Combinatorics
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Associated Series and Product of Species

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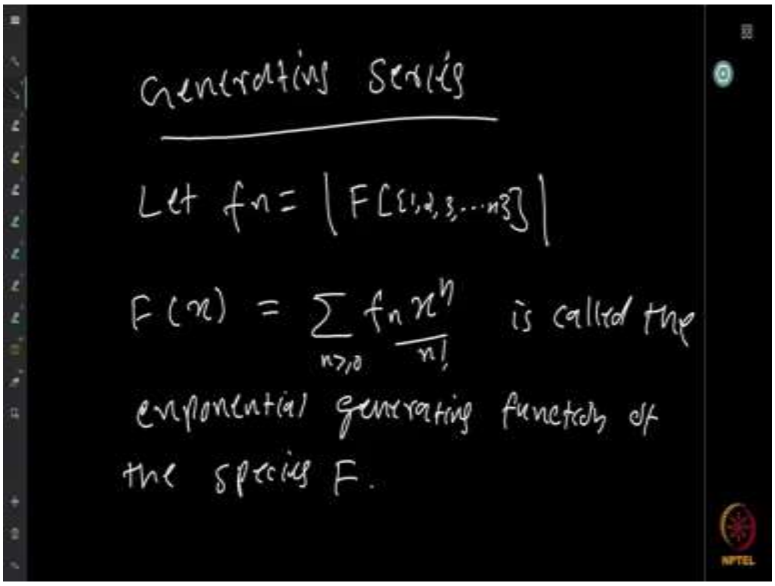

Now, let me define what are generating series. So, we can associate several series with species and we want to look at these series in detail. So, first we are going to define what is the exponential generating function of the species F, what is that? Given a species F we mentioned that, the cardinality of F[U], for any finite set U only depends on the cardinality of U.

So, therefore I can just talk about a fixed set {1, 2, 3, …, n}. The cardinality of F[U], for any n-elements set U is actually equal to the cardinality of F[{1, 2, 3, …, n}]. Because, that is also another n-element set, and this cardinality, the number of structures of type F on n-element set is denoted by f_n.

Now, to form generating series, we can basically multiply by x^n and then, depending on how f_n grows we will either normalize by $n!$ or something like that. So, here we are going to do that so we define $F(x) = \sum_{n\geq 0} \frac{f_n x^n}{n!}$ and this as we have seen in the case of the generating functions, is called the exponential generating function of the species F.

So, this is basically count f_n, and then if you want to recover f_n you have to just find out the coefficient of $\frac{x^n}{n!}$ in F(x).

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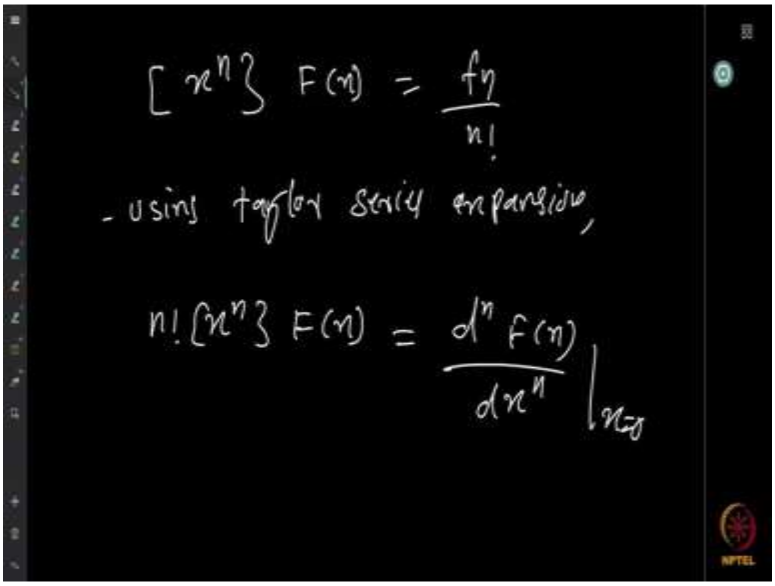

Using Taylor series expansion, whenever you are given the function F(x), you can use Taylor series expansion and then multiply by $n!$. Look at the coefficient of x^n in F(x) and by Taylor series it is $\frac{d^nF(x)}{dx^n}|_{x=0}$. So, if you want, you can say Maclaurin series, it does not matter. So, evaluate at x = 0 and that is going to give you f_n.

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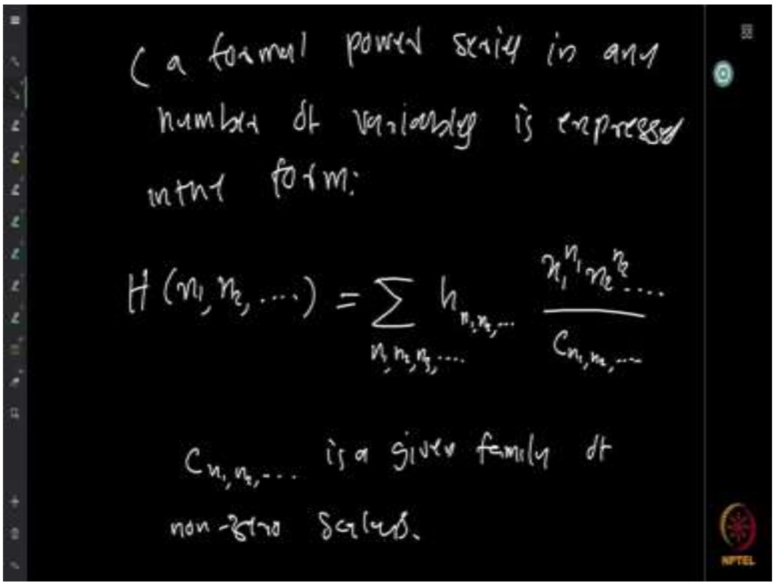

Now, a formal power series in any number of variables is expressed, so you can talk about formal power series in one variable two variables or infinitely many variables. So, if you have infinitely many variables you can just define it as follows.

$$H(x_1, x_2, \dots) = \sum_{n_1, n_2, n_3, \dots} h_{n_1, n_2, \dots} \frac{x_1^{n_1} x_2^{n_2}}{C_{n_1, n_2, \dots}}$$

Where $C_{n_1, n_2, \dots}$ is a given family of non-zero scalar. In our case it was n! here, it could be anything else here, so that kind of thing is basically a formal power series and you can generalize it to any number of variables as you please.

