Combinatorics

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Species of Structure – Definitions and Examples

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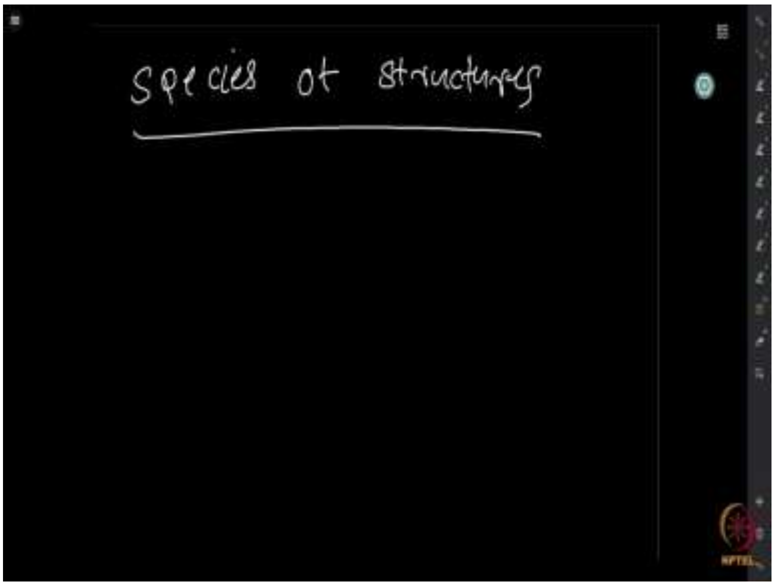

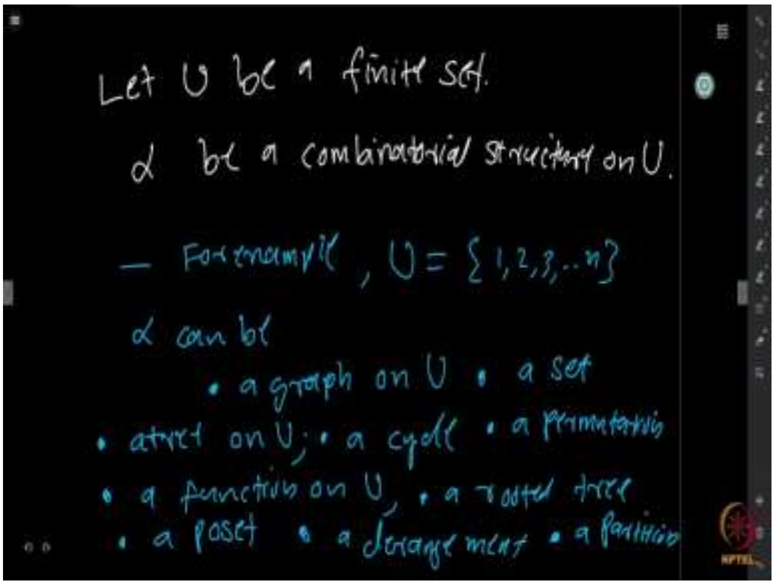

So we are going to start today the topic on species of structures. Let U be a finite set. So we are going to look at the topic on species of combinatorial structure. So, a combinatorial structure is basically any kind of construction that you make on a set. So, for example, if you have a set let us say a, b, c, then you can make a graph on these 3 vertices. For example, you can draw a graph with all the 3 forming a cycle that you have seen or forming a path or just 1 edge and 1 vertex.

Any of this is a graph on these 3 vertices. So you are basically putting a combinatorial structure on a set U, in the sense that you are constructing something out of this set. So, the construction can be many things, for example given a finite set let us say U as you see here you have this set U = {1, 2, 3, …, n} let α be the combinatorial structure on U. Then α can be a set,

a tree structure on the set, so you can take this and make a tree out of it by putting edges. You can make a cycle, putting them in some order and then connecting the edges. You can make a permutation, so just take any permutation of the elements, you get a permutation of that. You can define functions on U, elements basically function from endo functions or a rooted tree, like you can put a rooted tree. You can define one node as special, one of the vertices let us say 2 is special then it has 2 or 3 children say 1, 3 and 7. Then it has other children etc.

So, you can define rooted trees, you can define poset structure, you can put a derangement, derangement you have studied, are permutations which does not map elements to itself, there is no fixed point. Then you can define a partition, given a set you can partition it. So, all these can be examples of structures, I mean there could be many more. So, in short we are just constructing structures on a given set.

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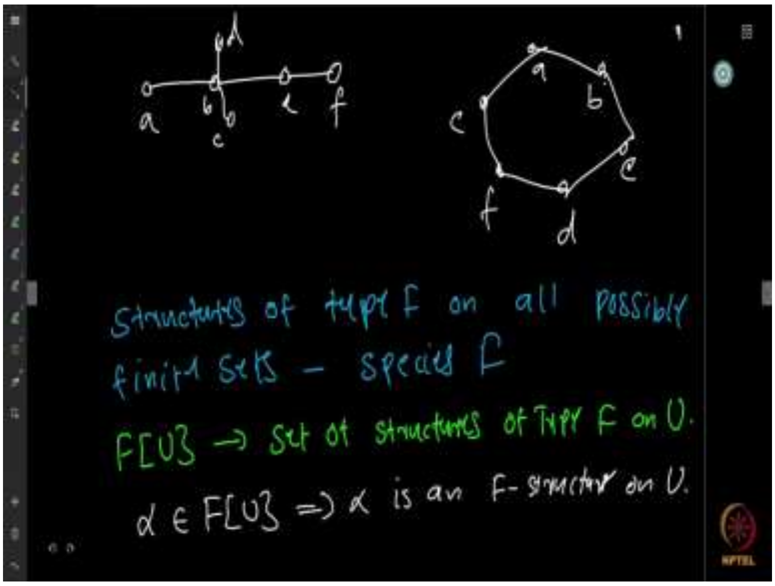

Now here is an example of 2 different structures on the set {a, b, c, d, e, f}. So {a, b, c, d, e, f} is given and then we put a tree structure on this so a to b is edge, b to c and b to d, b to e and e to f. On the same set I can put another structure for example c to a, a to b, b to e, e to d, d to f, and f to c which is the cycle.

