Combinatorics

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Polya's Theorem

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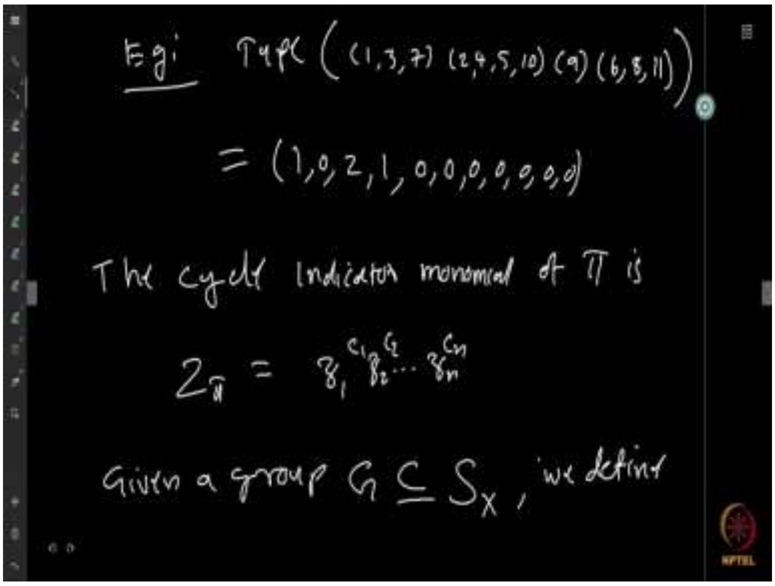

So, the type of the permutation given by the product of these cycles (1, 3, 7) (2, 4, 5, 10),(9) and (6, 8, 11) is of type (1, 0, 2, 1, 0, 0, 0, 0, 0, 0, 0) . So why is that? Well, there is precisely one 1- cycle, there is no 2-cycles in this, there are two 3-cycles (1, 3, 7) and (6, 8, 11) there is one 4-cycle (2, 4, 5, 10) and there is no other larger cycles. So, all these values up to 11 are going to be all 0, you cannot have anything larger than 11 because the number of elements is 11.

Now once you have such a type, then we can define the cycle indicator monomial of π given by $Z_\pi = z_1^{c_1} z_2^{c_2} \ldots z_n^{c_n}$ So, this tells immediately what is the cycle structure. Look at the exponent of z_i, that will tell you the number of i- cycles.

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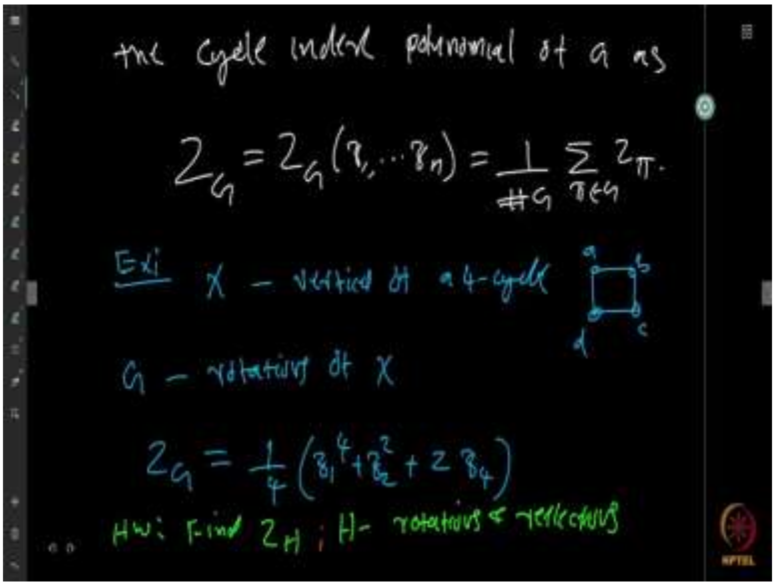

Given a group G of symmetries let us define the cycle index polynomial of the group as

$$Z_G = Z_G(z_1, \dots, z_n) = \frac{1}{\#G} \sum_{\pi \in G} Z_\pi$$

So, the sum of all these monomials normalized by the order of G. So, this is the cycle index polynomial of the group C.

This will be very useful we will see how this cycle index corresponds to the counting of colourings. So, as an example let us say that we have the 4-cycle X with the vertices {a, b, c, d} and look at the group of rotations of the cycle. So, what is the cycle index polynomial?

$$Z_G = \frac{1}{4}(z_1^4 + z_2^2 + 2z_4)$$

Here, why is it z_1^4? Because the identity has exactly 4 cycles. Then z_2^2 because you look at the the square of the generator you will see that you will have 2 cycles of length 2. And then you will get 2 times the z_4 because you will see that the generator and its inverse are 4 cycles there are 2 of them. So that will tell you this cycle index polynomial is $\frac{1}{4}(z_1^4 + z_2^2 + 2z_4)$.

