Combinatorics
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Basic Counting - the sum and product rules

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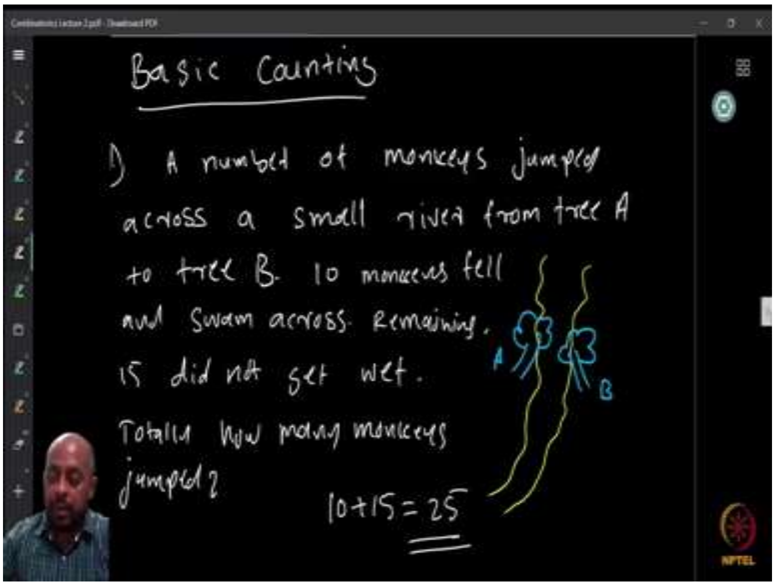

So, welcome to the second week of this course on Combinatorics and in the previous week we looked at the pigeonhole principle and some related the questions. This week we are going to look at some basic counting techniques which will be useful for developing new techniques as well as for any question that you might look at Combinatorics, any of these questions will most of the time need some of these basic techniques and their understandings.

So, we are going to start with a simple question and then we try to see, how to develop this into technique and then that is the idea. So, here is the question that, a number of monkeys jumped across a small river from tree A to tree B. So, we have these two trees, let us say A and B and then, the monkeys jumped from one three to other.

Now, when they were jumping, 10 of monkeys fell down to the river, then they got wet, then they swam across, what we, now further is where the remaining 15 monkeys did not get wet. Now, the question ask is to find what is the total number of monkeys that jumped?

Now this is the very easy question, any student of mathematics should immediately be able to tell that since, we have a, 10 monkeys who fell to the river and 15 who did not, we have 10+15 = 25. So, 25 monkeys basically jumped from river in one of the tree to the other on the river side. It is the answer that we all know.

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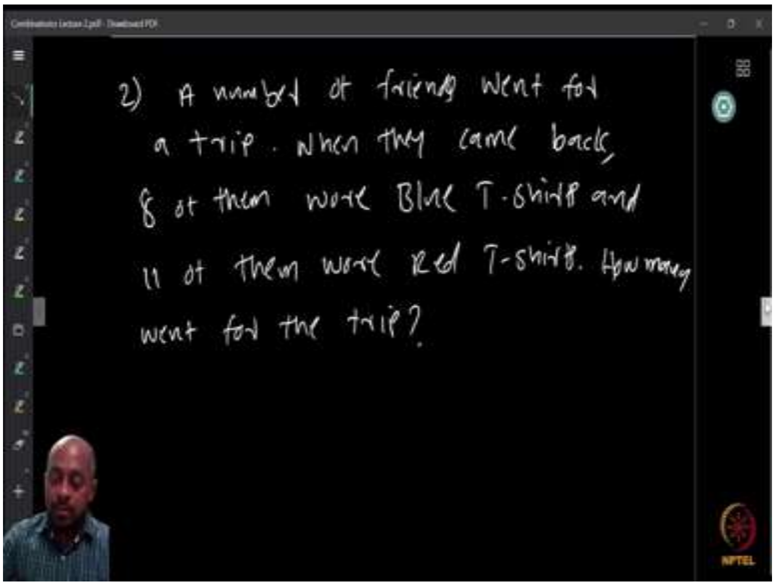

Now, let us look at a slightly different question, a number of friends went for a trip, somewhere. Now, when they came back, 8 of them wore blue T-shirts and 11 of them wore red T-shirts. Now, question is that, how many went for the trip? Now, is it possible, to answer that this is 11 + 8 = 19. But, in fact no, because, what we know is that, there were 8 people wearing blue T-shirts and 11 wearing red T-shirts. Now, we definitely know that there were at least 11 + 8, 19 peoples who went for the trip.

