parts then put G-structures on each and then on this G-structures you put an F-structure. So, the same thing.

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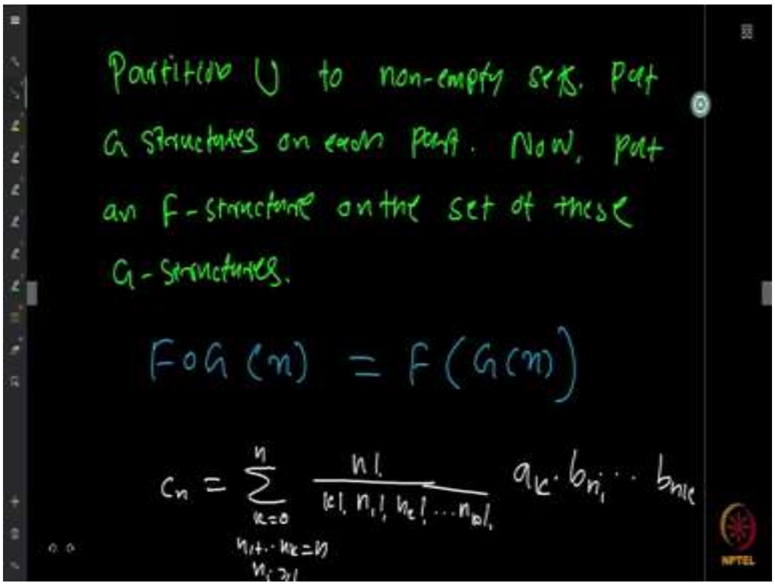

So, what we are saying is that partition U into non-empty sets, put G-structures on each part, now put an F-structure on the set of these G-structures, that is $F \circ G(x) = F(G(x))$. Now, it should be clear why we need that G of empty set to be empty. Because, if that is not the case we should be able to partition the set U to any number of any number of parts, because you can have empty sets as sets and on each of the empty set if you can put a G- structure.

Then, you do not have a bound on the number of $F \circ G$ structures that you can make because, you can make as many as many G structures as you want using the empty set and then once you have this you can use that to make the F-structure on this and the set cardinality keeps on increasing and therefore we do not want to allow that.

So, one can, verify that the number of $F \circ G$ structures on an n-element set U is given by this following formula.

$$c_n = \sum_{\substack{k=0 \\ n_1+\cdots+n_k \\ n_i \geq 1}}^{n} \frac{n!}{k!\, n_1!\, n_2! \ldots n_k!}\, a_k b_{n_1} \ldots b_{n_k}$$

So, this should be familiar to you because we discussed this when we are looking at the generating functions, composition of generating functions and it is essentially the same thing we are just looking at in terms of species now. And seeing that what is happening is precisely this.

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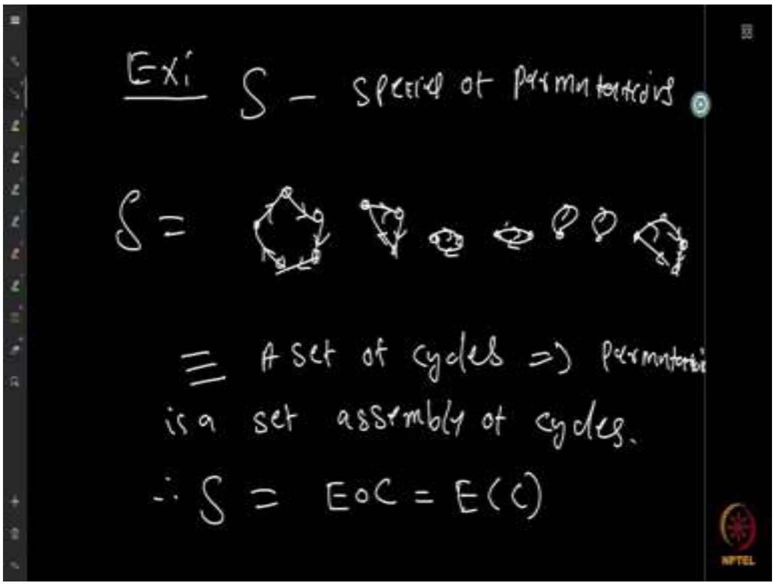

Now, as an example let us look at the species of permutations. We know that permutation is basically represented by cycles, so you can decompose the elements into cycles. Any permutation is basically a collection of cycles which is a set of cycles. So, you take the set, you partition, put cycle structure on each of them.

Put them together you get a permutation, because that precisely what a permutations is. And putting them together means that you are forming a set of things. So, basically it is a set assembly of cycles. So, therefore $S = E \circ C$, because it is set assembly of cycles, and which is equal to $E(C)$, the substitution. And if this is true, then it follows that.