Now as a homework you can look at the example to find Z_H where H is the rotations and reflections. So, find that as a homework.

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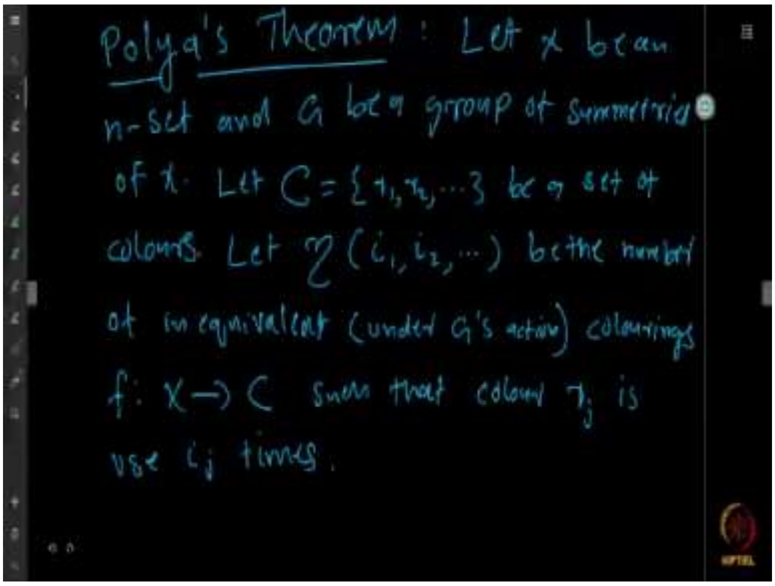

We come to the main theorem that we wanted to prove which is Polya's Theorem. So, Polya's Theorem again as we noted before, what we wanted to do was, to count inequivalent colourings where the number of colourings is given. So, let X be an n-element set and G be a group of symmetries of X. Now let us consider a set of colours $C = \{ r_1,\ r_2, \ldots\}$. C could be finite or infinite but we will just take it to be infinite at the time being. Now look at the number of inequivalent colourings under the action of G where the colour r_j appears exactly i_j times. So, that is denoted by $\eta(i_1, i_2,\ \ldots)$ So, $\eta(i_1, i_2,\ \ldots)$ says that colour r_1 appears i_1 times, r_2 appears i_2 times etcetera, r_j appears i_j times. So, look at such colourings and then see under the action of G how many inequivalent colourings are there.

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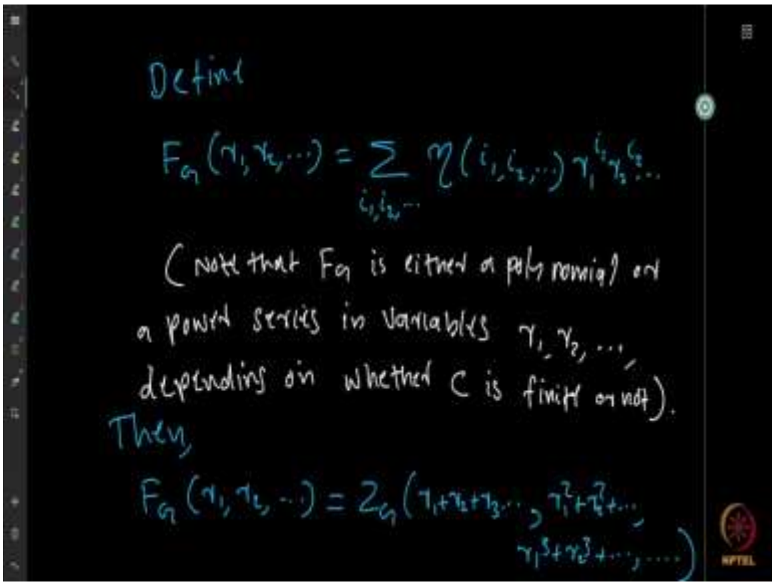

Now let us define

$$F_G(r_1, r_2, \dots) = \sum_{i_1, i_2, \dots} \eta(i_1, i_2, \dots)\, r_1^{i_1}\; r_2^{i_2} \dots$$

Which means that the coefficient of $r_1^{i_1} r_2^{i_2} \dots$ is the number of inequivalent colourings where r_j appears i_j times and sum over all these and you will get this polynomial.

If C is finite then it will be polynomial, if it is not finite then you will get a power series. So you will see that this is basically a generating series and then we can look at this and the coefficient of this term in the series will be precisely the number of such colourings, the inequivalent ones, that is the definition of the generating function. Then

$$F_G(r_1, r_2, \dots) = Z_G(r_1 + r_2 + r_3 + \cdots, r_1^2 + r_2^2 + \cdots, r_1^3 + r_2^3 + \cdots, \dots)$$

So, this is what the theorem says. The theorem says that there is a direct relation between the cycle index of the group and the counting of this inequivalent colourings where the number of occurrence of each colour is fixed.