But, now the question does not say anything about the other people who might be wearing let say yellow or green or other shirts. So, now the question does not list exhaustively all the people who went for the trip and therefore we really do not know whether, we have more than 19 or more. So, we cannot really answer it, except showing that okay there were at least 19.

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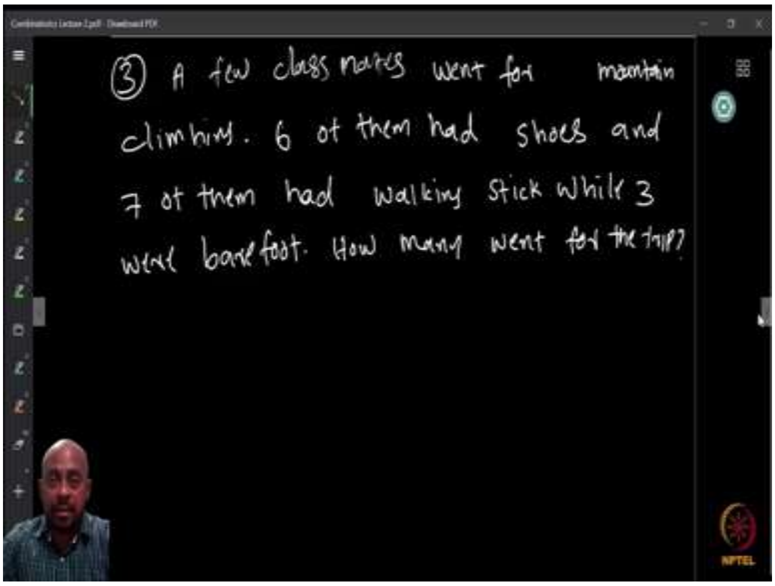

Now, let us look at a slightly more different question, we have a few classmates who went for a mountain climbing, another trip, 6 of them had shoes, they are wearing shoes and the 7 of them had walking sticks with them. While 3 were barefoot, now how many went for the trip? From here there are two problems, we do not know whether the question is exhaustive, whether it covers all the people but, even if let say that, suppose we are giving that the remaining 3 peoples were barefoot.

We still do not know, may be, the 7 of them who were having walking sticks, were precisely those having shoes and who were barefoot or may be some of them had something else, like boots or something. So, this question does not say, whether there could be intersection between those people or like whether there is some other thing, they were separately 7 peoples. So, this question we do not know so, therefore we cannot really answer anything about this. So, these 3 tells us something, in the first question we were able to answer precisely. Because, we knew that both the sets were disjoint and they were exhaustive.

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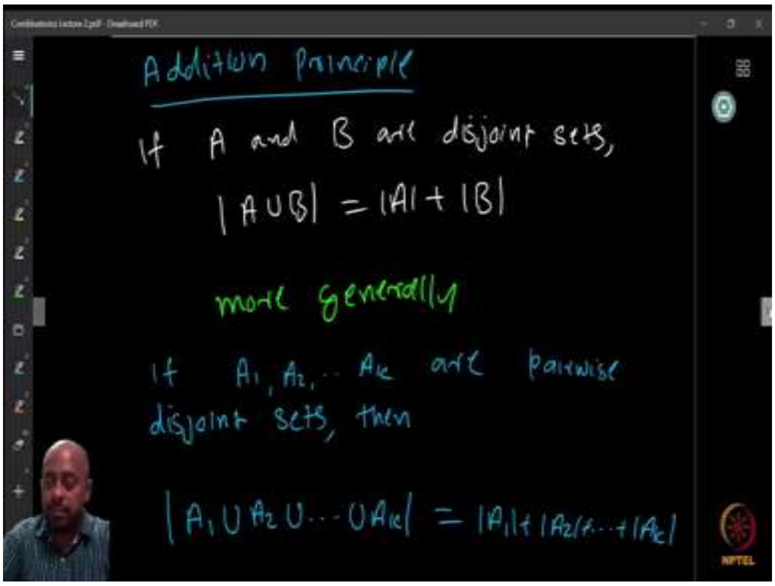

So, if we are given that, if A and B are disjoint sets, and then they are union, we can say, as cardinality equal to the sum of cardinalities of each other. That is $|A \cup B| = |A| + |B|$. So, this is called addition principle. So, if we want to use addition, to count things then we can use this principle but, we require that, they had to be disjoint. So, if A and B are disjoint then we can find $|A \cup B| = |A| + |B|$.

