Combinatorics
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Proof of Burnside's Lemma

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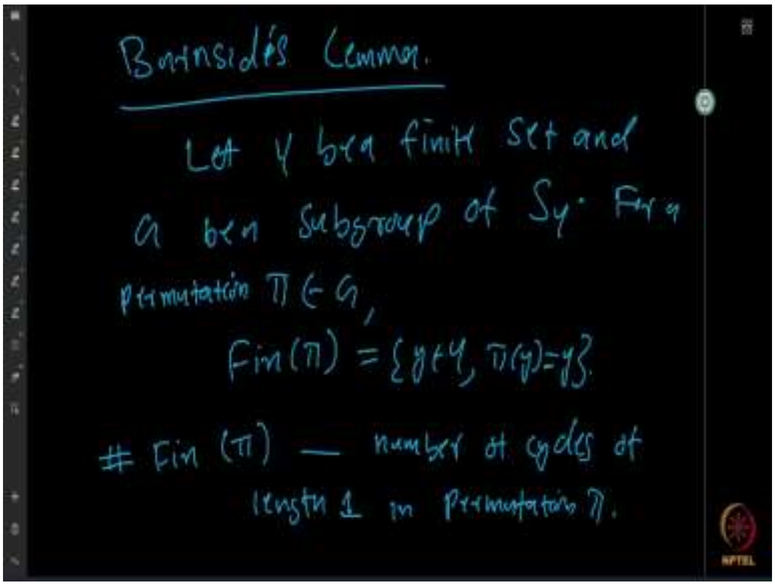

Okay! In the previous lecture we stated Burnside's Lemma which is the following. Given a finite set Y and a subgroup G of the symmetry group S_Y. Then recall that $Fix(\pi)$ is the set of all elements that are fixed by the permutation π and the cardinality of $Fix(\pi)$ is clearly the number of cycles of length 1 in permutation π.

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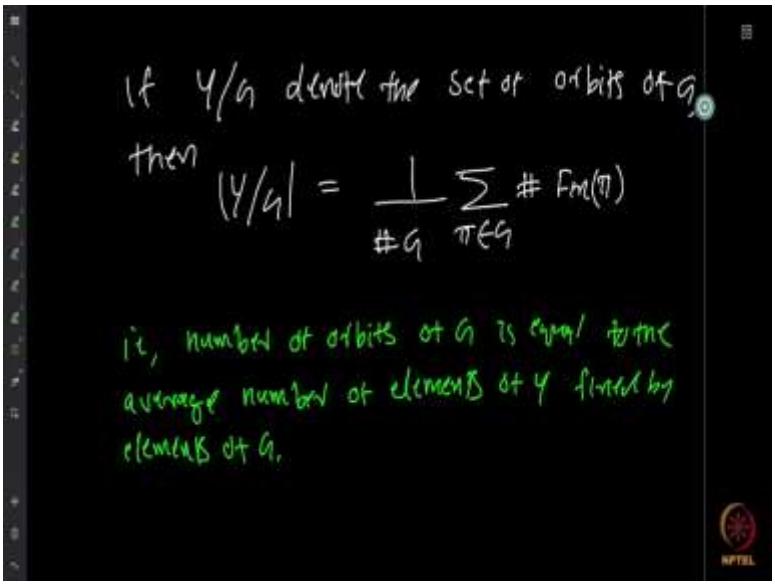

What is the orbit of group acting on a set Y? So, when you have the group acting on the set you get a partition into equivalence class because the G-equivalence is an equivalence relation and these equivalence classes are called the orbits. So, the elements which are taken by the action of a group like if I take 1 element it is going to be a subset of the elements and these subsets from an equivalence class and that is an orbit. So, we get these orbits.

Now the Lemma says that, if Y/G denote the set of orbits of G, then the number of orbits is given by the formula;

$$|Y/G| = \frac{1}{\#G} \sum_{\pi \in G} \# Fix(\pi)$$

So, the average number of elements of Y fixed by the permutations in G is actually equal to the number of orbits of G acting on the set Y. So that is Burnside's Lemma.