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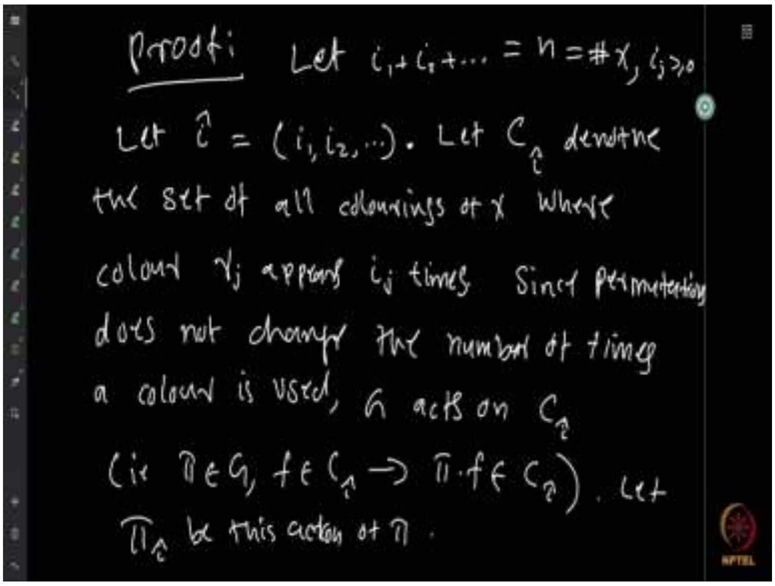

So, how do you prove this? To prove this suppose $i_1 + i_2 + \cdots = n$. So of course when we have only n elements we cannot use more than n colours even though the set of colours is infinite, the tuple that we are going to consider which is going to contribute anything is going to be always adding to n which is the cardinality of X.

So, let $i_1 + i_2 + \cdots = n$ and $i_j \geq 0$. Let $\hat{\imath} = (i_1, i_2, \ldots)$. Now let us say that $\widehat{C_\imath}$ denote the set of all colourings where the number of times the colour r_j appears is exactly i_j. So, as I mentioned before the permutations can only take such colourings in the colourings in C_i to colourings in C_i, because the number of times the colour occurs cannot be changed by the permutations.

So, therefore we can see that the group G whatever is the subgroup of the symmetric group we are taking, that group G acts on this restricted colouring $\widehat{C_\imath}$. That is any permutation π if you take and any colouring in $\widehat{C_\imath}$ you take, $\pi . f$ is again an element of $\widehat{C_\imath}$. So, this restricted action of this group, I mean of each of these permutations let us say it is denoted by $\pi_{\hat{\imath}}$

So, the action of π on $\widehat{C_\imath}$. is denoted by $\pi_{\hat{\imath}}$. For each $\hat{\imath}$, $\widehat{C_\imath}$ basically gives a partition of all such colourings and then for each partition the permutations acts within that partition. So, that is what we were saying.

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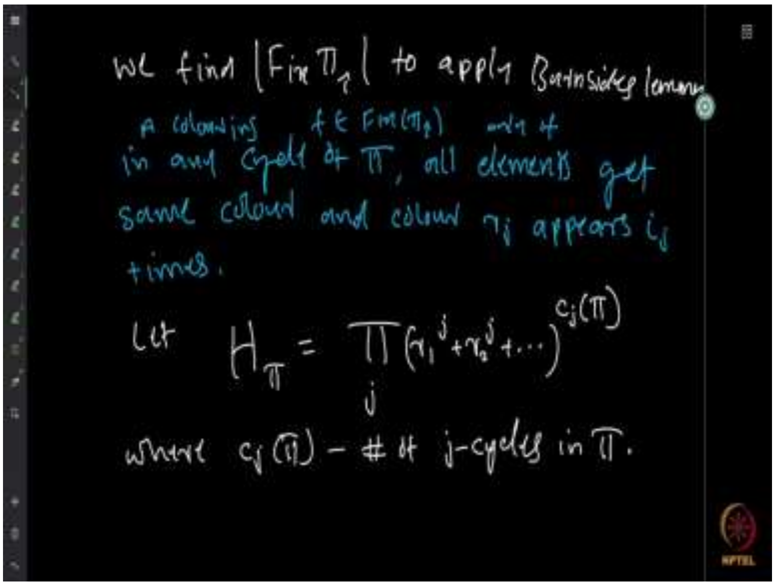

Now we want to apply Burnside's Lemma, so therefore we want to find the cardinality of $Fix\ \pi_{\hat{\imath}}$. Now how do you find the cardinality of $Fix\ \pi_{\hat{\imath}}$? If you look at any colouring f then this colouring is fixed by $\pi_{\hat{\imath}}$ only if in any cycle of π all the elements get the same colour by this colouring f.