On the other hand, if we have more we can still, now we can still use this but, we need the pairwise disjoint condition to be able to use this. So, suppose $A_1, A_2, \ldots, A_k$ are pairwise disjoint sets, then the union has cardinality equal to the sum of cardinalities of the individual sets. That is $|A_1 \cup A_2 \cup \ldots \cup A_k| = |A_1| + |A_2| + \cdots + |A_k|$. But, it is important to note that, $A_1, A_2, \ldots, A_k$ are pair wise disjoint. It is not sufficient to say that, let say that there are A_1, A_2 and A_3 and $A_1 \cap A_2 \cap A_3 = \phi$.

So, $A_1 \cap A_2 \cap A_3 = \phi$ says that A_1, A_2 and A_3 are all together disjoint. They do not have a common element. But, it does not say that $A_1 \cap A_2 = \phi$ and $A_1 \cap A_3 = \phi$ and $A_2 \cap A_3 = \phi$. So, this is required to be able to use the addition principle. So, we have pair wise disjoint sets, then we can also count. So, this is the generalized version of addition principle.

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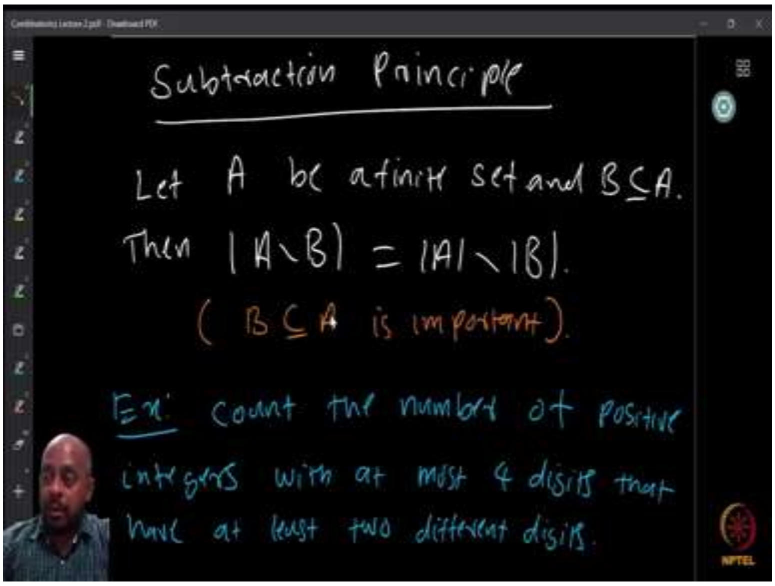

Now, so here is a subtraction principle, so what is subtraction principle? Says that if A is a finite set, and $B \subset A$, then $|A \setminus B| = |A| - |B|$. So, earlier we were looking it, when we can add. Now, we are asking when we can we can subtract?

Note that, $B \subset A$ is very important, if B is not subset of A, this principle cannot be directly applied. So, here is an example let a say count the number of positive integers with at most 4 digits that have at least two different digits. So, we have to count the number of positive integers with at most 4 digits that have at least two different digits. Now, one can directly count this, I recommend that you, to spend some time, finding out how to directly count this, rather than using subtraction principle.