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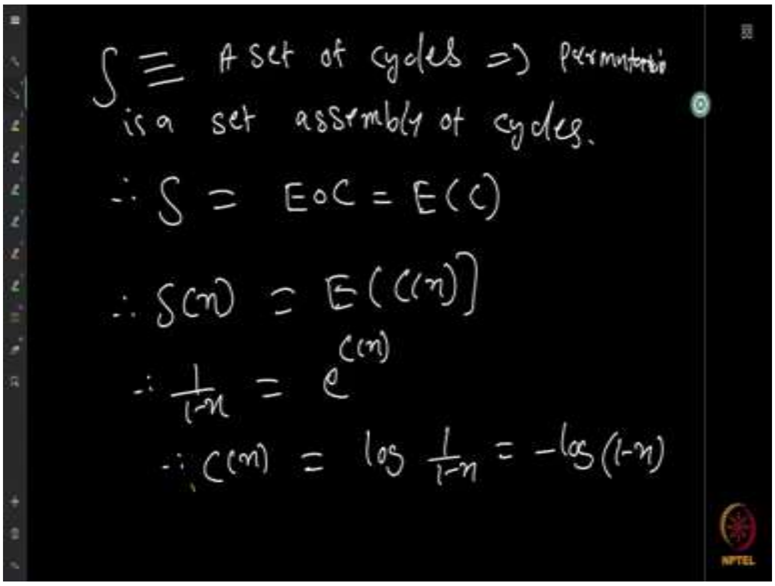

The exponential generating function $S(x) = E(C(x))$. So $\frac{1}{1-x} = e^{C(x)}$

And solving for $C(x)$ we get $C(x) = \log\frac{1}{1-x} = -\log(1-x)$.

So, we have a proof now. So, we can find out using just this observation that a permutation is basically a set of cycles, we can directly find the generating function for a cyclic permutation.

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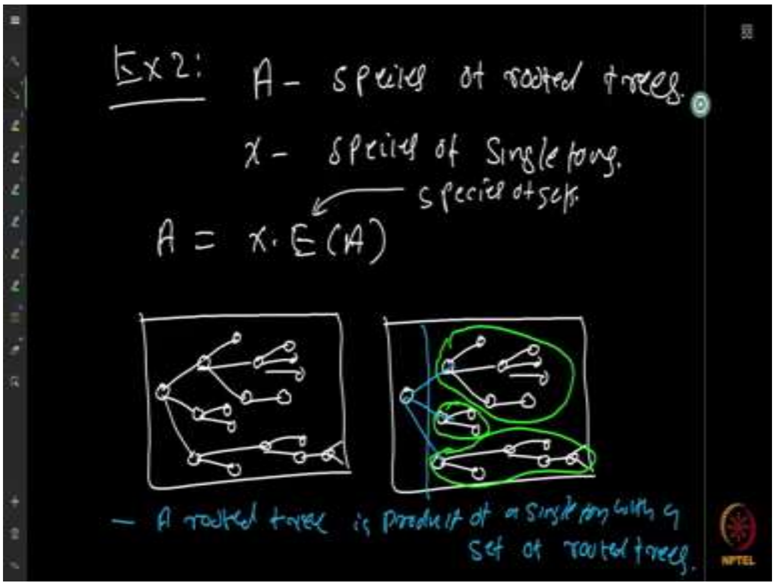

One more example. Let A be the species of rooted trees, and if you remember X is the species of singletons. Then the claim is that $A = X.E(A)$, it is the product of the singleton species into the composition of E and A, E is the species of sets. Why is this? Well, just look at this following figure that you have a rooted tree here, it is a rooted tree where this is the root let say.

So, you have a rooted tree and what is the rooted tree like, a rooted tree is basically, so you have a root, which is a special element, you have a root and then, once you remove the root what you get, if you remove this you will get several sub trees, several sub trees. And the sub tree is from a set, so it is a set of sub trees which is E(A), each of them is a root because this main root, that we removed is connected to these vertices, which makes them special.

They, are connected to it, therefore they are special vertices which means that they are roots, so therefore, we get a set of rooted trees which is E(A). But, now you have separated one vertex which is a special, which means that this vertex is a special vertex which can be thought of as neighbours of each of these roots, which makes it the new root. So, therefore a rooted tree, is the product of a singleton with a set of rooted trees.

So, therefore $A = X.E(A)$. Now, using things like Lagrange inversion because, the generating functions for E and X, using Lagrange inversion, we should be able to compute the generating function for A, which we are not covering because we are not looking at the Lagrange inversion or anything, in this course we will study that in a more advanced course.