If you are looking at a colouring f, then all elements of this particular cycle that we are looking at, π should be given the same colour by the f. Otherwise when you rotate it, it is going to give a different colour. So, therefore that will be the same and of course the colour r_j must appear exactly i_j times because by definition, in $\hat{\imath}$, the action of π and $\hat{\imath}$ is going to basically map a set of vertices with the same colour to some other set of vertices exactly the same number.

So, with this observation let us define

$$H_\pi = \prod_j \left(r_1^j + r_2^j + \cdots\right)^{c_j(\pi)}$$

where $c_j(\pi)$ is the number of j-cycles in π. So, if π is the permutation which is acting then correspondingly we define H_π to be the polynomial as above.

Now let us see what happens in this H_π. See, when you take the expansion of product you will get different monomials, that is what the polynomial is, it is the sum of monomials. Now how do we get one of these monomials? So, to get a monomial like this we have to choose some term let us say some r_k^j from each of the factors, so each factor in this product we have to choose some element.

So, we get this huge product of different sums and then a monomial in the whole product is going to come from by choosing some r_k^j from each of these terms.

Now what is the choice of r_k^j ? r_k is the variable which denotes the colour r_k. So, r_k^j says that some j-cycle is coloured with r_k. All the elements in some j-cycle is getting the colour r_k. Now we know that in this product you can take some r_i^j we can choose exactly c_j times.

So, therefore choosing a term of the type r_k^j from every factor says that we are going to colour the set X that we are colouring, such that every cycle is going to be monochromatic. That is what it comes to. So, all elements of the cycle get the same colour that every cycle is monochromatic. Now the product of these terms will be some monomial something like $r_1^{j_1}\, r_2^{j_2}$ ... where we have used the colour r_k number of j_k times.

So, this corresponds to whatever monomial we are looking at. Now it follows that the coefficient of this term that we are looking at $r_1^{j_1}\, r_2^{j_2}$ ... is actually equal to the number of elements fixed by the permutation $\pi_{\hat{\imath}}$. The action of π is $\pi_{\hat{\imath}}$.What is the $Fix(\pi_{\hat{\imath}})$ that is precisely the coefficient of $r_1^{j_1}\, r_2^{j_2}$ ...

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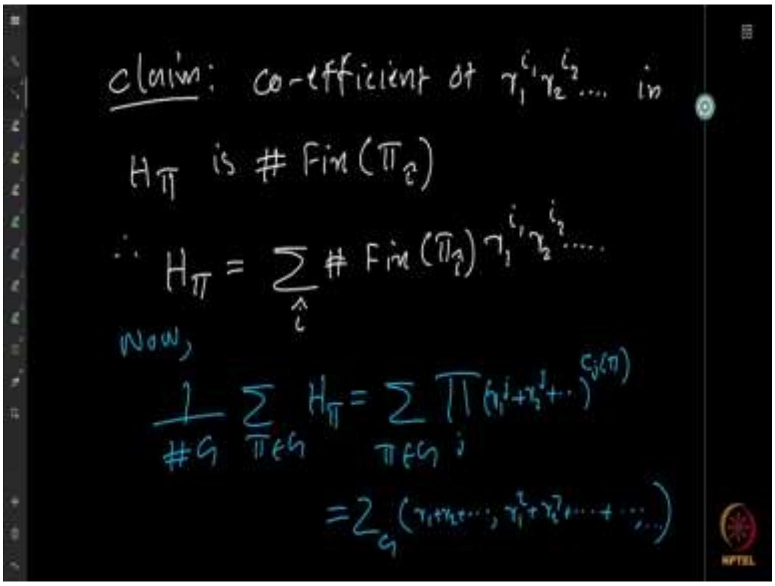

So, that is that was the claim that we want to make the coefficient of $r_1^{i_1} r_2^{i_2}$ ... in H_π is $\# Fix(\pi_{\hat{\imath}})$. Now therefore we can write

$$H_\pi = \sum_{\hat{\imath}} \# Fix(\pi_{\hat{\imath}}) \; r_1^{i_1} r_2^{i_2} \ldots$$

This is what we just argued before.