And then see, why it is going to be better to use subtraction principle here. So, how to we use the subtraction principle? So, what are we given? We are given that, we have the set of positive integers with at most 4 digits, that is what we are looking it. With some extra properties. So, we can consider your big set as a set of all positive integers with at most 4 digits, how many are there? Well, we know that, the smallest number with 5 digits is a 10,000 so that for up to 9999 right, 9999.We have integers,

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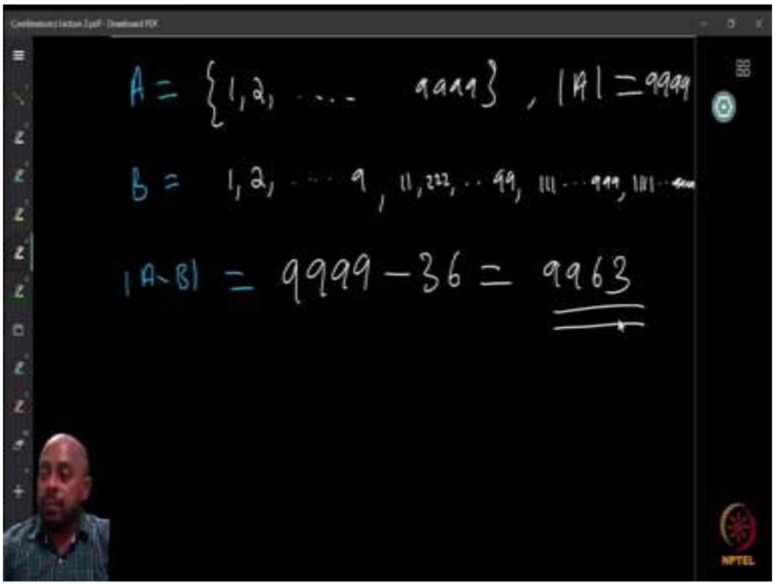

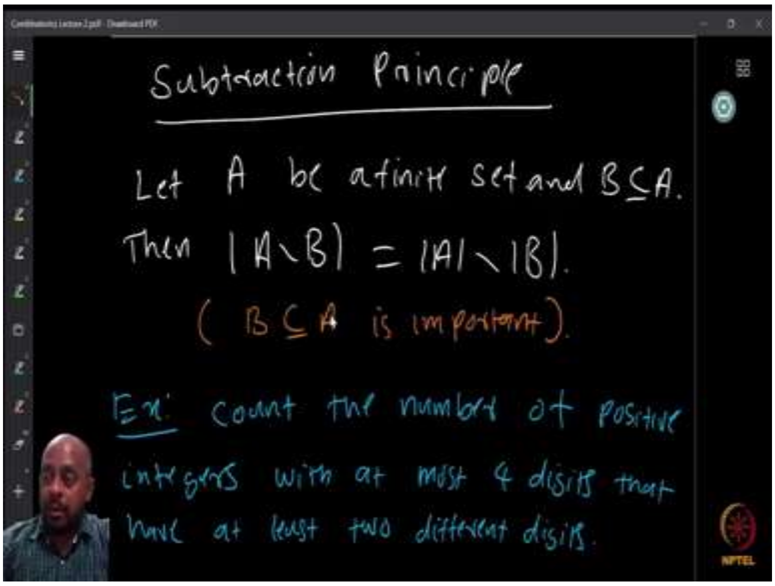

and they have a cardinality exactly, I think I made the wrong tool, so take 1, 2, etc, let us say 9999. Then this set has cardinality is equal to 9999. So, we have these many elements in our big set. Now, we want to be able to find another set B with the property that, if you subtract, a cardinality of B, we get what we want. We want to find out the sets. You know we want to find out the number of elements that have at least two different digits.

Now, so the natural set that we can look at, is all those integers with at most 4 digits, but having only one, there is no, there is no two distinct digits. So, exactly the same digit is repeated, for all these numbers. Now, this is very easy to count, so what is B consisting of it has all the numbers 1 to 9, write 1, 2 etc. 9. Then, it has a 11, 22, 33, etc., again there are 9 of them 99.

Then you can have 111, 222, etcetera, up to 999. Similarly, 1111 etcetera, 9999 with the precisely those with the 111, 222, etcetera., not all the numbers in between.