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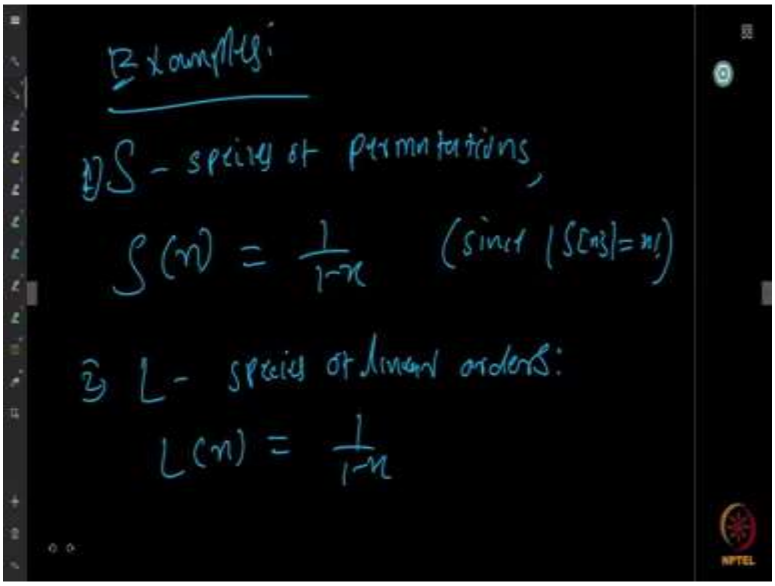

Now, here are some examples of species again, so S is the species of permutations and the generating function $S(x) = \frac{1}{1-x}$, because we know that the number of permutations on an n-element set is basically n!. So, because it is we have n! of them, $\sum \frac{n!}{n!} = \sum 1$

Then, if you talk about the species of linear orders, so we know that there is also n! many linear order, so $L(x) = \frac{1}{1-x}$, where L(x) is the exponential generating function for the species of linear orders.

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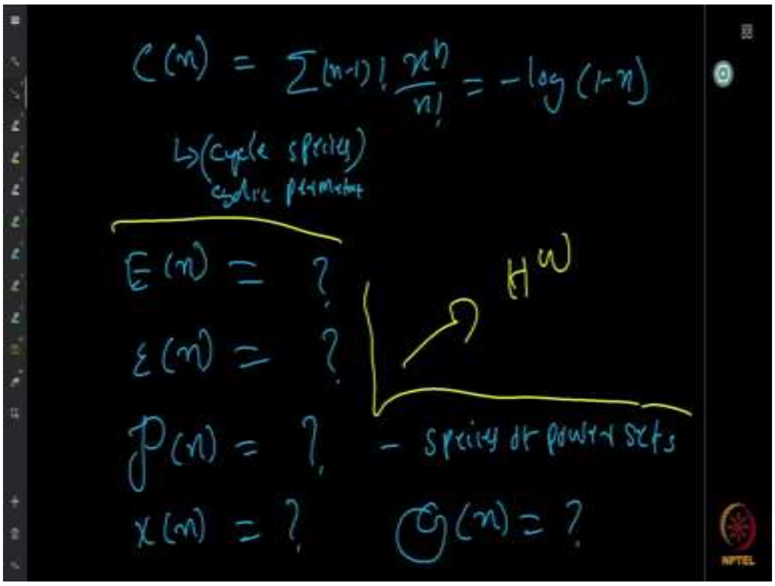

C(x) is the cycle species, cyclic permutations, then what is that it is basically summation, we know that there is (n-1)! of them, so therefore you will get $C(x) = \sum \frac{(n-1)!x^n}{n!} = -\log(1-x)$.

Similarly, can you find out the, the exponential generating function for the species E of sets, the species ϵ of elements, species P of power sets, and species X of singletons and species G of graph simple graph. So, take it as an exercise and try to find out what are the generating functions or generating series.

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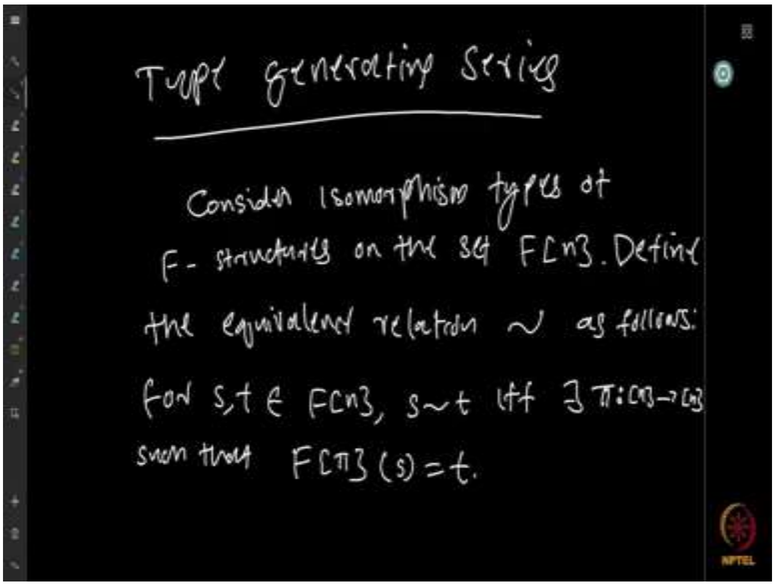

Now, so we were talking about so far the, counting the objects of a particular species, that you can make on a set U with let say n-elements. Now, there could be several objects when I use the set, the set relabeling, there could be several of them that may be isomorphic. So, there will be several isomorphic structures. When I talk about graphs on the set {1, 2, 3, 4} you will have several graphs but several of them could be isomorphic.

Now, I do not want to let us say that count all these isomorphic things separately, I want to just find out how many different inequivalent structures are there, so for that I can consider the isomorphism class, isomorphism types of F-structures, species F-structures, on the set let us say F[n]. So, take any n-element set, [n] here, look at F[n] and then we can look at the isomorphism types of this, this type on F[n], on this set.

Now, let me define an equivalence relation as follows. The equivalence relation $\sim$ it says that, for any two structures s and t in F[n], s $\sim$ t if and only if I can find a bijection $\pi: [n] \to [n]$ to such that $F[\pi](s) = t$. So, if that is true, so if you can find such a bijection π, then we can say that s and t are basically equivalent then they are isomorphic that is the idea.