So, now we can apply the result that is

$$\frac{1}{\#G} \sum_{\pi \in G} H_\pi = \sum_{\pi \in G} \prod_j \left(r_1^j + r_2^j + \cdots\right)^{c_j(\pi)}$$

$$= Z_G(r_1 + r_2 + r_3 + \cdots, r_1^2 + r_2^2 + \cdots, r_1^3 + r_2^3 + \cdots, \ldots)$$

$$= \sum_{\hat{\imath}} [\frac{1}{\#G} \sum_{\pi \in G} \# Fix\ (\pi_{\hat{\imath}})\ ]\ r_1^{i_1} r_2^{i_2} \ldots$$

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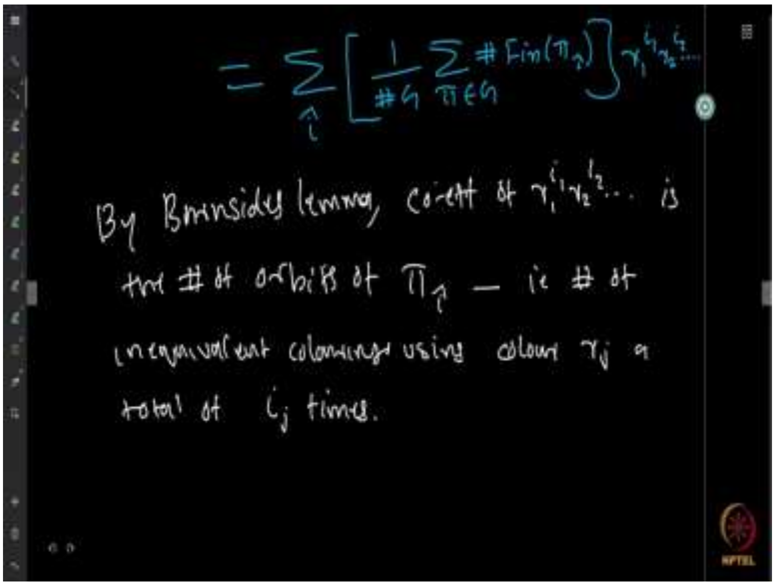

But now what is the coefficient of this term in this summation. This is by Burnsides Lemma, this is precisely the number of orbits of $\pi_{\hat{i}}$. That is how the Burnsides Lemma was stated. So, therefore this is precisely the number of inequivalent colourings using colour r_j total of i_j times. So, that is what the coefficient of the monomial means.

So, this is what we wanted to prove, so we proved Polya's Theorem as an application of Burnsides Lemma. What we proved is that the number of inequivalent colourings where the number of times the colour r_i occurs is given by the tuple i_1, i_2 et cetera, is obtained by looking at the cycle index polynomial of the group G, Z_G, and for the variable z_i, I replaced it by summation r_j raised to i_j.

So, this will give you the generating function of this kind of colouring. So, that is what Polya's Theorem is about. So, now let us look at some example, go back to this make sure that you understand the theorem well and then we can look at the examples.

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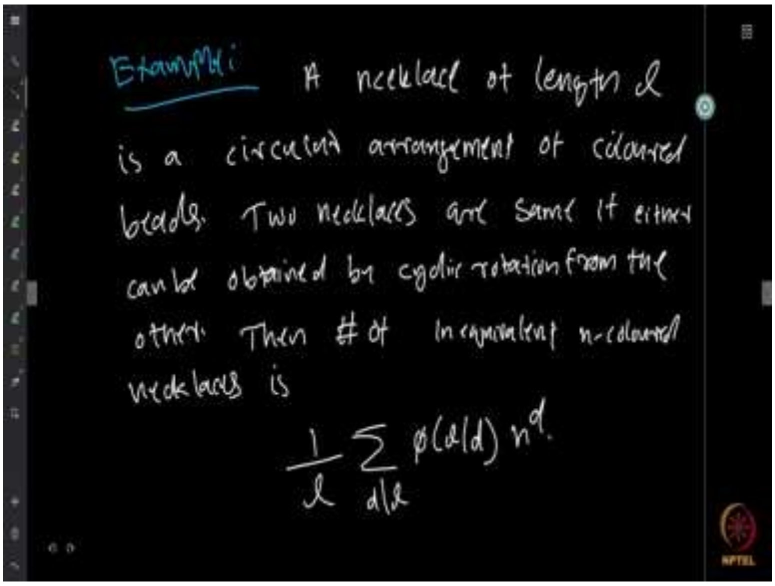

So, one of the standard examples that you give is that of counting of necklaces. So necklaces of length l is, I think we mentioned this in one of the classes long time before, it is a circular arrangement of let us say coloured beads. So, we were looking at another type of necklace at that time where instead of beads we are using kauri shells and something like that, which is slightly different in some sense but for the time being we have uniform circular beads.

So, we have a circular arrangement of this beads and we will assume that when we make the necklace the distance between any 2 beads is the same so that you get a regular polygon with the l vertices as the necklace. Now 2 necklaces are the same if one can be obtained by a cyclic rotation of the other.

So, this is something that we will assume, this is the group action that we have, the symmetry. Then the number of inequivalent n-coloured necklaces, if you are going to colour the beads with any of the n colours, it is given by

$$\frac{1}{l}\sum_{d/l}\phi\left(\frac{l}{d}\right)n^d$$

where ϕ is the Euler function, and the $\phi(l/d)$ is the numbers less than l/d and co-prime to it.