So, now these two are different type of structures, I mean one can say that, one is a cycle and one is a tree. So, you can now say that I am interested in making only cycles or you can say that you are interested in only making trees. So, if you decide that we are going to have only trees, then we say structures of type trees.

So, I can say that for any type of structure that we are looking at, let me say F, then you look at all possible finite sets, so U is one set that we consider like {1, 2, 3 etc n} or {a, b, c, d, e etc}. Consider any such finite set and then look at all possible structures of this type, for example trees you are saying, right? So, all possible trees you can make on this set of elements, collect them together for all possible finite sets is called as species of the type tree.

So, now similarly we can do this for any other object, for example I can say species of graphs or species of permutations which consists of all possible permutations you can make on all possible finite sets. Now you collect together the set of all permutations let us say, you can make on a fixed set U, so that I can denote by F[U] the set of structures of type F on U.

So, if it is the collection of permutations then it is the all possible permutations, whatever is the cardinality of U, that cardinality of U factorial many permutations are there on the specific set U. So, that is the set of structures of type permutations on the set U. So, F[U] in general denotes the set of structures of type F that is made on the set U.

Now when we say a structure $\alpha \in F[U]$. So it means that α is an F-structure on U. So, U is the set and α is the structure of the type F that we are talking about, whatever species, trees or paths or functions or posets or derangements whatever we are talking about that type of structure on U.

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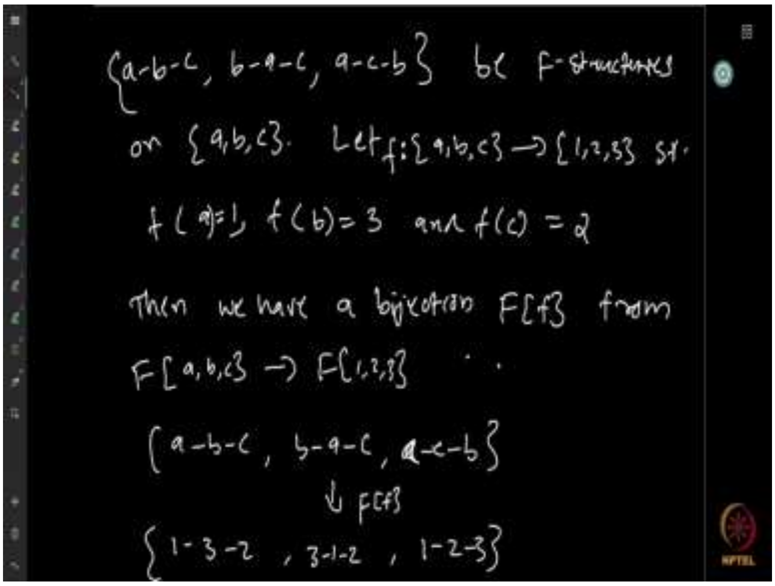

Now let us say that, you are looking at this example of paths {a-b-c, b-a-c, a-c-b} is a path. These be the F-structures on the set {a, b, c}. Now suppose we have a function f which maps let us say {a, b, c} to set {1, 2, 3} and f(a) = 1, f(b) = 3 and f(c) = 2. Now we know that if you look at the type of F-structures on the set {1, 2, 3}, there are corresponding objects there.

But now what we can observe is that, if for any bijection f from {a, b, c} to {1, 2, 3} we have a bijection from F[a, b, c] to F[1, 2, 3]. For example, a to b to c, b to a to c and a to c to b are there and what does F[f] do? That maps a-b-c to 1-3-2 because a is mapped to 1, b is mapped to 3 and c is mapped to 2 by f.

Similarly, b-a-c goes to 3-1-2 and a-c-b goes to 1-2-3. So, this is something one can intuitively feel. So, whenever we have a bijection from a set U to set V, we can think of a corresponding bijection from the set of all structures on U to set of all structures on V, where this kind of properties holds because f is taken by this, the corresponding let us say labels of these objects can be changed suitably to match the function. So, this is intuitively clear and one can in fact show it.

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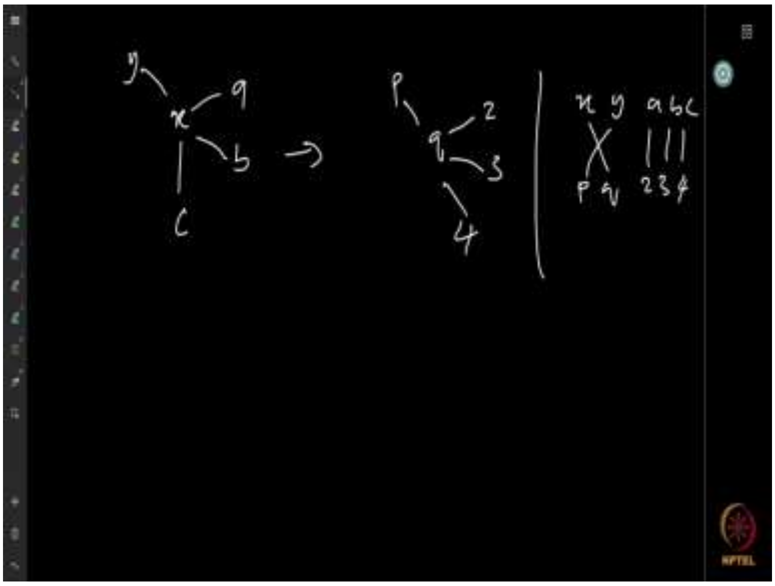

We will skip the details but let us look at one more example. So, you have this tree with a vertex x going to y, going to a, to b and c then, this can be mapped to let us say vertex q going to p, 2, 3 and 4 and this map between two structures can be represented by of course under the bijection between the corresponding sets {x, y, a, b, c} and {p, q, 2, 3, 4} where x is going to q, y is going to p, and a going to 2, b going to 3 and c going to 4. So, these kinds of things are intuitively clear.