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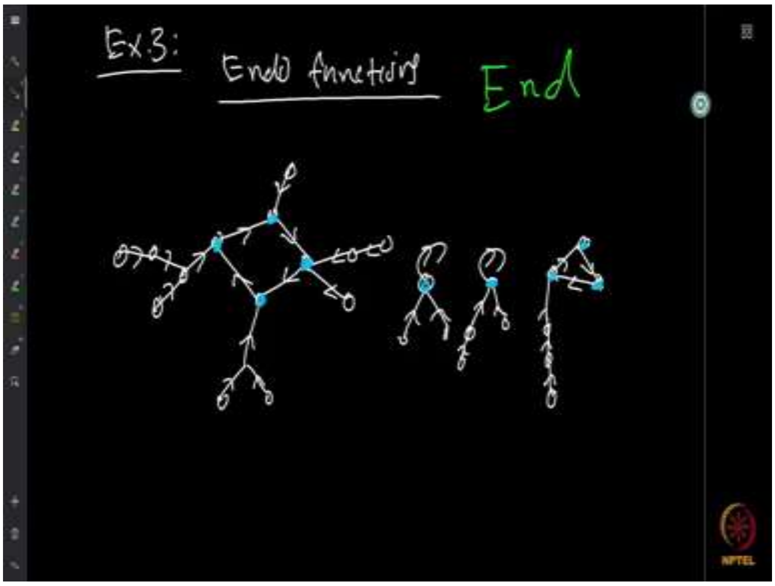

Now, another example is the species of Endo functions, we denote by End the species of Endo functions. If you recall from our earlier lectures on graphs, we came across Endo functions. Endo functions had this nice structure, that they have this kind of structure where, you have lots of rooted trees.

And these rooted trees were all going like this or we have this kind of graphs, we have this kind of graphs where all this direction is going in, because the graph of the Endo function of course has a property that every vertex has exactly one outgoing edge. Which implies that we cannot have more than one cycle, we have exactly one cycle because, all the element has some outgoing edge, therefore we will see that there will be cycles in each of the components.

And you will get something like this, now what is this, can you see it as some kind of operation on species? That is the question. So, the species of Endo functions can you think of it as a combination of species as a product or composition or something like that. What can you see here? So it turns out that if you look at each of these, this one is a rooted tree.

Because, all the arrows are going to this vertex from this part, similarly each vertex in the cycle has a property, that it is the root of some rooted tree. Each vertex has its property, even this vertex we have these two trees, it is the root of this tree and this tree. Or if you want, you can see it as this tree, and then you have this one, then here you have a tree with just one vertex, which is the root.

Here is another tree with one vertex, so, what you have is that, you take a collection of rooted trees, take the roots and put the roots into cycle forms like, group them together and put some cycle structure on that. What is this, putting them together and putting cycle structure on them, which is basically saying that I am making a permutation, I am looking at the permutations of the roots.

So, therefore it is basically a permutation of rooted trees. So, what we are saying is that we have permutations of rooted trees and which is actually equal to the Endo function.

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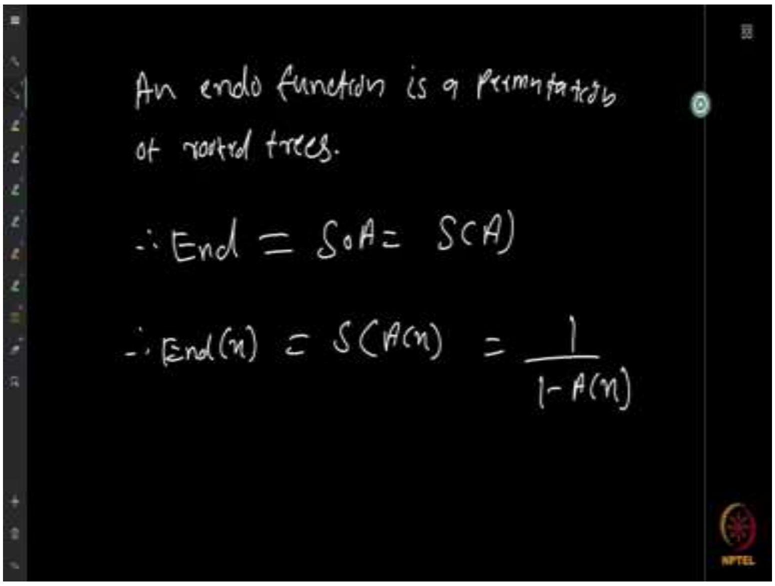

So, therefore we get Endo function is the composition of S and A. So

$$End = S \circ A = S(A)$$

$$End(x) = S\big(A(x)\big) = \frac{1}{1 - A(x)}$$

So, if you find out what is the A(x) from the earlier part using the Lagrange inversion, we can also find End(x) using this.