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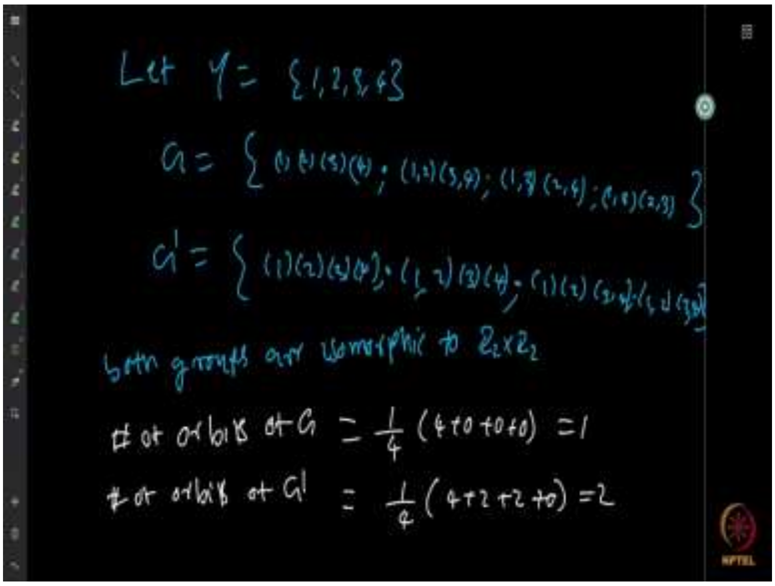

Now, let us look at an example. So, let Y = {1, 2, 3, 4} and the group G = { (1)(2)(3)(4); (1, 2)(3, 4); (1, 3)(2, 4); (1, 4)(2, 3) } and the group G′ = { (1) (2) (3) (4); (1, 2)(3)(4); (1)(2)(3,4); (1, 2)(3, 4)}. So as abstract groups both G and G′ are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

But as we have observed earlier, the action will be different because the cycle structures are different. So, the number of orbits of G is given by ¼ (4+0+0+0) =1 and on the other hand the number of orbits of G′ is given by ¼ (4+2+2+0) = 2. So, the number of orbits of G and G′ are different, for example, in the case of G you get only 1 orbit and in the case of G′ you get 2 orbits.

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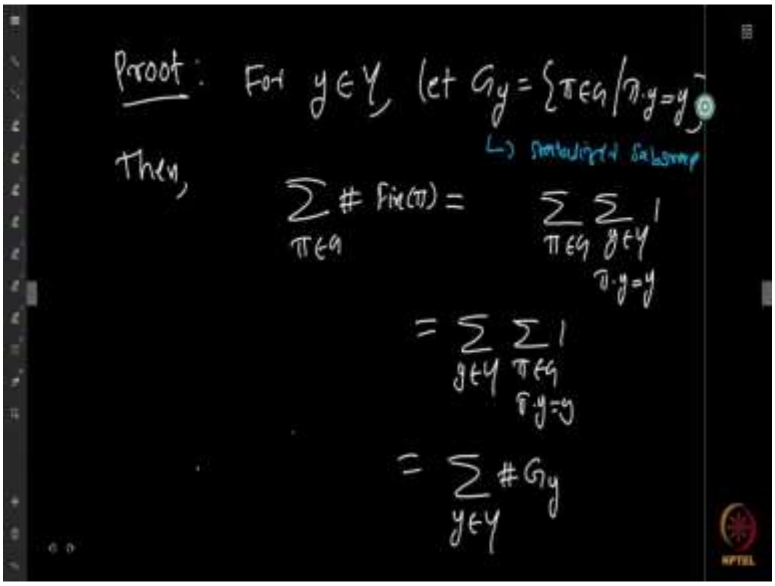

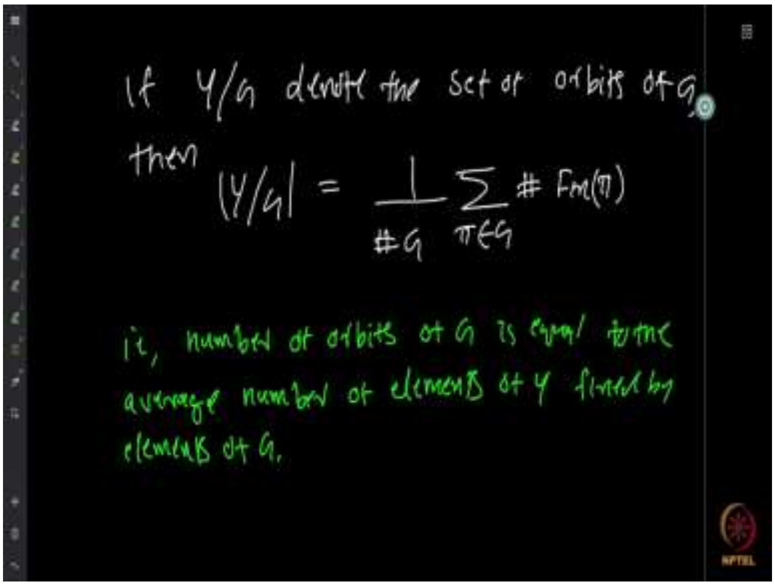

Now let us try to prove Burnside's Lemma. Now, for any element y in Y let us look at the set of all permutations which fix y, which is called the stabilizer subgroup of G, denote by G_y. So, $G_y = \{\pi \in G | \pi. y = y\}$. Then

$$\sum_{\pi \in G} \# Fix(\pi) = \sum_{\pi \in G} \sum_{\substack{y \in Y \\ \pi.y=y}} 1 = \sum_{y \in Y} \sum_{\substack{\pi \in G \\ \pi.y=y}} 1 = \sum_{y \in Y} \# G_y$$

In above summation, I changed the order in the second step because I can as well count for each element how many permutations fix this. Now that is precisely the number of permutations which fix the element y, which is cardinality of G_y, which is the stabilizer subgroup.