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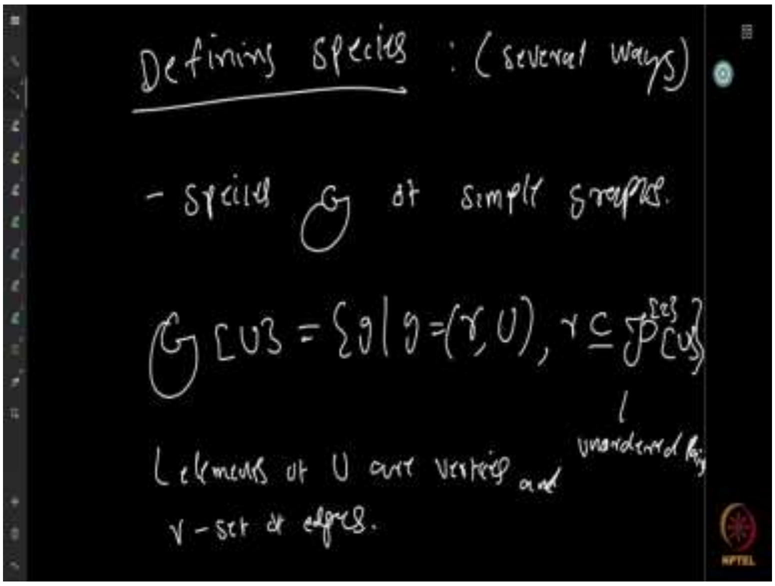

Now let us give an example for defining species. So we have several ways of defining species so let us look at one example in detail. So, we want to define the species $\mathcal{G}$ of simple graphs.

So, what is a simple graph? A simple graph, is a graph structure on a finite set U which means that there are edges and the edges are two elements subsets of U, unordered pairs of elements.

Now we define the species as follows, the structures on the set U, I denote by $\mathcal{G}[U]$ where $\mathcal{G}$ is the species. Therefore,

$$\mathcal{G}[U] = \{g \mid g = (\gamma, U), \gamma \subseteq \mathcal{P}^{[2]}[U]\}$$

So, one can see that gamma is basically the set of edges and elements of U form the vertices of the graph. So, we defined the species by this, whatever set you give me this is the definition of the set of all structures on this. So, therefore, since I can do it for any finite set, it tells me all about the species of graph or simple graph. So, given any set I can make, the set of all possible graphs on this or by this definition.

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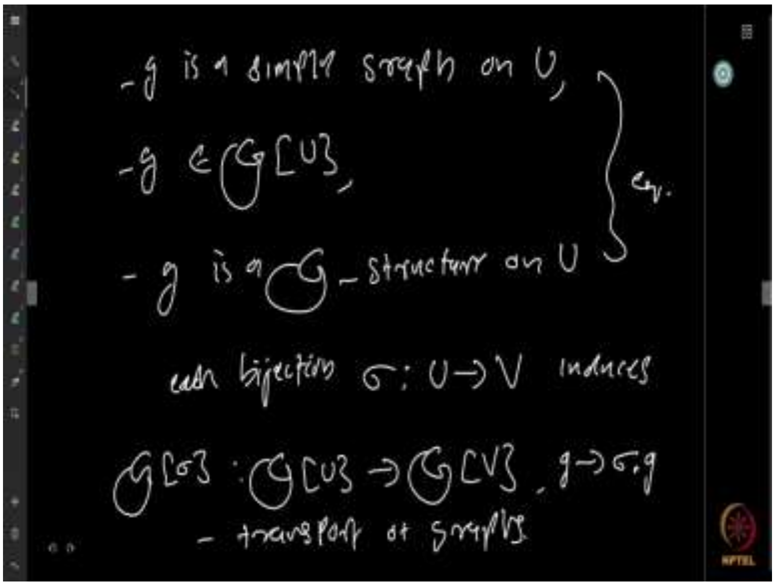

Now, what we can observe is that g is a simple graph on U and $g \in \mathcal{G}[U]$. And g is a $\mathcal{G}$-structure on U. All are equivalent statements and we observe that, each bijection $\sigma : U \to V$ induces another map $\mathcal{G}[\sigma]: \mathcal{G}[U] \to \mathcal{G}[V]$, where $g \to \sigma . g$.

So, this function $\mathcal{G}[\sigma]$ is called the transport of graphs. Basically, the graph structures on the set U is mapped to graph structures on the set V so therefore this is basically transporting structures which is defined on one set to those defined on another set.

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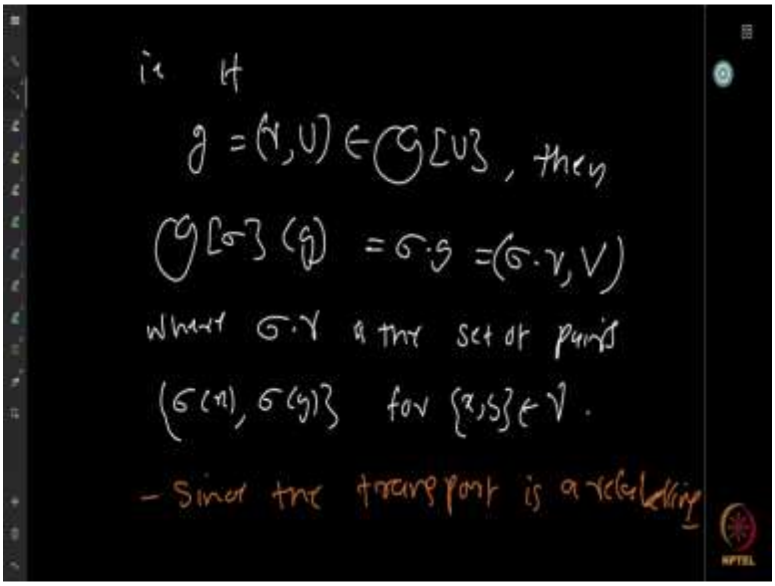

Now, if graph $g = (\gamma, U) \in \mathcal{G}[U]$, then $\mathcal{G}[\sigma](g) = \sigma. g = (\sigma. \gamma, V)$ where $\sigma. \gamma$ is the set of pairs $\{\sigma(x), \sigma(y)\}$ for $\{x, b\} \in \gamma$.