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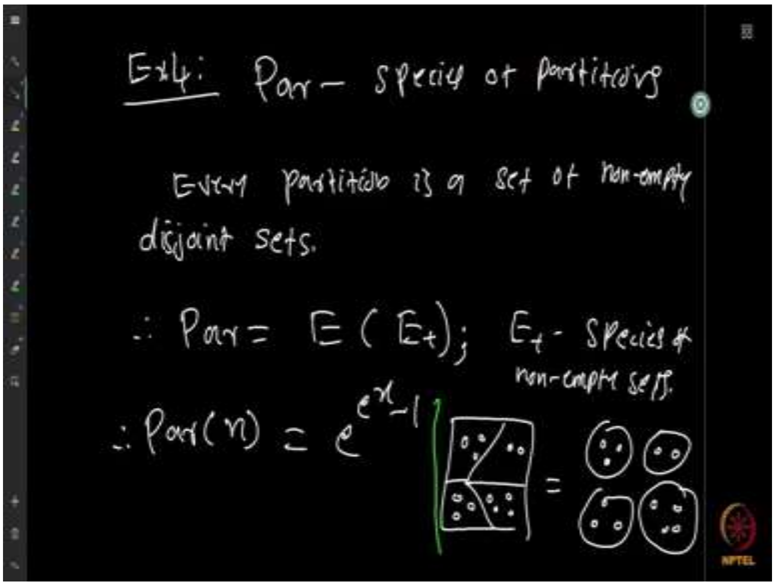

Look at the species of partitions and what is the partition, every partition is basically a set of non-empty disjoint sets, so given a set you are going to partition into non-empty disjoint sets, that is why it is called a partition I mean you cannot have two of the parts intersecting and you do not want to have empty parts in the partition. So, therefore for us the partition is a set of non-empty disjoint sets.

Now since, the sets are going to be non-empty sets. We look at the species of non-empty set which is E_+, E is the species of sets. E_+ is the species of non-empty sets, you just remove the first term or whatever is corresponding. Then, partition species is actually a set assembly of non-empty sets, because it does not matter what order is here. So basically a set of like when you are partitioning you are saying that “Okay”, you have this parts U_1 , U_2, U_3, U_4, U_5. So, whichever order you put them it is going to be a partition.

Now, if you remove just the first element from E(x), it is basically, the first term is going to be one in.. $e^0 = 1$. So, the generating function for E is e^x, so therefore it is e raised to x but, for x equal to 0, it is basically 1, I mean, so, not x equals 0 for, for the sets on empty sets, there is there is only one empty set. So, therefore E plus minus the case where you are looking at the empty set is basically, $e^x - 1$.

That is generating function for E_+ and generating function for E is e^x , so therefore we get $Par(x) = e^{e^x - 1}$, which is the generating function for the partition species. So, this thing shows us, that the method of species can be quite convenient, to come up with this kind of generating functions where if somebody is asked, to just find out the, the generating function for the species of partitions, it may not be as easy as this.

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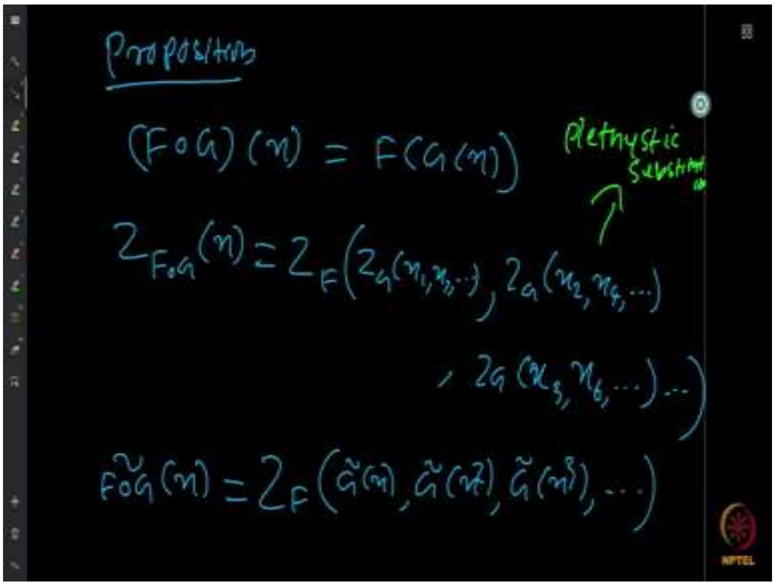

Now, here is a proposition which I am not going to prove this can be done in a more advanced course. That if you take the substitution, of F and G then $(F \circ G)(x) = F\big(G(x)\big)$.

So, we are substituting G(x) for x in, F(x). On the other hand, if you look at the cycle index series, the cycle index series is obtained as follows,

$$Z_{F\circ G}(x) = Z_F(Z_G(x_1, x_2, \dots),\ Z_G(x_2, x_4, \dots), Z_G(x_3, x_6, \dots), \dots)$$

Now this kind of substitution is called a Plethystic substitution, it has a name again we will not go into any further details on this, but it should be clear what we are doing here. And the claim is that this is precisely the cycle index series for the substituted species F ∘ G.

And from this one can immediately reduce that,

$$\widetilde{F \circ G}(x) = Z_F(\tilde{G}(x), \tilde{G}(x^2), \tilde{G}(x^3), ..)$$

So, these are something, if you are interested you can try to see, why, but it requires a little more work and we do not want to do it in this course.