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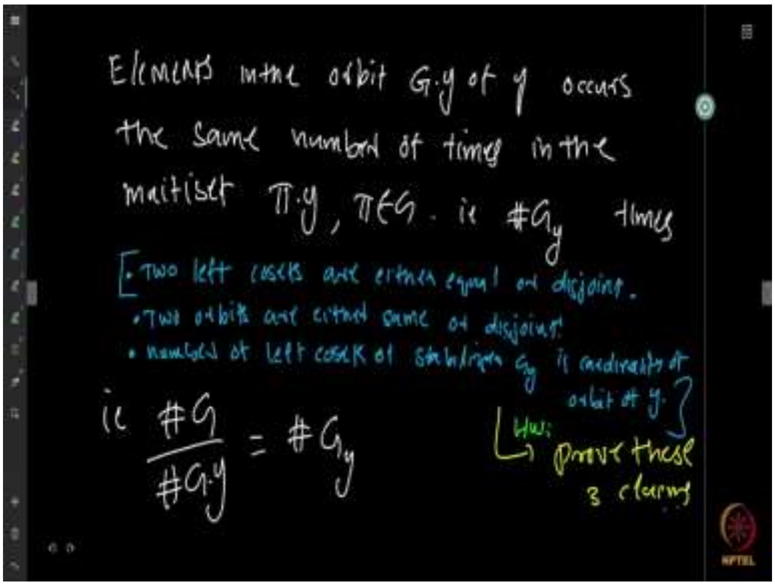

Now let us see, can we look at cardinality of G_y in another way. So, when we look at the action of the group, the elements in the orbit let us say G_y, so I denote by G_y, the orbit of y because, what are basically the orbits? So, if I take an element and look at all the other elements to which this element is mapped by elements of G. So, this form the orbit of y, so therefore it makes sense to denote it by G_y or G.y.

Now elements in the orbit G_y of y occurs the same number of times in the multiset $\pi.y$ set of all permutations $\pi.y$ for π in G. So why is that? And, how do you prove this? So, if you have the following 3 easy results like 2 left cosets are either equal or disjoint; this is something I asked you to go through or prove, then another easy result that we can show that 2 orbits are either the same or disjoint. And the number of left cosets of a subgroup is the cardinality of the orbit of y.

So, stabilize of G_y is the cardinality of the orbit of y, how many elements are there in the orbit, that is the number of left cosets of the sub group G_y. Now if you prove these 3, it will be immediately clear that the elements in the orbit occurs exactly cardinality of G_y times. So, this will give you the identity that the cardinality of G_y is actually equal to the number of elements in G by the number of elements in the orbit of y.

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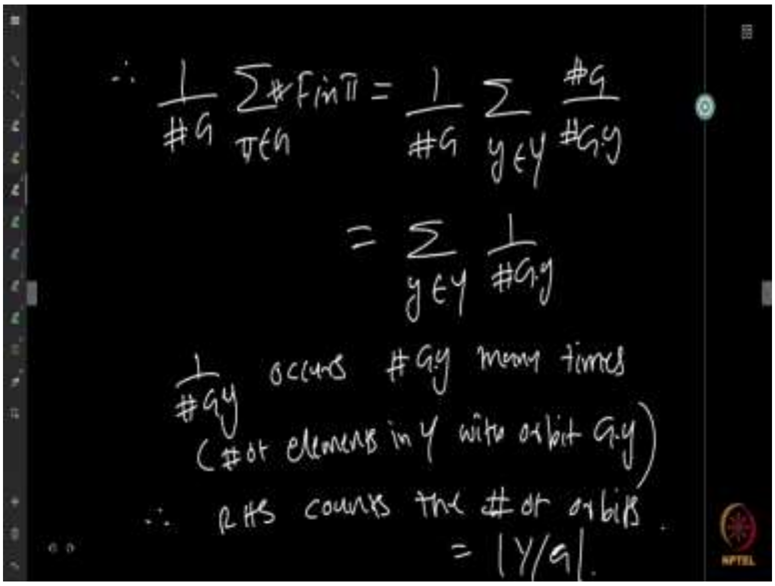

So, once you have this, we can just substitute so we get,

$$\frac{1}{\#G}\sum_{\pi\in G}\#Fix(\pi) = \frac{1}{\#G}\sum_{y\in Y}\frac{\#G}{\#G_y}$$

$= \sum_{y\in Y}\frac{1}{\#G_y}$, since order of G is fixed we can cancel it.