So, whenever there is an edge {x, y}, $\sigma(x)$ and $\sigma(y)$ will be the corresponding pair of elements that is going to make the edge in the transported structure. Now one can see that the transport is just a relabelled from. From this example one can see and in general one have an intuition that the transport of structures is just a relabelling.

And because of this we should be pretty clear that, if U and V are two sets and $\mathcal{G}[U]$ is the set of all $\mathcal{G}$-structures on U, $\mathcal{G}[V]$ is the set of all G-structures on V. If there is a bijection from U to V, then there must be a bijection from $\mathcal{G}[U]$ to $\mathcal{G}[V]$ in the sense that $\mathcal{G}[U]$and $\mathcal{G}[V]$ must have the same number of objects.

If U to V there is a bijection then it is kind of intuitively clear why it should be the case. Therefore, one can feel that $\mathcal{G}[\sigma]$, the transport of sigma is basically a bijection also and this also tells you that the cardinality of F[U] for any species F. Cardinality of F[U] only depends on cardinality of U. It does not depend on what are the elements of U, it does not matter what the names of the elements of U are, what are the elements of U but you have U and V, $\mathcal{G}[U]$ and $\mathcal{G}[V]$ depends only on cardinalities of U and V.

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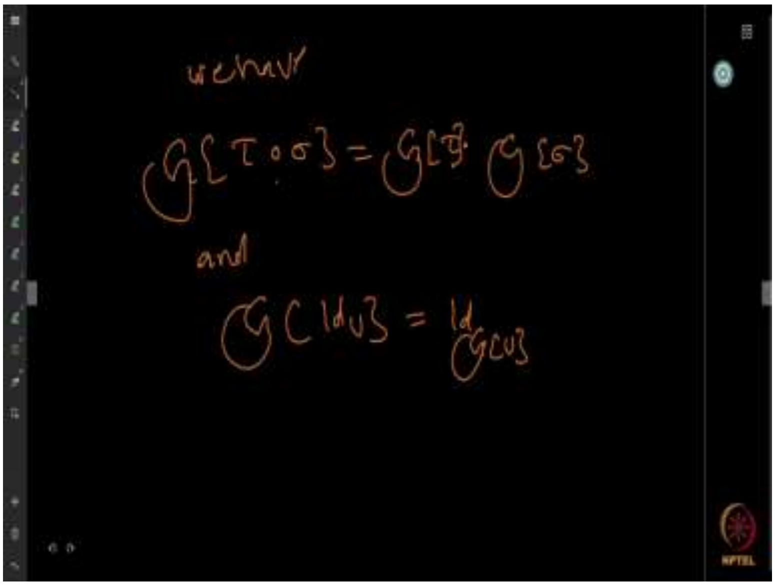

Because the transport is just a relabelling, we can also see that for two bijections τ and σ

$$\mathcal{G}[\tau \circ \sigma] = \mathcal{G}[\tau] \circ \mathcal{G}[\sigma]$$

Let Id_U be the identity map on U, the transport of Id_U must be the identity in $\mathcal{G}[U]$. That is,

$$\mathcal{G}[Id_U] = Id_{\mathcal{G}[U]}$$

And since the bijection σ is just identity the transport also must be identity on the corresponding set. If this is not true then we can see that like something will go wrong in our relabelling. So, I am not going into the details of this, we want to finish this as a just quick review of these topics, just an introduction. So, I will not go into details but you can take it as homework to prove this.

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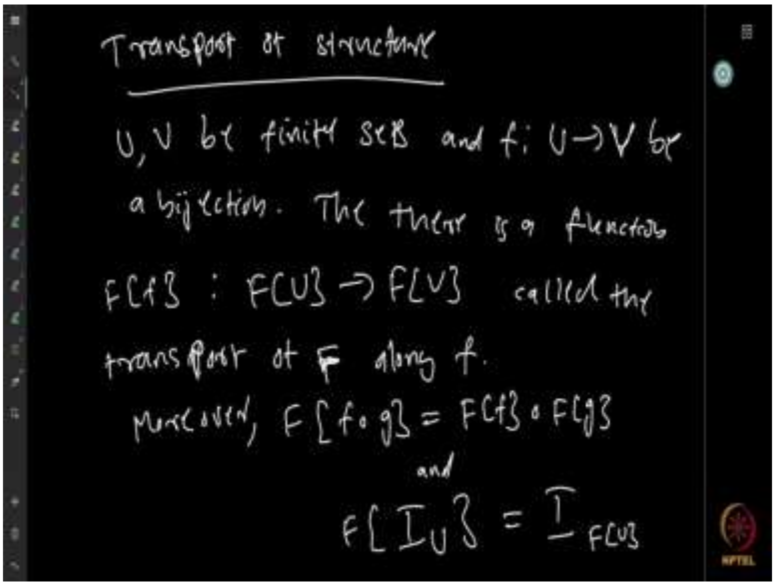

Now more formally the transport of a structure can be defined as follows. Let U and V be finite sets and $f : U \rightarrow V$ be a bijection. Then there is a function $F[f]: F[U] \rightarrow F[V]$ called the transport of F along f. So, for different bijections F you have different transports, that is clear.