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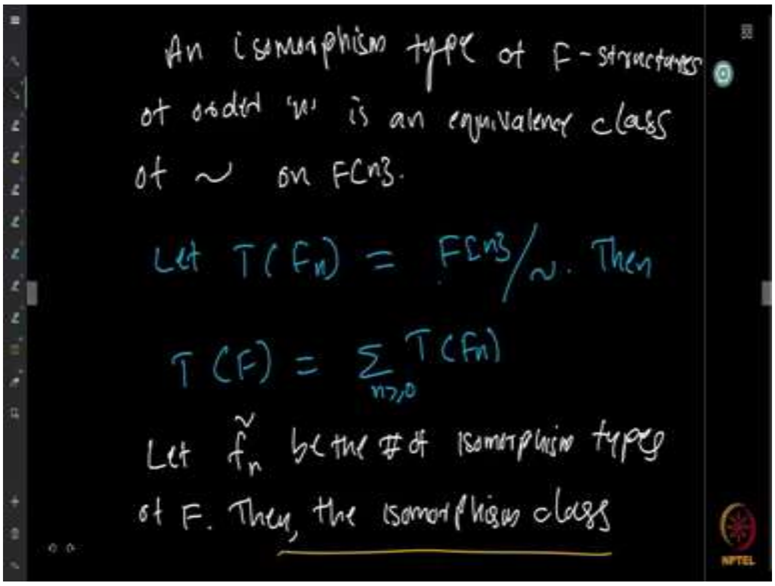

Now, an isomorphism type of F-structures of a fixed order let us say an n-element set is an equivalence class of the equivalence relation $\sim$ on F[n]. So, the equivalence classes are called isomorphism types, so it is basically

$T(F_n) = F[n] /\sim$. Then $T(F) = \sum_{n\geq 0} T(F_n)$

So, for all possible structures we have T(F). So for n greater than or equal to 0, you take $F_0, F_1 F_2$ etcetera which are structures of type F on an n-element set and then, you are looking at the, the equivalence classes of these structures. Now, let me denote by small $\widetilde{f_n}$ as the number of isomorphism types of F, how many distinct type of structures are there.

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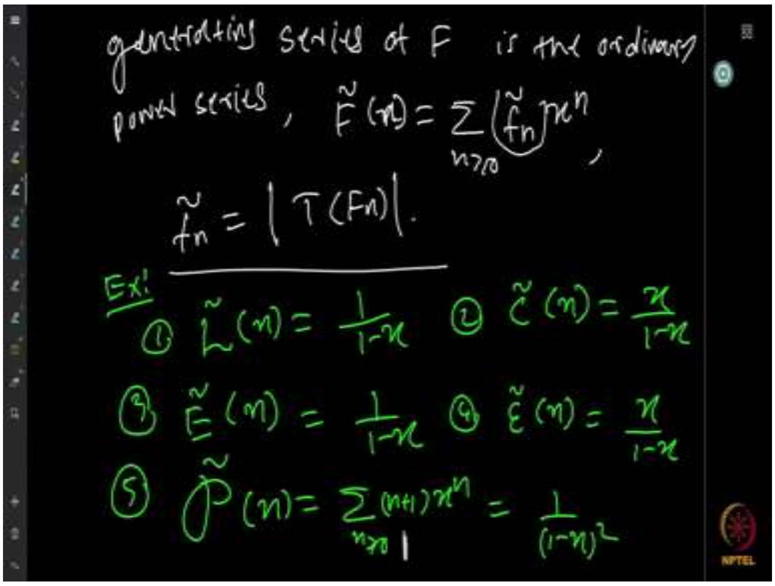

Then the isomorphism class generating series which is called $\widetilde{F_n}$ is the ordinary power series

$$\widetilde{F_n} = \sum_{n \geq 0} \widetilde{f_n} x^n, \quad \widetilde{f_n} = |T(F_n)|$$

This the ordinary power series not the exponential power series. Because, when you are talking about isomorphism types, we are just counting the class, how many classes are there.

Usually the numbers grow much smaller, so, we do not want to use exponential generating series. Then we have the ordinary power. For example, .

Now you are looking at linear orders, $\tilde{L}(x) = \frac{1}{1-x}$, because, when we are arranging objects you have n! many ways to do that but, if you consider all of them to be identical, the labeling is not important then whichever order that you are going to put in a line it is, exactly the same line.

So, there is only one way to do that and because of this summation x^n and reduces to $\frac{1}{1-x}$. Therefore, we get that the isomorphism type generating function for the linear orders is $\frac{1}{1-x}$. Now, if you look at the cycle permutations, $\tilde{C}(x)$, can you show that, $\tilde{C}(x) = \frac{x}{1-x}$.

So, it basically is a shift by 1 for $\tilde{L}(x)$, and can you connect this and see why, why this is this. Similarly, if you look at the $\tilde{E}(x)$, E is the species of sets, so what is $\tilde{E}(x)$? So I am putting a set structure on U. So, it does not matter then there is only one so therefore, it does not matter whether it is labeled or not.

So, you will still get a set with just one element and the count is going to be exactly 1. So it will also be $\frac{1}{1-x}$. What is the species of elements, again can you show that $\tilde{\epsilon}(x) = \frac{x}{1-x}$. If P is the species of power set what is $\tilde{P}(x)$? Show that, $\tilde{P}(x) = \frac{1}{(1-x)^2}$.

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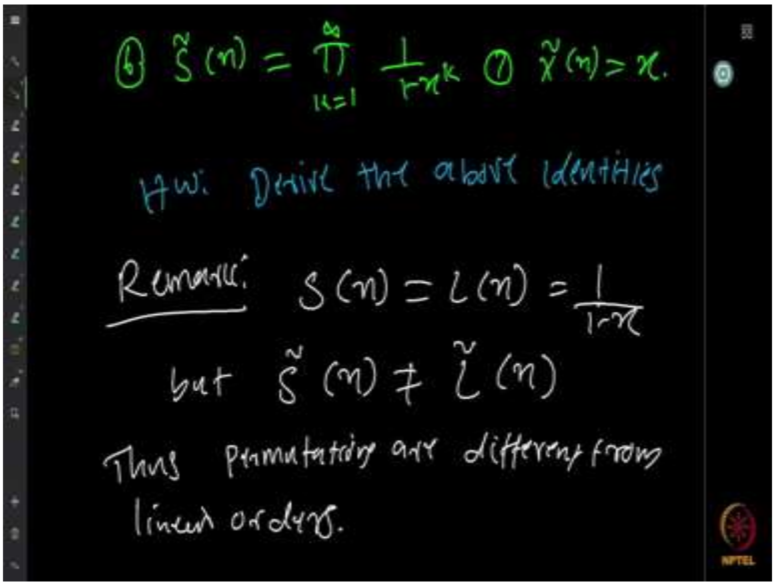

Now, even more interesting is that, $\tilde{S}(x)$, what was S, S is the species of permutations and we can show that $\tilde{S}(x) = \prod_{k=1}^{\infty} \frac{1}{(1-x)^k}$ .Now, what you can see from this is that $\tilde{S}$ was this function, on the other hand $\tilde{L}(x) = \frac{1}{1-x}$.

Now, when we are looking at linear orders and permutations, we can all we often have this tendency to see them as kind of the same thing. We are permuting objects and we are arranging them in n! different ways.