So, the Euler function ϕ is the coefficient of n^d. So, this is what we want to prove. How do you go about proving this. So, if you want to apply Polya's Theorem we want to look at the cycle index of the group, so let us find the cycle index of the group that we are looking at. The group is the group of rotations, so for the l-cycle.

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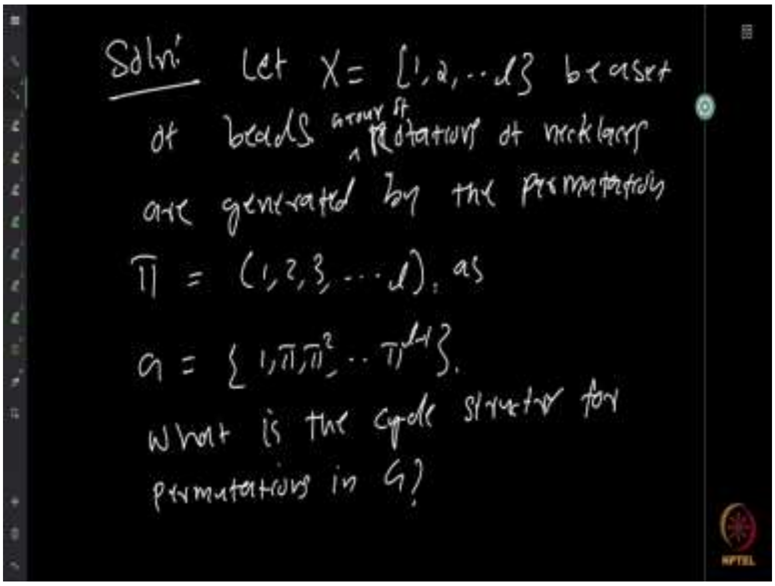

So, let us say that our set is X ={ 1, 2, …, l} is the set of beads and the group of rotations of the necklace are generated by the permutation $\pi = (1, 2, \ldots, l)$. So, if this is the generator then the group G is given by $G = \{ 1, \pi, \pi^2, \ldots, \pi^{l-1}\}$. Now what is the cycle structure for permutations in G? This is a very interesting example. Can you think of some nice properties of the cycle structures of the permutations in G. So, if you are looking at the group of rotations what can you say about cycle structure.

So, all the cycles in each of the permutations will have the same length that is what I wanted to say. So, all the permutants will have the same length. So, can you think of why? Or can you prove this is true? For every permutation in G all its cycle lengths will be the same. So, if you write any permutation in G as a product of cycles, then all the cycles will have the exactly the same length. So, prove this.

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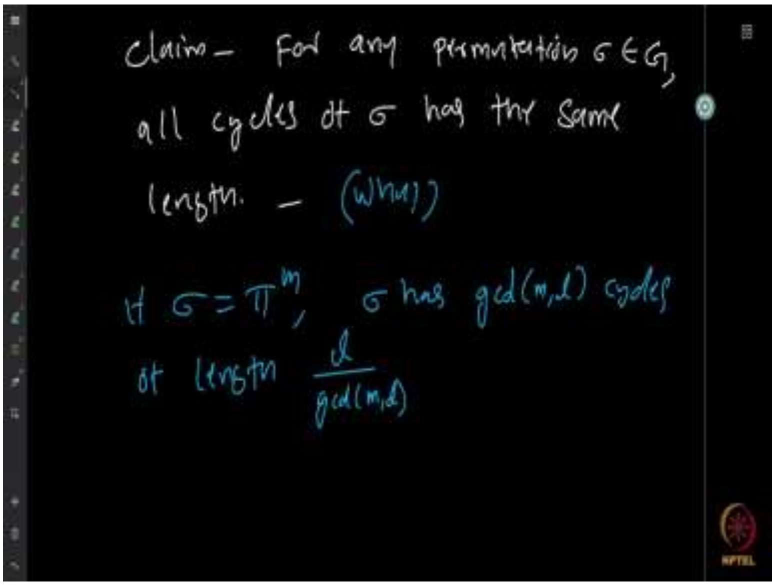

So, why is this? Well if you look at the rotation and then try to represent it as using some modulo arithmetic you can easily prove this. And observe that if $\sigma = \pi^m$, where π is generator, then σ has exactly $gcd(m\,,l)$ cycles of length $\frac{l}{\mathrm{gcd(m\,,l)}}$. So, this is an even more refined claim so prove this. Once you have this we can apply Polya's Theorem. Polya's Theorem says that once you have the cycle structure we can directly apply this.