Moreover, the transport of F should satisfy the following. For any two bijections f and g

$F[f \circ g] = F[f] \circ F[g]$ *and* $F[I_U] = I_{F[U]}$. This is something we observe.

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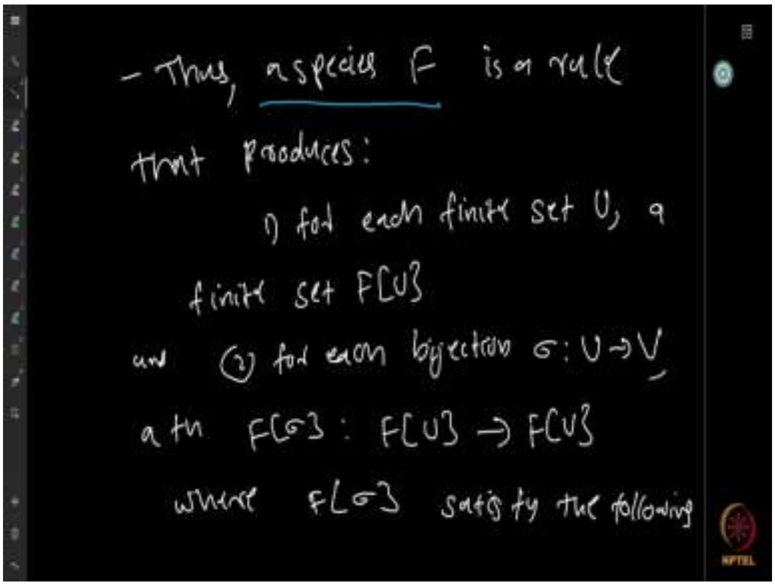

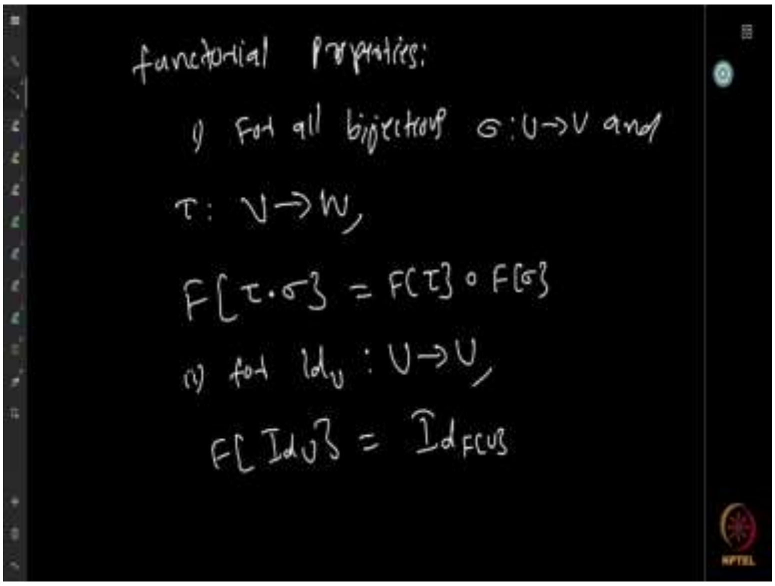

So, in short, a species F is basically a rule we can define species also as a rule that produces: for each finite set U, a finite set F[U]. This is something that we need to know. When we say I am looking at species of all graph, you should know what are the possible graph structures on a finite set, without that we cannot talk about the species.

So, species is a rule that produces: for any finite set U, a finite set F[U] and for each bijection $\sigma: U \to V,$ you can find a function $F[\sigma]: F[U] \to F[V]$, where $F[\sigma]$ satisfies the following conditions which are called functorial properties:

(1) For every bijections $\sigma: U \to V$ and $\tau: V \to W,$

$$F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$$

(2) For $Id_U : U \to U,\ F[Id_U] = Id_{F[U]}$

So, when we talk about categories and functors in category theory the functor basically satisfies this condition. So, that is called functorial property, we are not going to go into details of category theory, that all comes in a specialized course on this.

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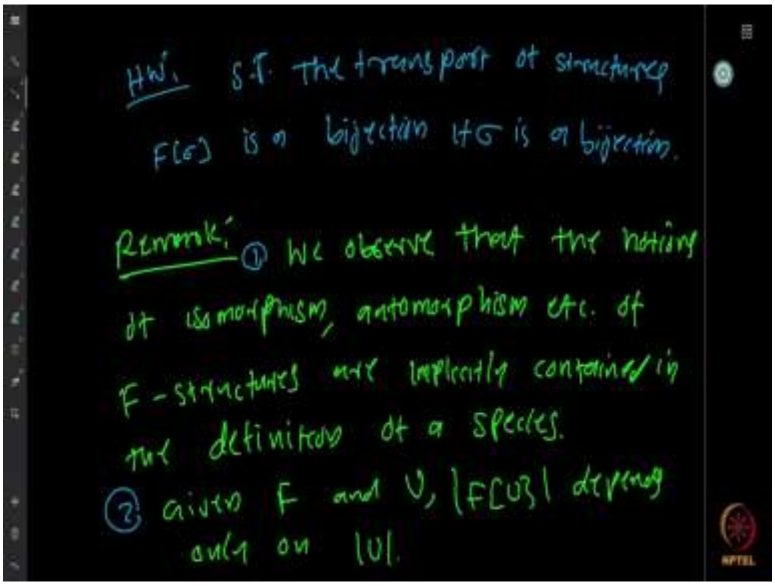

Now as a homework, show that, the transport of structures $F[\sigma]$ is a bijection if σ is a bijection. So, for every bijection the transport is also a bijection. It is intuitively clear and it should not be very difficult to prove also. Now let me remark that one can observe that the notions of isomorphism, automorphism etc of F-structures, whatever is the species are implicitly contained in the definition of species.

This is something that you can observe by the way we have defined these bijections. So, these all comes for free, like isomorphism, automorphism etc comes for free. Now given let us say a species F and set U, the cardinality of F[U] depends only on cardinality of U because what are the labels does not matter.