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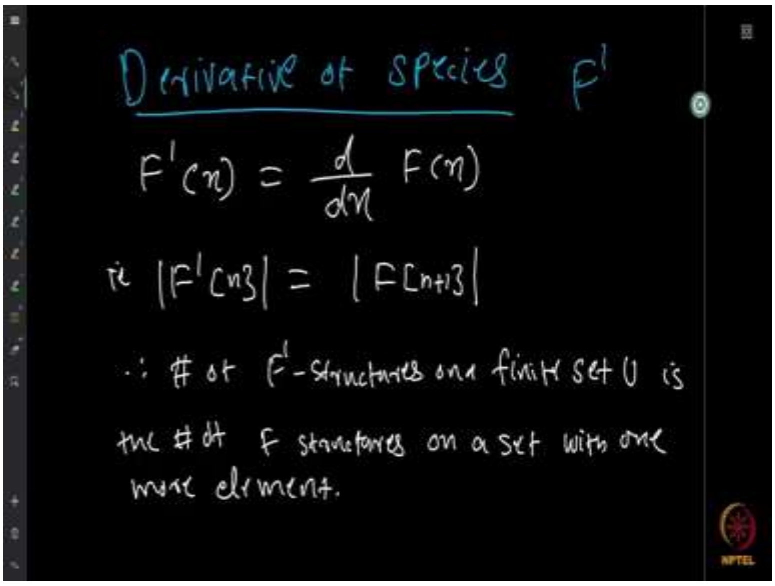

Now, let us talk about the derivative of a species. So given a species F, I want to look at the derivative species F′. Now, what we know about the derivative of the generating function, so we start with the exponential generating function, look at the derivative of that.

$$F'(x) = \frac{d}{dx} F(x)$$

So, what I want is that, I want the generating function for the derivative specie F′, to match with the derivative of the generating function of the species F. Now, from this equation, we know immediately that $|F'[n]| = |F[n+1]|$.

Now, therefore what we want is that, we want to make sure that this property holds then only the generating function here, will satisfy this equation. We want to make sure that, this number, game is perfectly matched. So, that is the number of F′ structures on a finite set U is the number of F structures on a set with one more element because we are looking at the set with one additional element.

So, the derived species can be thought of, so derived species on a set U can be thought of as the species F on a set where we have one additional element, which is a special element but then we do not have this element in our set, so therefore you put an F-structure on this set and then remove this. Or forget about this element, then you look at only the remaining element and what kind of structure is that forms.

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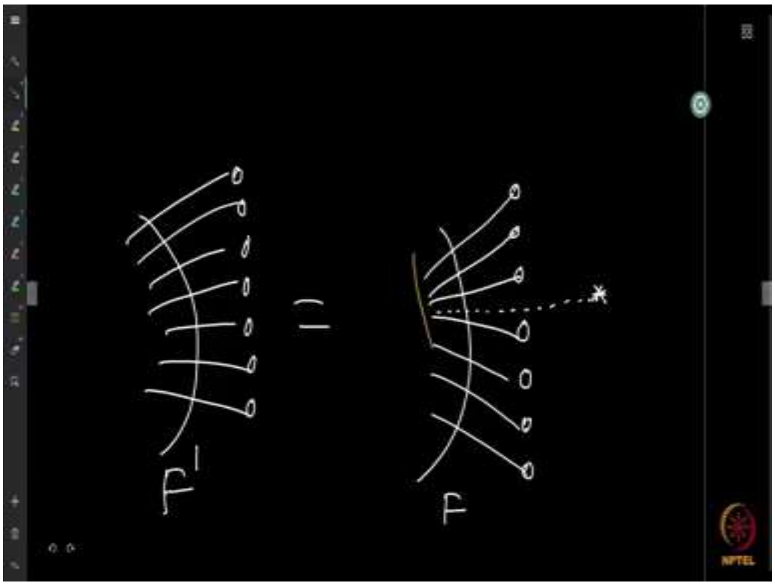

So, let us look at the, picture. The F′ structure, on a set is basically an F-structure on the set together with a new element, where the new element is discarded after making the structure.