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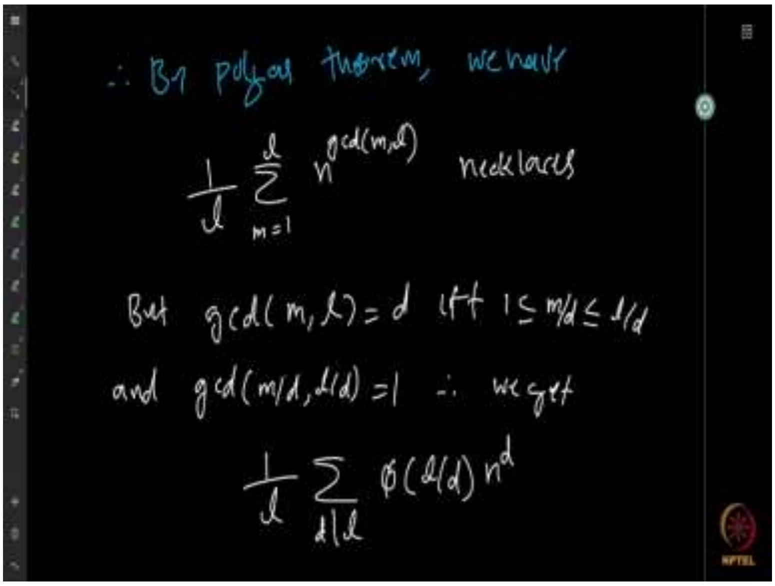

Therefore, by Polya's Theorem, we have, $\frac{1}{l}\sum_{m=1}^{l} n^{\mathrm{gcd(m\,,l)}}$ inequivalent coloured necklaces. But what we know is that $\mathrm{gcd(m\,,l)} = \mathrm{d}$ if and only if $1 \le \frac{m}{d} \le \frac{l}{d}$ and gcd $\left(\frac{m}{d}, \frac{l}{d}\right) = 1$

Using this we can write this same summation $\frac{1}{l}\sum_{m=1}^{l} n^{\gcd(m,l)}$ as $\frac{1}{l}\sum_{d/l}\phi\left(\frac{l}{d}\right)n^{d}$. So, therefore we can write as in this form. So, this is what we were asked to prove.

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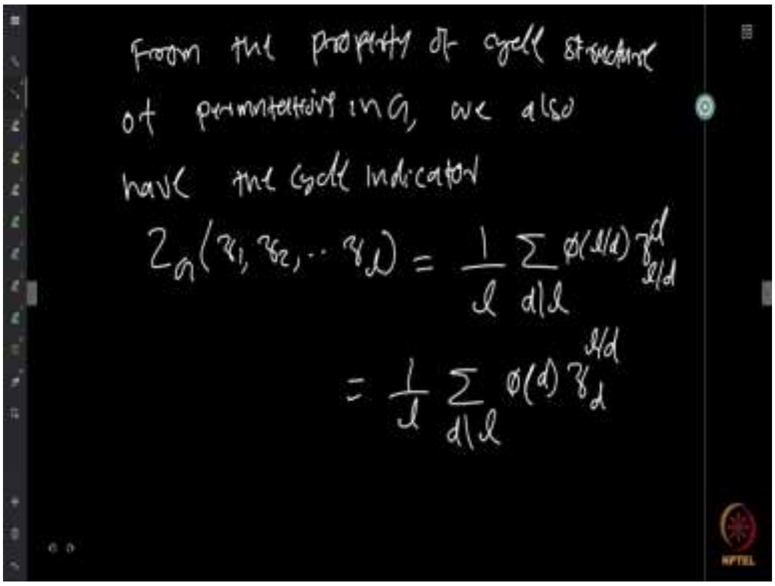

Now as a further observation, the property of the cycle structure of the permutations in G, we have the cycle indicator

$$Z_G(z_1, z_2, \dots, z_l) = \frac{1}{l}\sum_{d/l}\phi\left(\frac{l}{d}\right)z_{\frac{l}{d}}^{d}$$

This again directly follows from the property of the cycle structure because all the other terms will be disappearing, they will not be here. So, therefore we will get exactly this. And but now I can write it also as $Z_G(z_1, z_2, \dots, z_l) = \frac{1}{l}\sum_{d/l}\phi(d)z_{d}^{\frac{l}{d}}$, because I am just exchanging d and l/d because they are just divisors. we will see that the connection between ZG and this holds.

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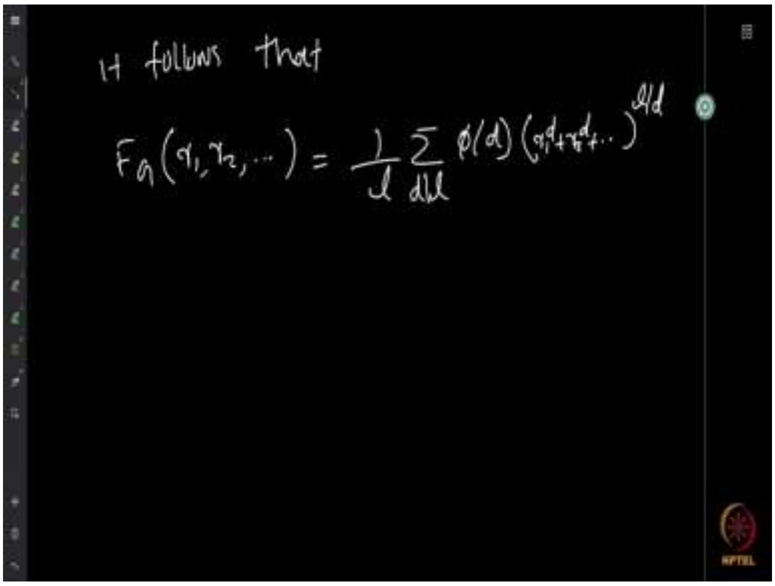