Since, there is a bijection between these two, we think that, they are kind of the same thing. Often, there is a mistake came that linear orders are the same as permutation. on the other hand. This tells you that, when you look at the unlabeled structures, the isomorphism classes, we see that, they are in the different objects because $\tilde{S}(x)$ is actually very different from $\tilde{L}(x)$.

Then, for the singletons $\tilde{S}(x) = x$, these things, you can try to derive this identities, as homework. Now, some remarks. So, this is something that I already remarked, that

$S(x) = L(x) = \frac{1}{1-x}$ but $\tilde{S}(x) \neq \tilde{L}(x)$. Thus, permutations are actually different from linear orders.

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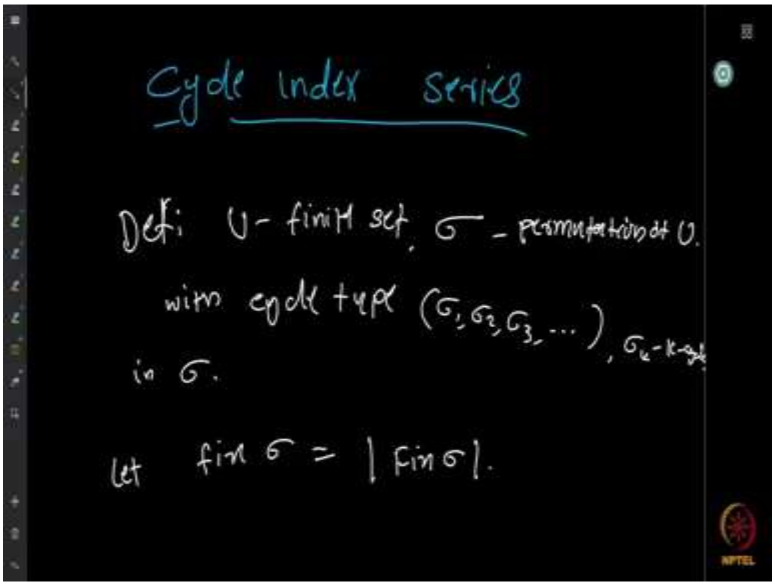

Now, let us look at the cycle index series. So maybe, I should just mention one more thing, see S(x) and L(x) was $\frac{1}{1-x}$, but you see if you look at L(x), L(x) was $\frac{1}{1-x}$ then $\tilde{L}(x)$ was also $\frac{1}{1-x}$. This should not confuse you.

Here, L(x) is 1 by $\frac{1}{1-x}$. Because it was the exponential generating function, and we were actually dividing, by n!. On the other hand here we have the ordinary generating function, where we are not dividing by n! therefore, the count is 1. And there it was n factorial we made it 1 by dividing by n!.

So, these two are indeed very different. L(x) and $\tilde{L}(x)$, one is exponential generating function, which is $\frac{1}{1-x}$, by chance and $\tilde{L}(x)$ is also $\frac{1}{1-x}$. But it is the ordinary generating function. Now, because these are different we see that, permutations are different from linear orders as species.

Now, finally I want to introduce one more series associated with the species which is called cycle index series. So, to define that, let us first define and look at the following. So U is a finite set and σ is a permutation of U, where σ the permutation has the cycle type so, we studied this, in our class earlier cycle type $(\sigma_1, \sigma_2, \ldots)$, where σ_k in general is the number of k-cycles in σ.

Now, if you are given an n-element set U then we will see that this the cycle type will all be 0 after the nth term. σ_{n+1} onwards, will all be 0. This is something immediately clear there cannot be a cycle of larger length. Now we denote by $fix\ \sigma = |Fix\ \sigma\ |$ where Fix σ as we saw earlier again, it was the, the number of elements fixed by the permutation Sigma, which is actually equal to σ_1.

Basically, the number of unit cycles, singleton, single cycles in the permutation, the representation of permutation of cycles.

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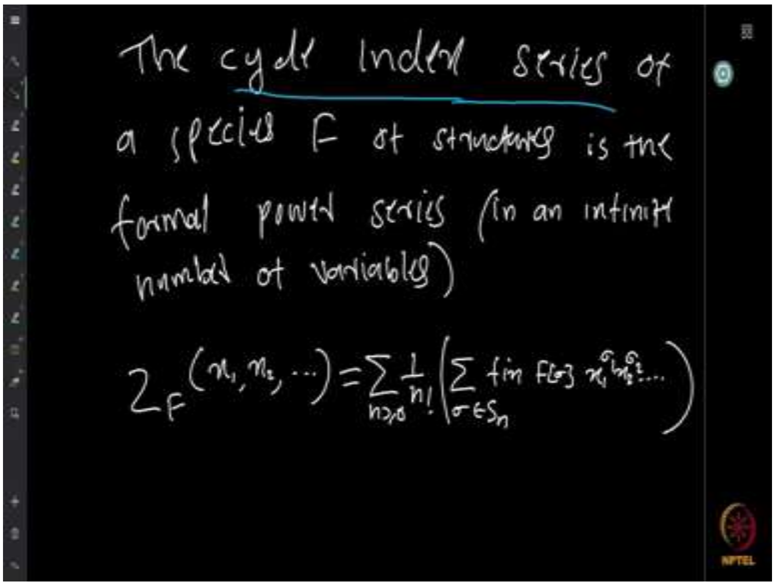

Now, the cycle index series of species F of structures is the formal power series in an infinite number of variables

$$Z_F(x_1, x_2, \dots) = \sum_{n \geq 0} \frac{1}{n!} (\sum_{\sigma \in S_n} fix \ \ F[\sigma] \, x_1^{\sigma_1} \ x_2^{\sigma_2} \dots)$$

So, it should be pretty clear where, these objects are coming from what this objects mean from our short excursion we did to poly's theory. So we do not go into the details further. Think about this and try to understand if you have any questions come back to me.