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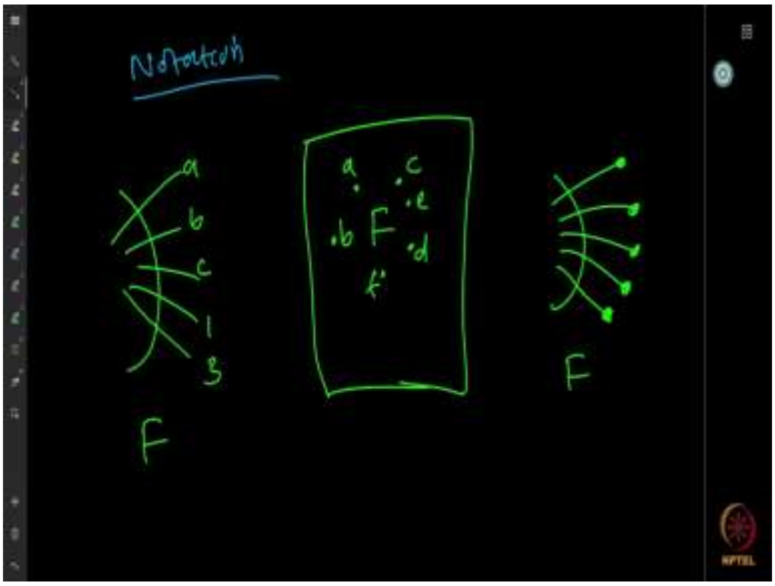

Now we denote there is an F-structure on a set as follows. Let us say that a, b, c 1, 3 is a set and I am going to make some object of type F. So, I am looking at the F-structures on the set. I am going to denote it in the following way, so I write a, b, c 1, 3 here and then I put this lines and then by this arc I mean that I am putting some F-structure on this set.

So, it could be any of the structures. Therefore, this represents that we are talking about all possible F-structures on this particular set. The same can also be represented by putting in a box all these points and putting just F to say that I am basically talking about all possible structures that you can make on this particular set which is of species F.

I do not want to talk about specifically what are the elements, I mean I can also just not write the set elements when the set is clear. I can just put this without the labels and this also tells that, whatever is the set having this many elements, I am going to put the structures and these are just notation that we can use to describe type F structures on any set.

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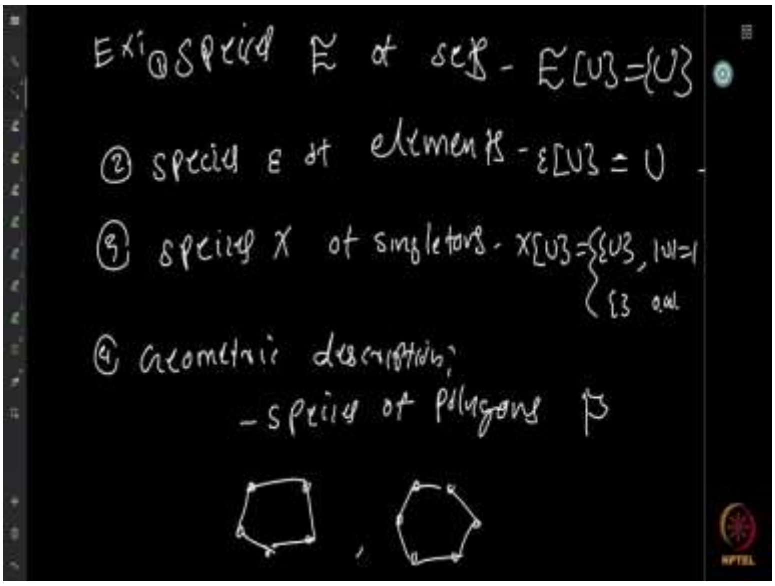

Now let us look at some examples, so a species E of sets. We are talking about the species of sets, so given any finite set U, what is E[U], we are putting a set structure. So, when I say E[U], it means that I am talking about structures of type E which are sets on the set U. So, I want to put a set structure on U. How do we put a set structure on U?

Just look at the set U, so there is only one way to put a set structure on U, basically that set itself . So elements of this E[U] are the sets on U. There is only one set on U because U is a set and then I want to make a set structure using all the elements. If I am using a small subset of the element then that is not the substructure on U, it is a substructure on the subset.

So, therefore, E[U] = {U}, where the element within this U are the objects of the type that we are talking about.

Now, the species of elements, $\epsilon[U]$. Given a set U, I can talk about the elements.

So, $\epsilon[U]$ objects of type elements of the given set U and that structure on U. So, we are talking about all possible elements of U. Now this forms the set of all this element in the set U itself because I am talking about the elements of U and forming the set, I get U itself. So, therefore $\epsilon[U] = U$.

I can talk about the species of singletons, one element sets. So, what is X[U]? Given any set U, X[U] is basically {U}, if there is only one element in U. If there is more than one element, I cannot put any structure. So, therefore this empty. So, that is the species of singletons.

I can define species by geometric descriptions now. So, I can talk about species of polygons, let us say P as follows. I just define a set let us say 5 elements here put a structure and say that this is a structure on 5 elements, this is a structure on 6 elements etc. So this geometric descriptions tells you how to make more polygons given any set, how do you make polygon? So, that tells you how to do it and therefore this also can be used to define a species.