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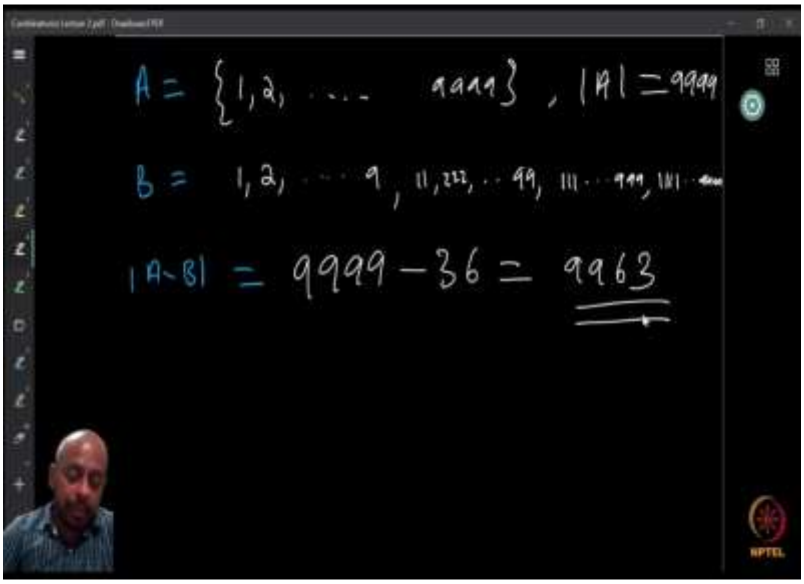

So, here we have exactly 9, here exactly 9, here exactly 9 and here also exactly 9. So, this is that we have exactly 36 elements in B. So, 36 numbers has its property that, it uses only one digit, repeatedly but only one distinct digit is appearing, I mean unique digit is appearing. So, therefore, we can find out the $|A\backslash B|$, which says, so $A\backslash B$ is the set of numbers with the property that we were looking. Less than 9999, less than 10,000 and having at least two different digits.

So, what is this? This is $9999 - 36 = 9963$ and what is this number? Whatever it is 63, 9963. So, we used subtraction principle to count this. But, again you can verify that if you can count without using subtraction principle directly and may be you can see that like this is much more efficient way to do the counting.

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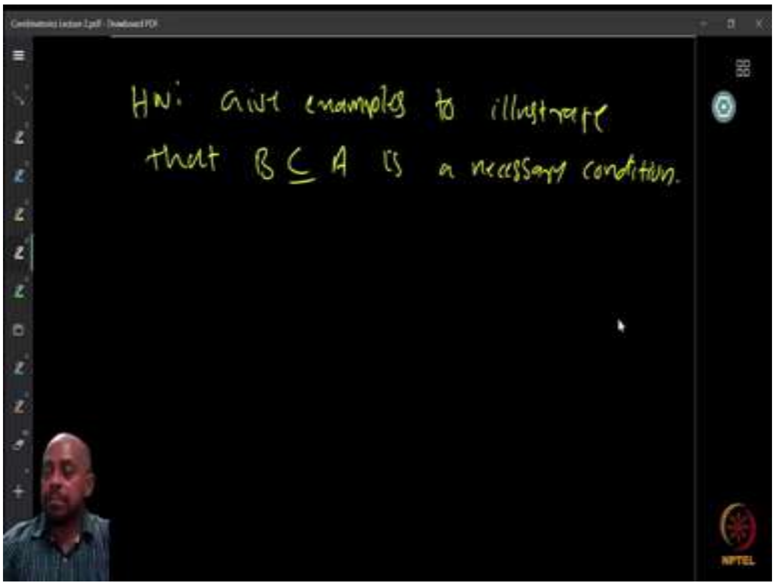

So, now as I told you, we want the condition that $B \subset A$, to be able to use the subtraction principle. So, I want you to come up with some examples that illustrate that $B \subset A$ is necessary. Other ways the counting will go wrong. So, come up with some examples. It is very easy to come up with but, still think about it, come up with the couple of different examples. So, this is homework.

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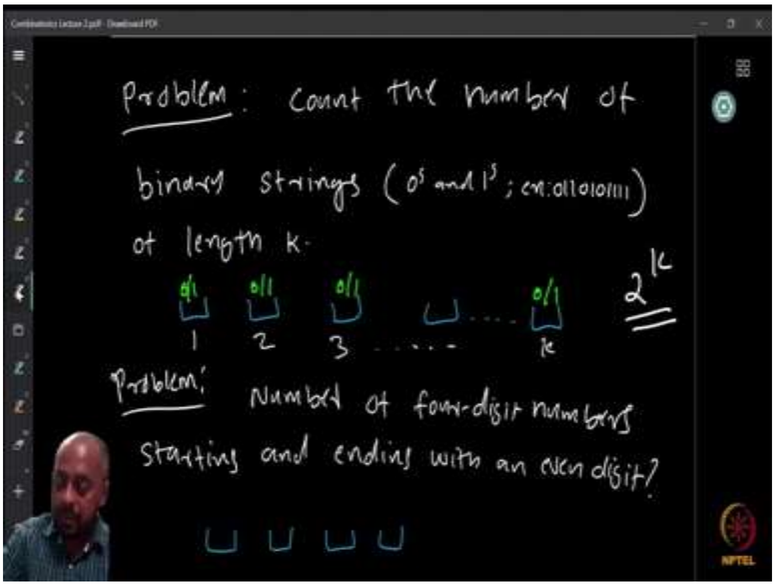