So, we wanted to find out FG.

$$F_G(r_1, r_2, \dots) = \frac{1}{l}\sum_{d/l} \phi(d)\left(r_1^d + r_2^d + \cdots\right)^{\frac{l}{d}}$$

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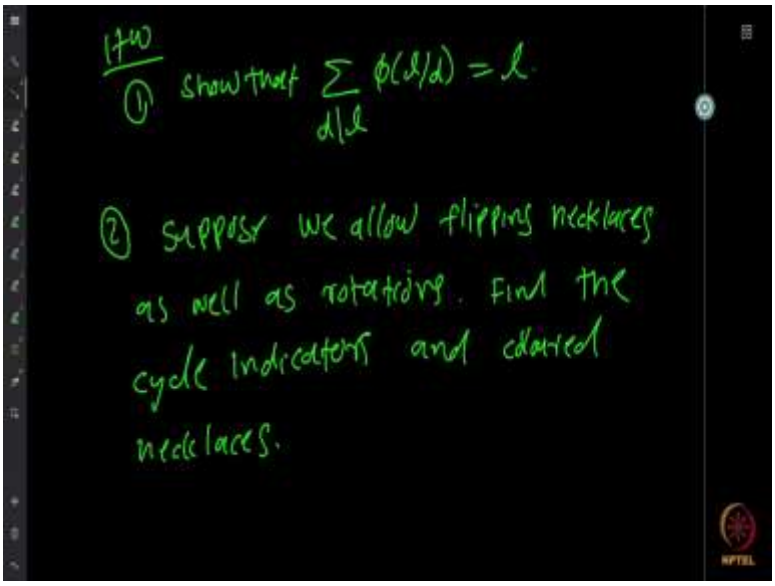

So, this is how we use Polya's Theorem. Here are some homework questions.

(1) Show that $\sum_{d/l} \phi\left(\frac{l}{d}\right) = l$.

(2) If we allow flipping of necklaces as well as the rotations, instead of the just rotations we also allow flipping or taking mirror images. Find the cycle indicators and the coloured necklaces. So, try to solve these question.

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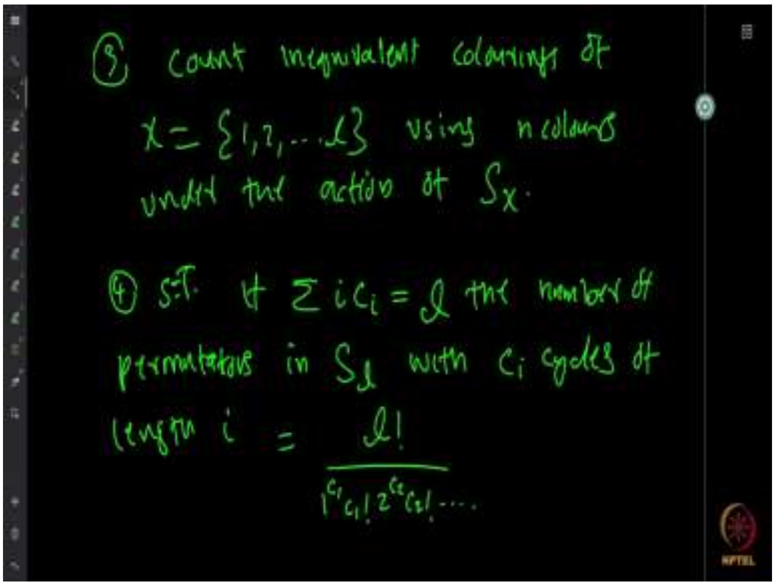

(3) Count the inequivalent colourings of a set of l elements using n colours under the action of the symmetry group S_X.

(4) Then show that if $\sum ic_i = l,$ then the number of permutations in S_l with the c_i cycles of length i is given by $\frac{l!}{1^{c_1}c_1!2^{c_2}c_2!\ldots}$

So, these questions are your homework questions. So with that we will windup the topic on Polya’s theorem. We can do more on this but let us stop with this for the time being. I will try to give you more questions if you want, there are several interesting questions, but at the moment these are your homework questions and then I will come up with more questions and send you soon.