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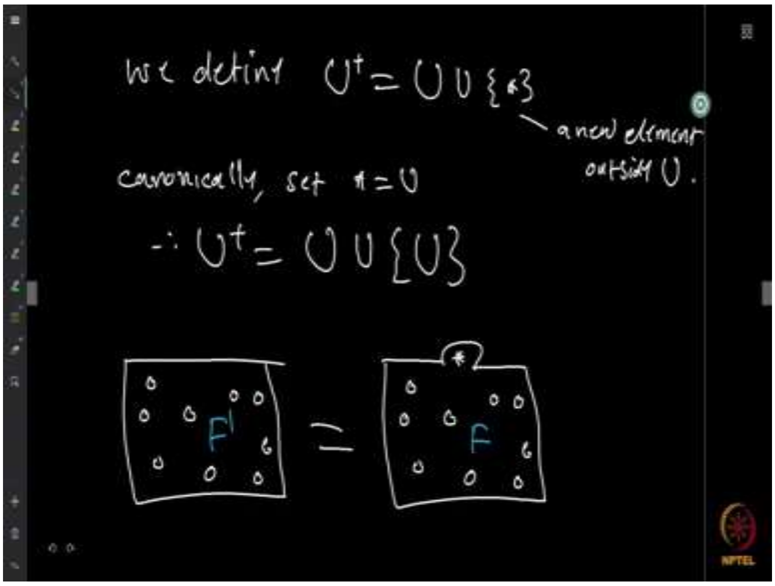

So, we define $U^{+} = U \cup \{*\}$ where * is a new element that is not in U.

So another picture representation is the following that, an F′ structure on a set U is basically, an F structure on the set U with a new element where the new element is sitting outside. You put that structure; forget about this thing, whatever it is doing there.

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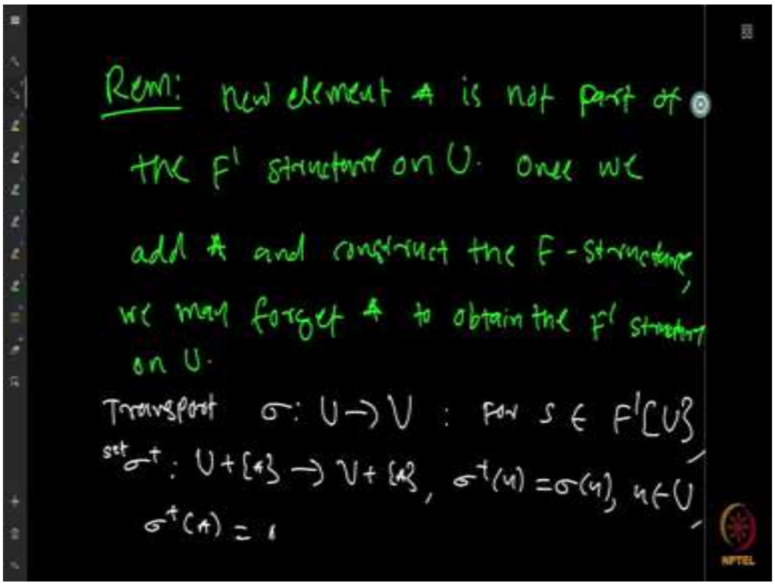

So, let us look at an example before that maybe I will just make a remark, that the new element is actually not part of the F′ structure on U, because it is never part of U. Once we add * and construct F structure, we can forget a * to obtain the structure. Now, one question that may arise is that, how do you define the transport of structures, so σ is mapped from U to V.

Now, we have to make sure that, this σ is not affected when we extend it. So, we are going to extend σ to, let set say σ^+ . Where, σ^+ goes from U + {*} to V + {*}, you add * to each U and V such that $\sigma^+(u) = \sigma(u),\ \sigma^+(*) = *$.

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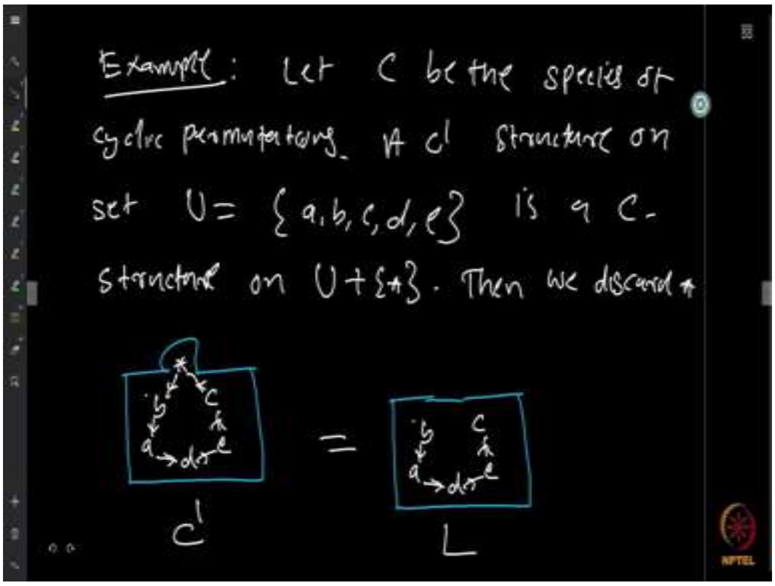

Now, here is a nice example. Let C be the species of cyclic permutations, now we want to look at the derivative C′. C′ structure on U let say U is the set {a, b, c, d, e} is as we said is a C structure on U +{*}. So, basically we are defining C′ as a cycle structure on {a, b, c, d, e, * }. So, you put this structure, but now forget about star which means that we are going to get, a linear order on the elements b a d e and c forms stays in a linear order. Because, it had originally a cyclic order, where we removed the star, now, making it just a linear order. So, the derivative of C′ is the species L of linear orders.