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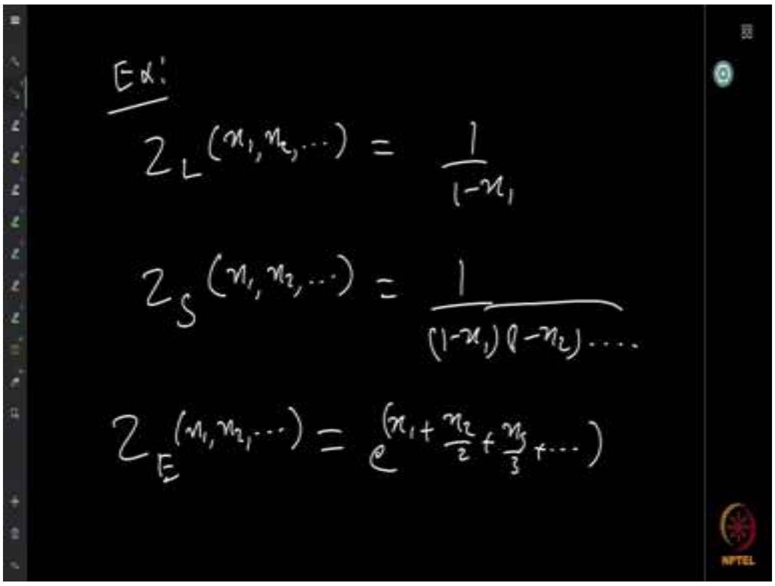

Now, here is some examples I am going to give, I am not going to go into details how I obtained this or how we obtained this, it takes a lot of time to explain many of these things but, you should be able to work out the details for at least some of these examples. So, as a homework you can try to figure out why they are the same? So, if you look at the cycle index series of the linear structures like linear order.

$$Z_L(x_1, x_2, \dots) = \frac{1}{1 - x}$$

Now, on the other hand, if you look at the cycle index series of the species of permutations that is actually

$$Z_S(x_1, x_2, \dots) = \frac{1}{(1 - x_1)(1 - x_2) \dots}$$

What is the cycle index series of the species E of sets?

$$Z_E(x_1, x_2, \ldots) = e^{x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \cdots}$$

So it should be pretty clear once you understand what is this exponents meaning, and how you connect it with the cycle types that we are talking about earlier, $\sigma, \sigma_1, \sigma_2$, etcetera. And if you do that you should be able to figure out why this indeed makes sense.

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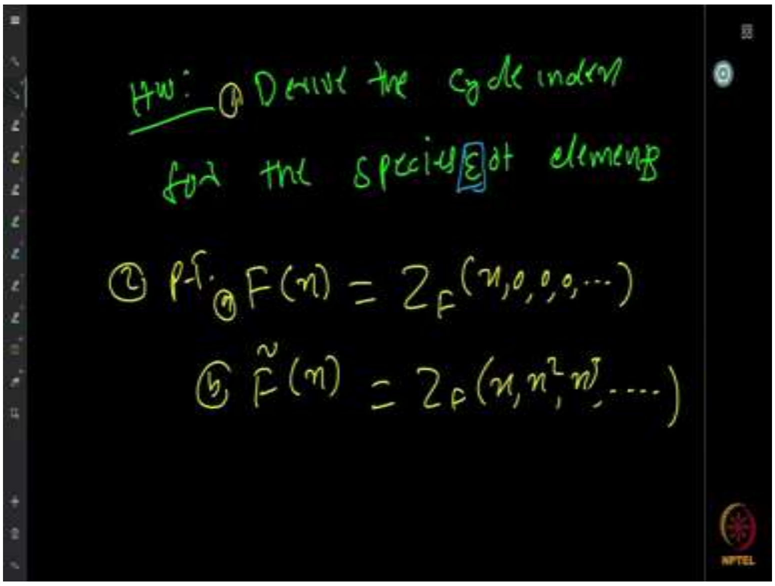

Then as homework derive the cycle index for the species ϵ of elements. Second question is to prove that, F(x) = $Z_F(x, 0,0, \ldots.)$.Let us look at one example here.

If you substitute for, this was just x, so I will get 1 by 1 minus x and that is precisely exponential generating function for the linear orders. Then $\tilde{F}$(x) also you can obtain from the cycle index. So cycle index contains much more information than, what is contained in the exponential generating function and the type generating function.

Because, you can get both of this directly by just simple substitutions, for example, $\tilde{F}(x) = Z_F(x, x^2, x^3, \ldots)$. So, you just replace x_i with x^i in Z_F and you will get $\tilde{F}(x)$. So, the type generating function can be computed, in fact to compute the type generating function for most of the species, you need to go through finding the Z_F and then do this.

It is not immediately clear, how to count the type energy function without using the help of the cycle index, I mean for some things like linear orders and all we know immediately, but otherwise it is not very direct. So, prove these two statements, and these are not very difficult just substituting this and you will see, what is happening there and using some of the observation that we have before, you should be able to do this.

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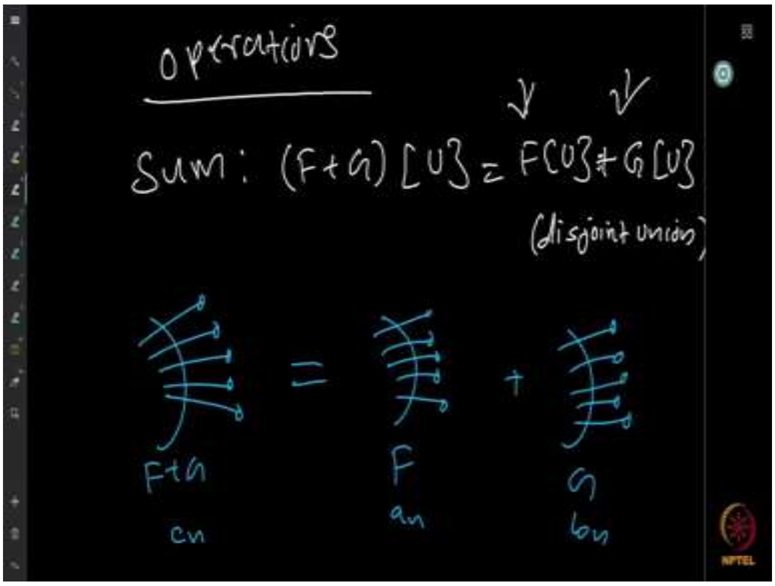

Now, let me define operations on species, so what do you mean by operations on species. I am talking about addition or some products or taking derivatives or exponentiations, compositions and all kind of things we can talk about.

So, what is the meaning of sum of two species? So given speech is F and G, (F + G) [U], so see this is the basic way to define any object on a species, given any finite set U the species tells you, the rule to generate species of U for any finite set.

So, therefore (F + G)[U] should tell you how to do that, and what is this I define this as the disjoint union F[U] $\dot{+}$ G[U]. So, (F + G)[U] = F[U] $\dot{+}$ G[U] , where, I consider the disjoint union, in the sense that, it is possible that the species F and G are such that for some elements for example on the empty set or something else, it might create the same kind of objects but I want to distinguish them by marking it as the elements coming from the species F and the elements coming from G. So, I want to make sure that the union is disjoint, so I can do this by specially marking each type F structure with a marker that it is coming from F and each G type structure with another marker to say that it is coming from G. So, this way I can make the disjoint union.

And what is the disjoint union looking like it is basically, so F + G on this structure is in the picture, pictorial manner it is basically an F structure on the same set, plus a G structure on the same set. So, all possible F-structures here you can make and all possible G-structures you can make on the same set. And all of them put together is basically the F + G structures on the set, we are looking at.