Now, let us look at another problem so, we already studied two different principles, addition principle and subtraction principle. Now, let us look at another problem, to count the number of binary strings of a length k. The binary strings are the strings that we create with just 0s and 1s. So example is like 011 01, or 11 00 111 so, all these are binary strings.

Now, we want to look at strings of length exactly k. How many are there? Now how do you, how do you count something like this? So, we can try to find out like for like 1 for example 2 so for 1, if the length is exactly 1, we have either 0 or 1. There are two possibilities and that is it.

If you are looking at length 2, then you have 00, 01, 10 and 11, there are four of them. Now, once you do this, we can try to come up with an idea and then try to use some other like induction or something to do this that is one possibility. And now, let us look at another way to look at this. So, what we know is that we are going to look at strings of length exactly k. So, we have a string of length k means that there are k positions, position 1, position 2, position 3 etcetera, position k. So, we have strings with the k positions.

Now, if you look at any position, what are the possible letters that can appear here? Either 0 or 1. So, at each position we have only the possibility to write either 0 or 1, 0 or 1, 0 or 1, 0 or 1. So every position has exactly these two possibilities. Now, I can put 0 or 1 at the first place and without worrying about what happened in there, I can put 0 or 1 at the second place one of them, exactly one of them I can put at each of the k different places.

So, this says that I have exactly two choices, two possibilities for the first position 0 or 1, exactly two possibilities for the second position and the first position whichever I choose, I can still choose the second position without worrying about what happened there. So, they are basically independent, so I can decide this independently like what to write here. So, therefore what we know is that? If I have two choices here and two choices there, then I have $2 \times 2 = 4$ possible choices for the strings of length 2.

So, I can basically multiply the choices available at the position 1, position 2 etcetera, position and multiply them together, to get the number of strings. So, the number of strings of length k is going to be $2 \times 2 \times \ldots \times 2$, k times. So, that is 2^k so we immediately get that the number of binary strings of length k is equal to 2^k.

So, now let us look at another question, we want to find the number of 4 digit numbers, starting and ending with an even digit. So, how will you calculate this? Again I want you have to pause the video and think about it for a few minutes if you want, maybe you can get immediately but, think about it and then find your own answer, and then if then you proceed. So, how do we do this? So, we have 4 digit numbers, and then it can start and end with even digit. So, this put some restrictions on what are the possible numbers that you can give, for example, if you are looking at the first position.

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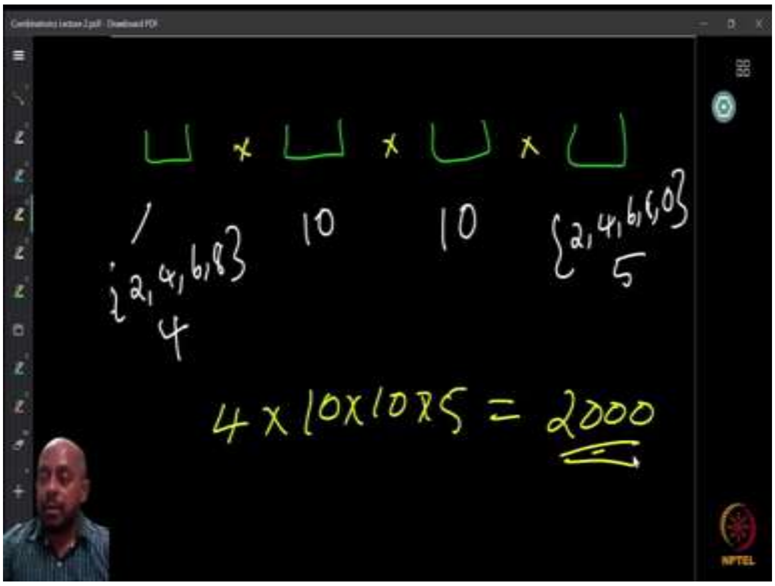