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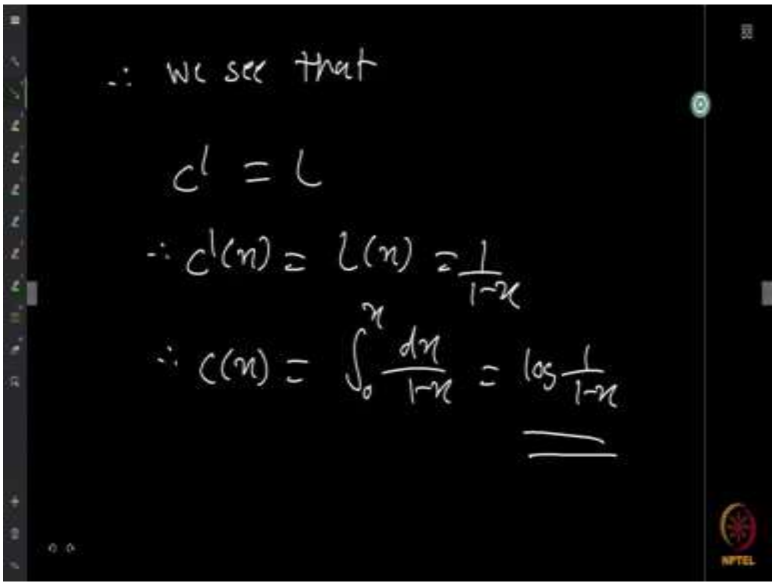

So, from this we can see that $C'(x) = L(x) = \frac{1}{1-x}$

Therefor $C(x) = \int_0^x \frac{dx}{1-x} = \log\frac{1}{1-x}$

Again, we obtain the same result that we proved earlier using another method, using the derivative.

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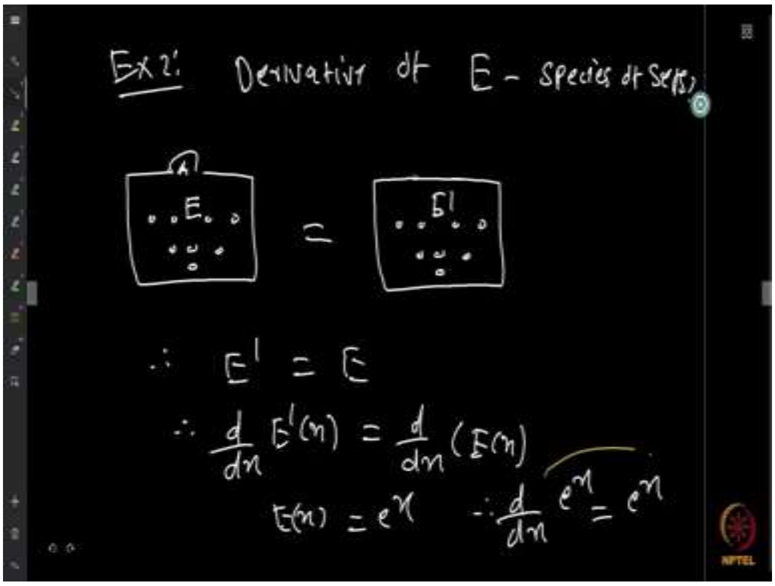

One more example, so look at the derivative of the species of sets, E is the species of sets, so you are making the set, taking the set U putting an E structure on the set U + {*}. Then you remove the element *. You get the structure E′, but it is again just a set without the *, but it is the set structure on U itself. So, therefore we see that, E′ is actually equal to E itself. So

$$\frac{d}{dx}E'(x) = \frac{d}{dx}E(x)$$

We have $E(x) = e^x$. So we get $\frac{d}{dx}e^x = e^x$

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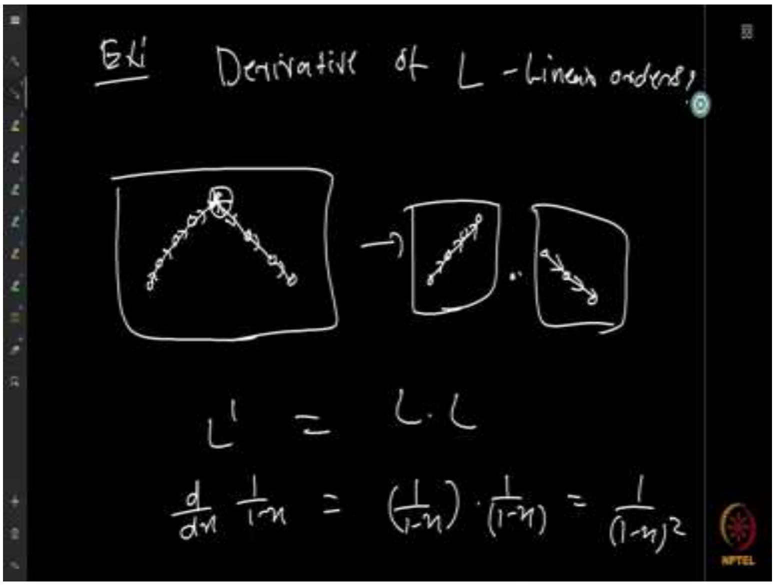