And if c_n is the number of success on n-element set of type F + G and a_n and b_n are the corresponding number of type F elements and G elements on the same set. Then, $c_n = a_n + b_n$ because if you are doing this, this should be the case.

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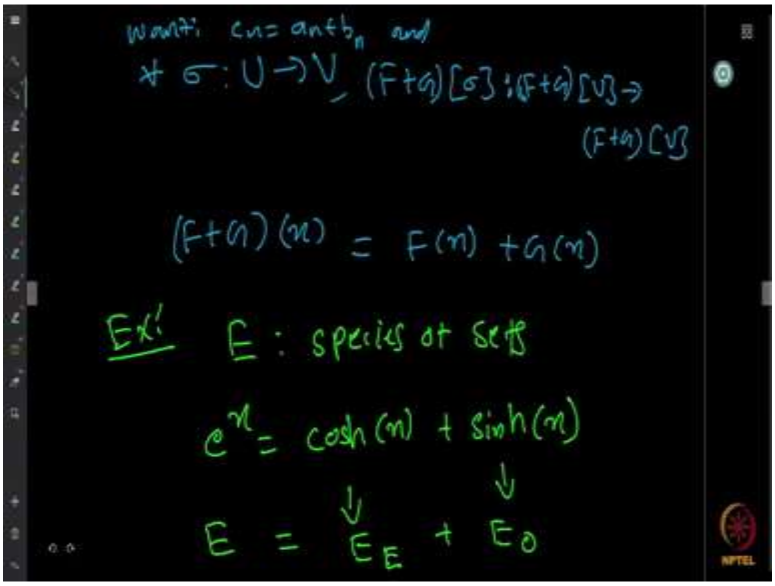

So, what we want is that $c_n = a_n + b_n$ and for every function and every bijection σ from U to V, we should have $(F + G)[\sigma]: (F + G)[U] \to (F + G)[V]$. This should be again intuitively clear, so we want this also. And we also want the functions to be adding, the generating function should be added the exponential generating functions should be added together.

So, an example is the species of sets E. What is the generating function for E? which is e^x, we saw this or you can prove this. Now, we know that $e^x = \cosh(x) + \sinh(x)$ something that we have studied in calculus. But what is even more interesting is that, now the species E can be written as the sum of two species, which is the set of all even sets and set of all odd sets, so, the species of even sets and species of odd sets. $E = E_E + E_O$

Now, what is the generating function for the species of even sets that is actually cosh (x) and the generating function for the species of odd sets is sinh (x). Any set is basically either an even set or an odd set and the generating function for even set is this, and generating function for odd set is this.

We should have this identity, $e^x = \cosh\ (x) + \sinh\ (x)$. So, if you find it easier to prove that E_E is this, and E_O is this then you have another proof of this identity or using this identity you have a proof of this I mean all these things one can do.

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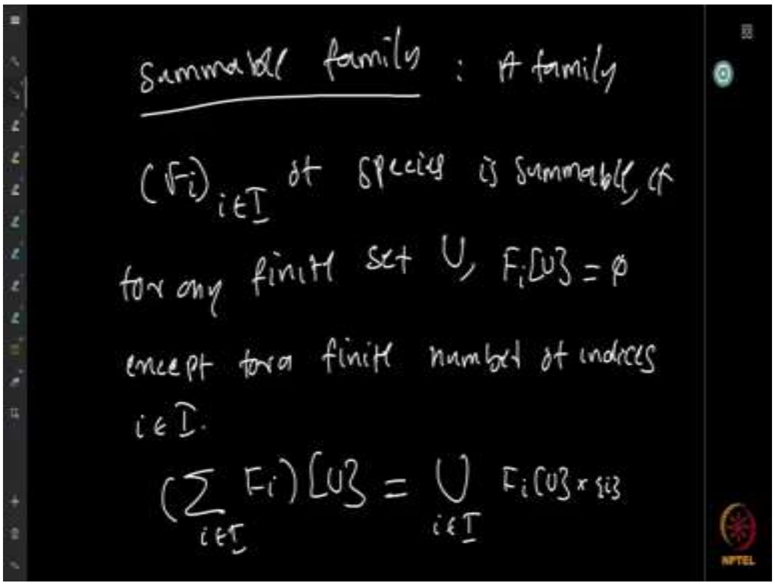

Now, a family F_i of species is said to be summable family, if for any finite set U, $F_i[U]$, the F_i type of objects on U is empty, except for a finite number of indices $i \in I$. So, if there are infinitely many, so I could be an infinite family but if only for finitely many such i's, F_i can be defined and for everything else it is basically empty.

Then, we say the family is summable, because when we sum over all these things we want to still remain in the domain of finite numbers. We are always talking about finite sets and the finite set of objects that you can make out of this finite set of objects. So, therefore we want to always stay in the finite domain so, we will say that "Okay" a family is a summable if in that family indexed by i only finitely many i's define any objects. And everything else produces empty.

Then we have that, $(\sum_{i \in I} F_i)[U] = \bigcup_{i \in I} F_i[U] \times \{ i \}$

So this is basically used to make sure that, we are looking at disjoint union. That is, the only purpose of that and whenever this is empty we do not have anything here in the product.

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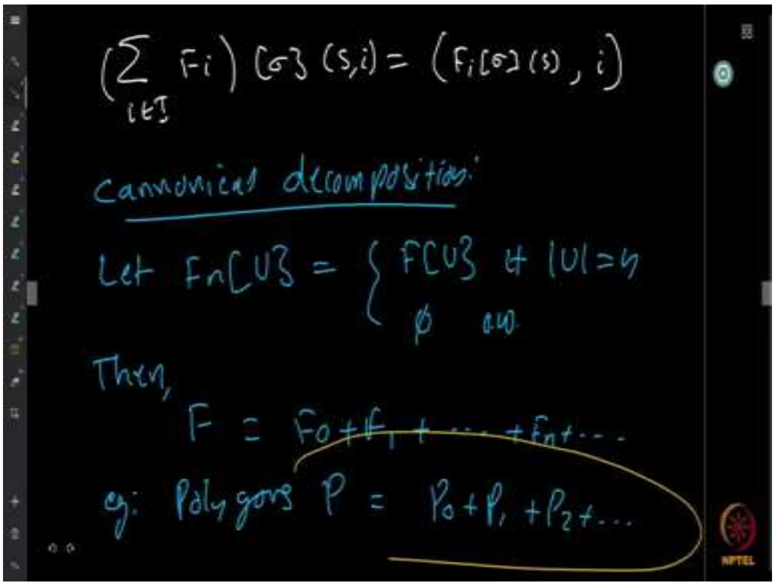

And this also satisfies that, if whenever there is a bijection σ, $(\sum_{i \in I} F_i)[\sigma](s,i) = (F_i[\sigma](s), i)$.