The first position, can you use, let us say, you are using the 4 positions are here, now in the first position, can you use like you are only allowed to use even digits so, therefore what are the even digits like 0, 2, 4, 6 and 8. But, 0 cannot be used in the first digit because, we are talking about numbers, it is not string. So, therefore we are not allowed to use 0 so, we can use 2. 2 comma 4 comma 6 comma 8 any of these numbers we can use here. Now, it should start and end with even digits but that does not say that, the middle digits can be anything. So, middle digits can be any of the possible numbers 0 to 9. So, therefore there are 10 possible choices here.

Similarly, third digit can also be any of the 10 possible digits, what about the last digit it must be an even digit so, what are the possibilities? 2, 4, 6, 8 but, here we also can have 0. So there are 5 choices here, 4 choices here. But, now these can be independent, no matter what I choose at the first place, I can choose any of the 10 digits in the second place. So, I have 4 choices here and independently I have 10 choices here so, 4 into 10. Similarly, another independent 10 choices here, I have another into 10 and into 5. So, therefore I have total number of choices is 4 into 10 into 10 into 5 which is equal to 2000. So, we have exactly 2000 numbers with this property.

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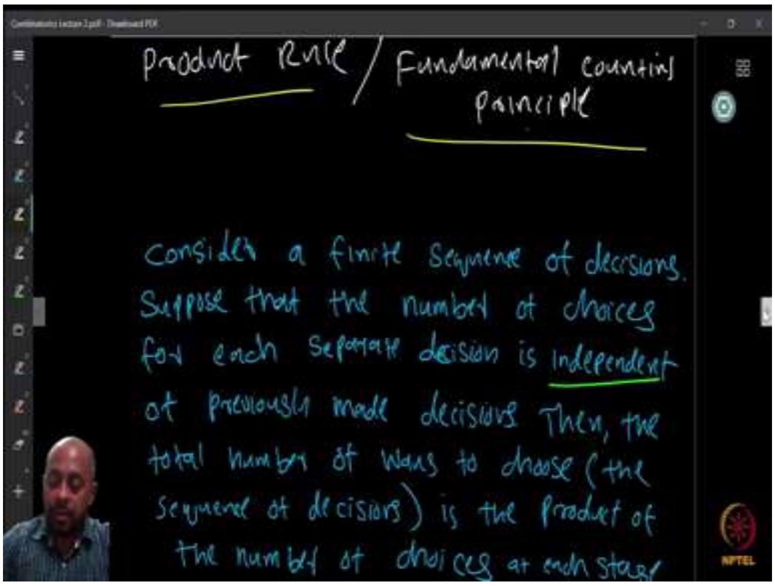

Now, this tells us a general principle, which we call the product rule or fundamental counting principle. So, what does this exactly state? So, we can state the fundamental counting principle as follows. So, consider a finite sequence of decisions so each, so sequence means that, first time you make a decision, then there is another place where you can make another decision this way.

So, consider a finite sequence of decisions, and suppose that the number of choices for each separate decision is independent of the previously made decisions. So, the decisions are made in some sequence and I make the first decision, then second decision, when I take the third decision, I want to make sure that all the previous decisions does not affect the choices here. So, the choices are independent of the previous decisions.

Then the total number of ways to choose the sequence of decisions is the product of the number of choices at each state. So, we have a finite sequence of decisions and the choices, the number of choices for each separate decision is independent of the previously made decisions, then the total number of ways we choose, the sequence of all the decisions is the product of the number of choices at the each stage, so this is the product rule.

Now, we already saw, two applications of product rule but let us look at the few more. Because, this is a very, very important rule, and we will need the lots of applications of this in the following days.