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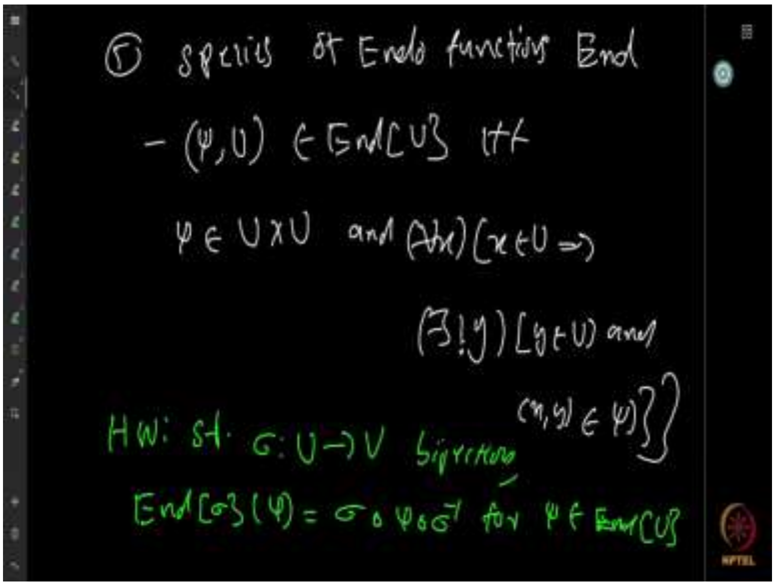

Now species of endo functions. So, what is the endo function? Endo functions are functions from a set to itself. So, species of endo functions which I denote by End is basically the pair let us say $(\psi, U) \in End[U]$, if and only if $\psi \in U \times U$ and $\forall x \, (x \in U \Rightarrow (\exists! \, y)[y \in U \; and \; (x, y) \in \psi))$.

So, this is a set theoretic description of the species of endo functions. Now as a homework you can show that if $\sigma: U \to V$ is a bijection then the $End[\sigma](\psi) = \sigma \circ \psi \circ \sigma^{-1}$ for $\psi \in End[U]$

So, for any endo function $\psi \in U$, it is transported to $\sigma \circ \psi \circ \sigma^{-1}$ by any bijection σ. So, this you can take it as a homework.

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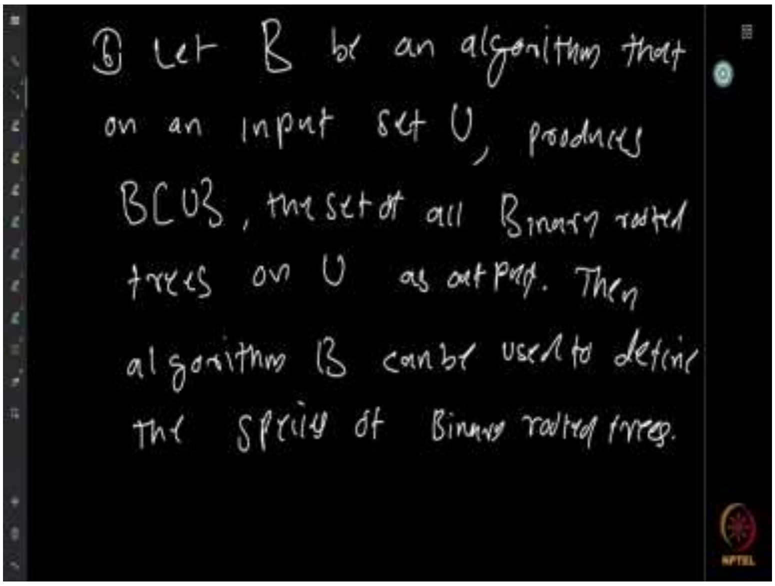

Now another way to describe species is by using the help of algorithms because sometimes it may not be easy to define an object. So, you may be talking about for example the outputs of an algorithm. Some algorithm is there that can produce, given a set it can produce objects of a particular type whatever we want and these outputs are let us say objects of interest.

So, I want to talk about the species of all objects that is created by this algorithm. So, given any finite set U the algorithm produces a set of all possible structures of a particular type whatever it is and I can now define the species by a looking at the outputs of algorithms. What are the things that is produced by this algorithm that forms a species?

So, let B be an algorithm that on an input let us say set U produces B[U], the set of all binary rooted of tree is on U as output. Then the algorithm B can be used to define the species of binary rooted trees.

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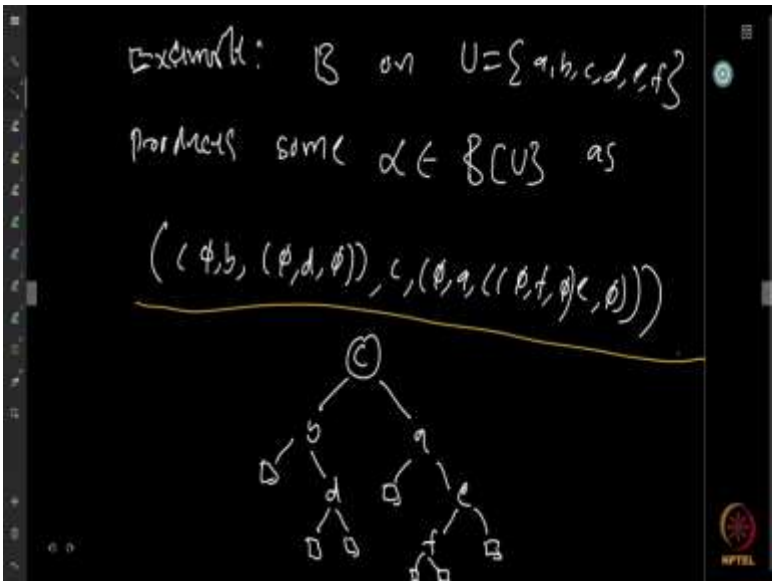

For example, B can be defined on the set U = { a, b, c, d, e, f } and produces some set of all structures and in particular one of the output let us say $\alpha \in B[U]$ was the following. This was the output, so it says $((\emptyset, b\ , (\emptyset, d, \emptyset)), c, (\emptyset, a, ((\ \emptyset, f\ , \emptyset), e, \emptyset)))$.

Now when you are talking about let us say this structure, this structure is a binary tree if you look at this, the way to produce this is by looking at this nodes, so each of these elements denotes nodes and whenever a pair is there that forms the edge relation.