Now, derivative of the species of linear orders, what is that, so you take the linear order species. So therefore you put, you take the set U add a new element star. Then you put a linear order on that, once you put the linear order you remove star, you will get two linear orders, it is basically like partitioning the set into two parts, putting an l structure on this and putting an l structure on this.

Therefore, it is basically the product L . L. Therefore, $L' = L.L$ and this should be true, because $\frac{d}{dx}\frac{1}{1-x} = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$.

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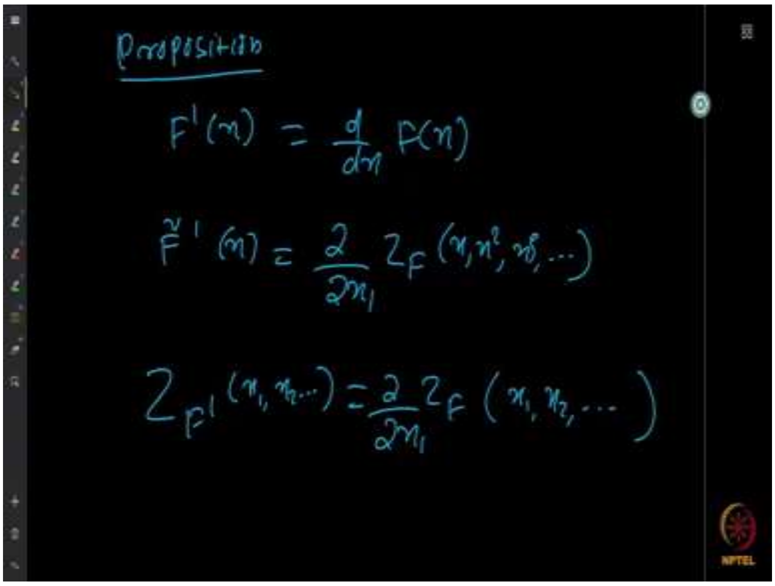

Again a proposition with out proof. The, derivative satisfies the following identities.

$$F'(x) = \frac{d}{dx}F(x)$$

$$\tilde{F}'(x) = \frac{\partial}{\partial x_1}Z_F(x, x^2, x^3, \dots)$$

Then the derivative has the following cycle index series

$$Z_{F'}(x, x^2, x^3, \dots) = \frac{\partial}{\partial x_1}Z_F(x_1, x_2, \dots)$$

So, from this if you just substitute with x, x^2 you will get this one naturally. So, again this we are not going to prove, but just keep it in mind that, this can be rigorously proved.

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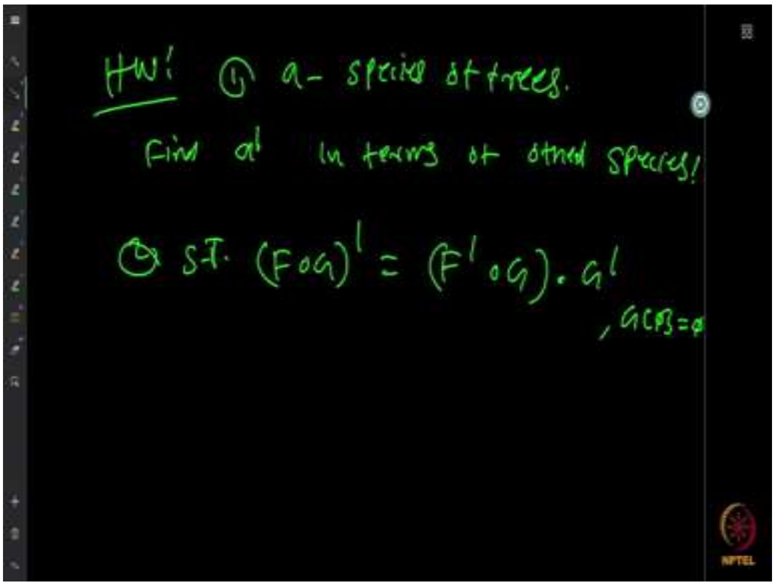

As a homework, prove the following. Take a to be the species of trees, then find the derivative a′ in terms of other species. Can you, represent the derivative of the species of trees in terms of some other species.

Show that $(F \circ G)' = (F' \circ G) \circ G'$. Where, of course G of empty is equal to empty.