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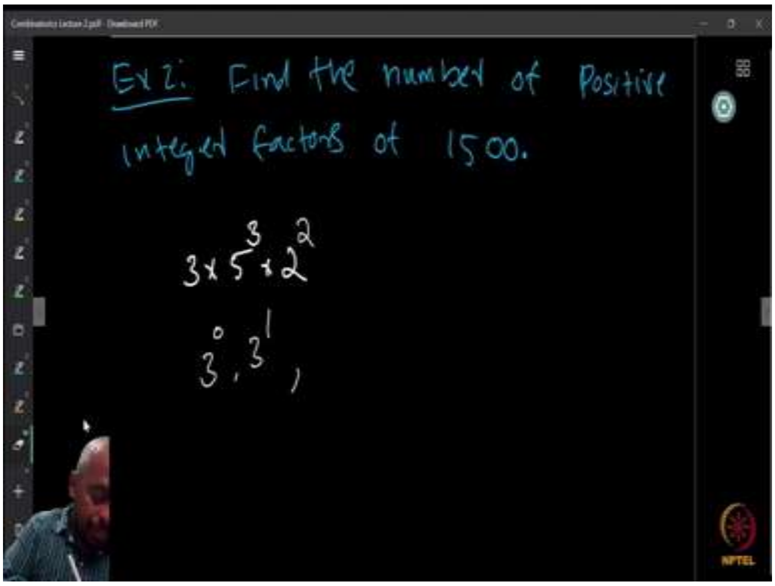

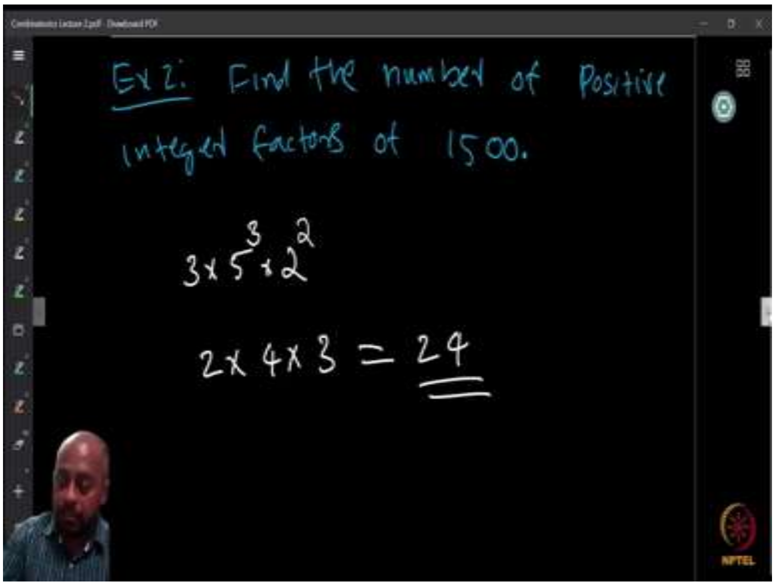

So, it is another example, find the number of positive integer factors of 1500. So, how can you use product rule, to find out the number of factors of a number given like 1500 here. So, again, try to work it out yourself, stop the video, watch and then get back. So, how you are going to do this? Now, what we know about 1500? 1500 I can write it as it 3 into 5 raised to so, how many so I basically, I look at the prime factors here.

So, I know that there is a 5 appearing 3 times, 5^3, and then how many of times 2 is appearing? 2^2 so, 25 × 4 = 100 then, 3 × 5 =15. so, $3 \times 5^3 \times 2^2$ is the prime factorization of 1500.

Now, how can you use the prime factorization, now to find the number of positive integer factors? So, if you look at any factor, we know that the factor is basically a product of the possible subset of the prime factors. So, therefore we can just look at how many such possibilities are there? So, since 3 is appearing only once, my choice is to have either 3 to appear in the factor or not. So the power of 3 in the factor can be 0 or 1. So, I can have 3^0 or 3^1 as the possible factors.

Similarly, I had, I can have $5^0, 5^1, 5^2, 5^3$. There are 4 possible choices for 5. And then I have 3 possible choices for 2, like 0, 1 or 2. So, therefore what can I say, I can say that like if this number is given, then I can find out, there is a $2 \times 4 \times 3 = 24$ possible factors. So, these are the number of possible integer, positive integer factors of 1500.

So, here again we used product rule because, the choices were independent, the 3, I am choosing 3 to be one of the factors or not, is independent of my second choice, how many 5's I am going to take, or how many 2's I am going to take. So, therefore I can use this so I get the answer 24.