So $\emptyset, b$ basically becomes an edge from b to the empty set $\emptyset$, which is represented by box here and similarly if you follow through entire thing you will see that this basically represents this binary tree and c will be the root of this so you will see all this thing if you work through the example more carefully.

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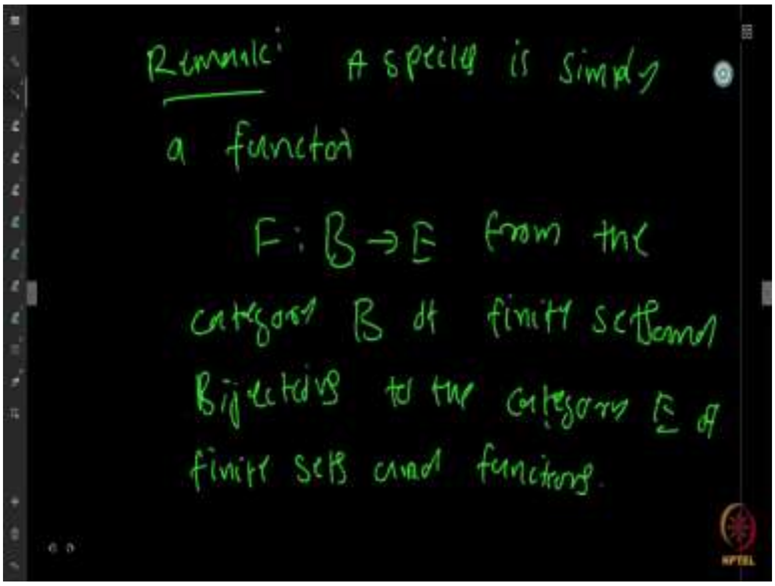

Now as a remark I want to say that a species is simply a functor, let us say F taking elements from B to E where B is the category of finite sets and bijections and E is the category of finite sets and functions. So, species is just a functor, if you are talking in terms of category theory.

We are not going into category theory because if we even define all these things it will take time and we do not have that time. So, if you are not familiar with that just forget about it. It is just a remark the species is just a functor from the category of finite sets and bijections to the category of finite sets and functions.

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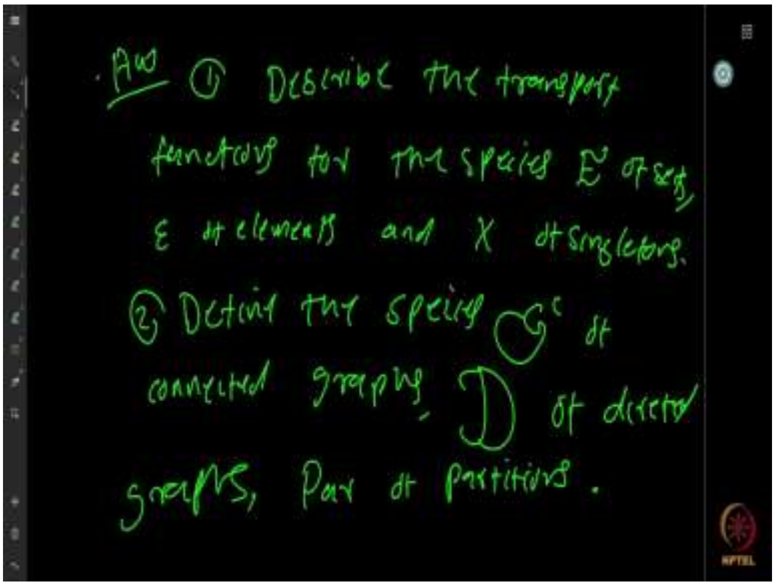

Now some homework questions, describe the transport of functions for the species E of sets, species is ϵ of elements and species X of singletons. So, basically you have to see what happens whenever a bijection is given and what will be the transport corresponding. For example we gave the example of graphs and species $\mathcal{G}$ of graphs and then what happened to the transport for any σ ? We mentioned it in the example that we were looking at with graphs.

So, here we said what happened to $\mathcal{G}[\sigma]$, so what was the transport. So, this way we have to mention what happens to the transport for each of the species in the question. Second question is to define and the species $\mathcal{G}^C$ of connected graphs, the species $\mathcal{D}$ of directed graphs and species Par of partitions. All these species you can define in your own words and let us see how good that is.

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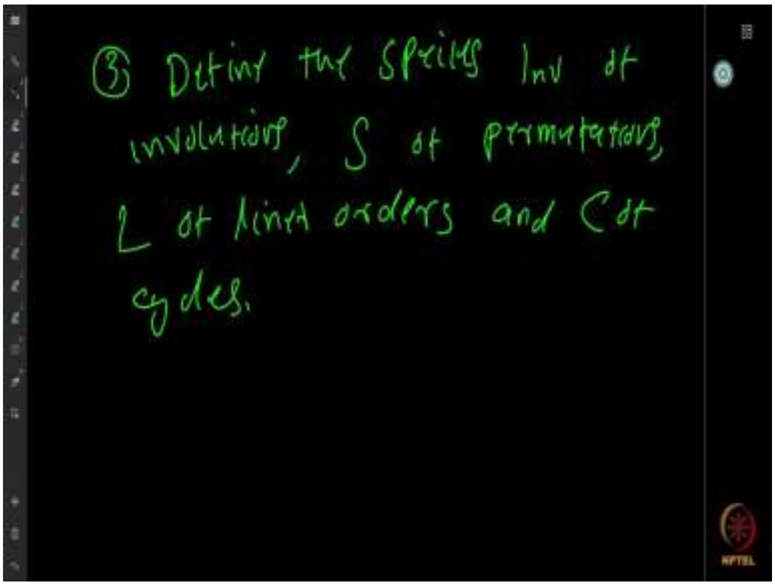

Third question is to define species Inv of involution, species S of permutations and species L of linear orders and species C of cycles. So, we have already come across most of these things earlier, so I do not want to go into any more details of what are these things and all.