Now, this is the general fitting and for the special case, we have this canonical decomposition when we want to separate the elements that we are making the structures that we are making on sets of different cardinalities. So, when you are looking at the n-element set U and so, what we want to say is that, like F_n is basically going to look at the type F-structures that you make on n-element sets only.

So, $F_n[\,U] = F[U]$, if |U| = n. And if U has more or less elements there is nothing that you can depend on, that belongs to F_n. So, this way we can always decompose so, the family $F = F_0 + F_1 + \cdots + F_n + \cdots$, where F_i is the species made out of an i-element set. The species are the structures of type F made out of i-element sets. So, only i-element sets can produce elements in F_i.

Because of this we know that there is only finitely many indices and therefore, we directly follow from the above property, that we have the summability and therefore, I can write it as the disjoint union of F_0 plus F_1 plus etcetera. So example is, you are looking at polygons, so polygon with 0 elements, 1 element, 2 elements, 3 elements etcetera and then all of them together from the species of all polygons.

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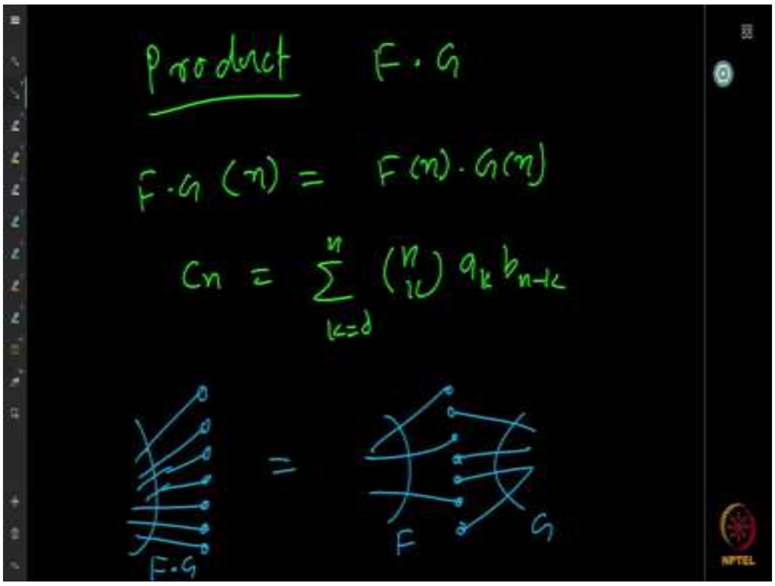

Now, we quickly look at the product of two species so, F and G are species I denote by F . G the product species, FG. So what is the product species FG, its generating function FG(x) is the product of the generating function. So, we want this and we are going to define the product as something that works out for this property to hold. So, what is C_n, C_n is the number of elements of type F.G on an n-element set.

$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$, so these all we saw when we define the product of exponential generating functions. And what is this so the product of two species is denoted in pictorial way as follows, you have this set and you are putting an FG- structure on this, which is equivalent to taking the same set finding some subset putting an F-structure on this subset.

Looking at the complement of this and putting a G structure on the remaining elements. So, this picture tells much more things that, one can express in words, how the product species looks like is very clear from the picture.

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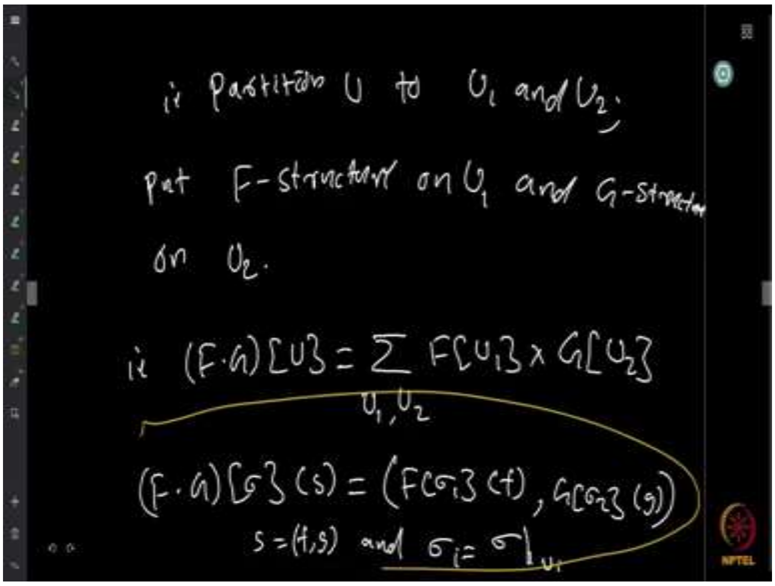

So, basically, when we are looking at the product it is basically partitioning the set U to two sets U_1 and U_2, put an F-structure on U_1 and the G-structure on U_2. And then, because of this, we know that, what happens to FG on U is basically, the different ways you can do this to U_1 and U_2, $F[U_1]$ and $F[U_2]$, you are looking at the Cartesian product. And this will be our structures. The structures will be of this type $F[U_1] \times G[U_2]$

And sum over all these U_1 and U_2, which are basically partitions of U to 2 sets and that should give you the species FG. And what happens to a permutation σ or a bijection σ? The bijection σ is taken by FG the product. The transport of σ under this product, is basically what happens to an element s, an object of type FG. Now, what is s, s is an object of type (f, g), but we know that any such s is basically of this type, $F[U_1] \times G[U_2]$, so therefore it has actually two components.

So, let us say, that is basically f and g. So, if s is (f , g), then this Sigma must act as separately on f and g, where σ_i is the restriction of σ to just U_i. So, σ_i acts on the entire $U_1 \times U_2$, but the restriction of σ to U_1 or U_2 will give you σ_1 and σ_2.

So, $(F.G)[\sigma](s) = \big(F[\sigma_1](f),\ G[\sigma_2](g)\big)$.

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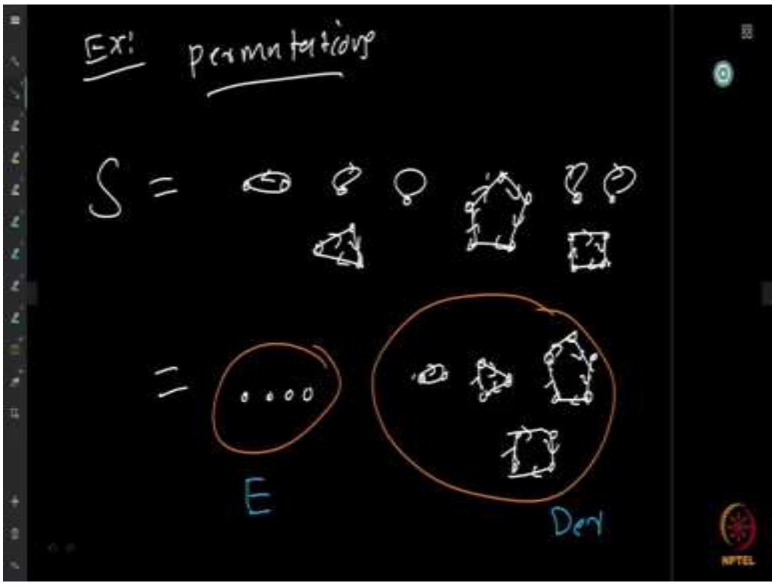

So, let us look at a very interesting example of permutation. The species of permutations is denoted by S and we know that permutations are basically cycles. So you have several cycles here and this cycles, define the permutations. Now, we observe that like in this permutation, we have these cycles with more than one element.

And there could be some cycles with just one element. So you collect all the cycles with one elements, put them together since the 1-cycle does nothing, it maps to itself, these are basically elements which does nothing else. So, I collect them together it forms a set, so it form a set partition U_1 and then, then you have the cycles where you have at least two elements.

Now, where all the cycles have two elements, the product of such cycles will be a permutation where no element is fixed, so therefore they are called the derangements. Derangement is a permutation without any fixed points. So, we can see any permutation, as basically a possibly empty set of elements together with derangements.

So, we see that, the species of permutations is basically the product of the species E of sets and the species Der of derangements. Because, given any permutations I can do this, I can basically look at the 1-cycle form a set out of these elements. And then the remaining larger cycles defines a derangement and on the other hand, given any set and any derangement of the remaining elements, I can put them together, and it defines a permutations, of all the elements.

So, therefore there is one to one correspondence and then we see that, species of permutations is the product of species of sets and derangements. Now, this helps us to compute for example, the generating functions for derangements, if we know what is the generating function for S and E. So, we computed this earlier.

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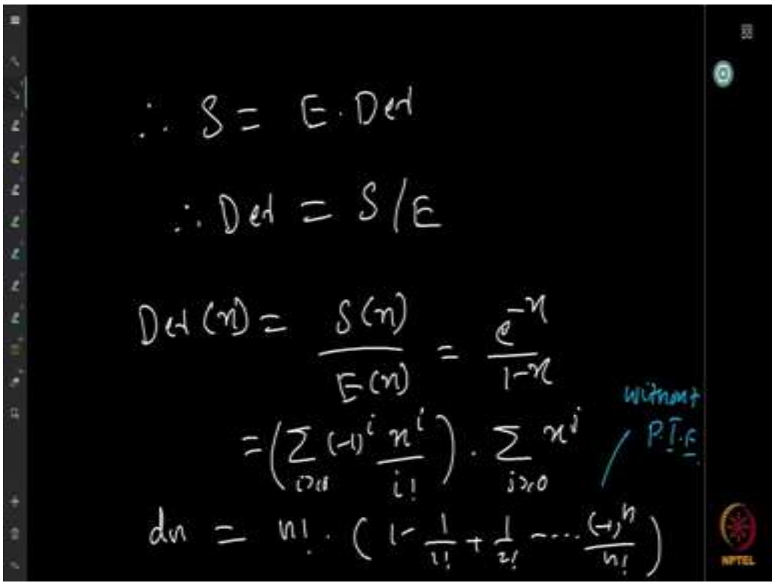

$$S = E\,.Der$$

$$Der = \frac{S}{E}$$

$$Der(x) = \frac{S(x)}{E(x)} = \frac{e^{-x}}{1-x}$$

$$= \left(\sum_{i\geq 0}(-1)^i \frac{n!}{i!}\right)\sum_{j\geq 0} x^j$$

this is the geometric sequence.

Now, the product of these two, we can immediately find because, the coefficients here are just 1's. So therefore what is the product of these two and you are looking at the coefficient of x raised to n, by n factorial that is basically n factorial into, look at the terms which are going to contribute to x raised n, I have to take some element from here.

And the remaining numbers should come from here, so that j + i must be equal to n. So if I collect them, altogether I know that, there could be at most n + 1 terms. So

$$d_n = n!\,(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots \frac{(-1)^n}{n!})$$

And this is the formula for d_n, we calculated earlier using principle of inclusion and exclusion. But, now here we without using any such thing, we can directly compute the formula just by looking at the generating function for the permutations and sets. So, this is a nice way to compute other generating functions which are not easy to combine.

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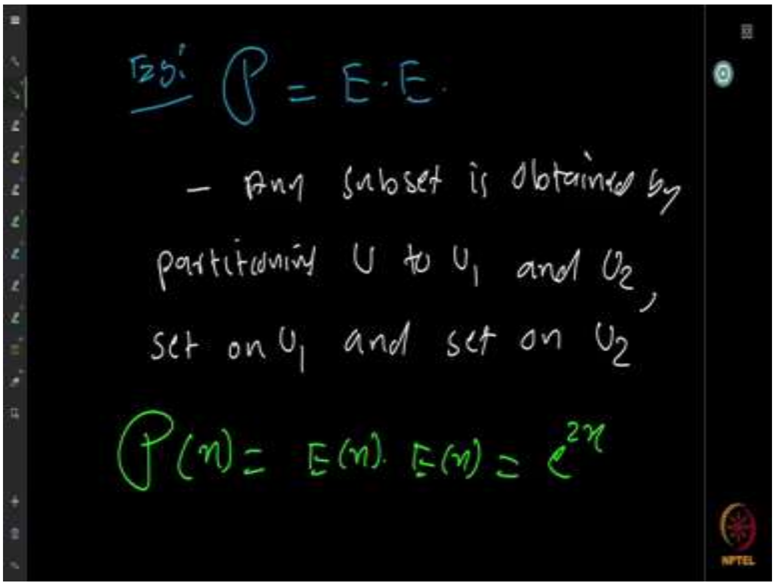

Another example is the power set. set of all power sets on set U is basically the product of E with itself. So E is the species of sets, so species of sets with itself. And why is this, this should be clear because, any subset is basically partitioning a set U to U_1 and U_2, one is to keep and the other one is to throw out.

So, that forms a subset. So basically power set of X is basically the product of E with itself, and that should also make sense because, we get e^{2x}, the number of objects of this type is basically, if you look at the coefficient of x^n it will be 2^n. On the other hand, when it is just e^x, this is just one that is only one such structure. So you will get this directly from here.

So, with that example I will stop for today, then, we will continue with more examples, more examples of more operations on species and with that let me stop today.