Now how many times this particular term occurs in this sum? So, for every element in the orbit of y, the corresponding term will be $\frac{1}{\#G_y}$ because elements in the orbit stay together under the action of the group. Now the elements in that orbit are taken to the other elements in the same orbit, it does not go out of it.

So, therefore you will see that exactly the number of elements in the orbit that many times the term $\frac{1}{\#G_y}$ occurs, which means that the contribution of each orbit will be exactly 1, because each orbit has exactly that many elements. So, therefore this is actually counting the number of orbits, the contribution of each orbit is exactly 1.

So therefore, it counts the number of orbits in the action of G on Y. So, that is what we wanted to prove,

$$|Y/G| = \frac{1}{\#G} \sum_{\pi \in G} \# Fix(\pi)$$

So, that is the Burnside's Lemma.

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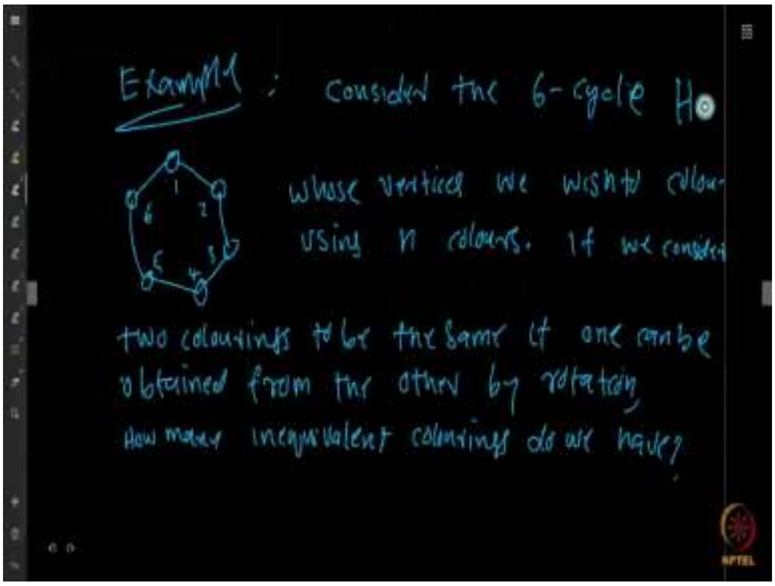

Now, let us look at an example, so let us look at the 6-cycles, let us say H, whose vertices we want to colour using n colours. So, you are looking at a regular hexagon where all the sides are equal, so that there is the symmetry. Now we want to colour the vertices using a set of n colours. Now, we say that two colourings are same if one can be obtained from the other by rotation.

So, if I can just rotate the regular or hexagon, so that the colourings remain in the position and then I rotate it then I get other colourings. So, these colourings are all considered equivalent, basically I do not distinguish between colouring obtained by just the rotation of this. Now, how many inequivalent colourings are there under this? How do you use this? So, we want to find the cardinality of $Fix(\pi)$ and then we will get the formula.

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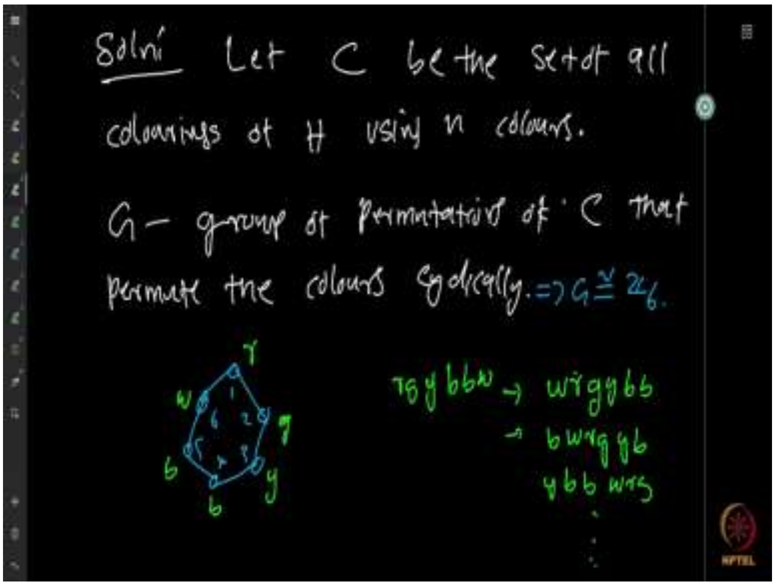

So, let us look at the set of all colourings of the hexagon using let us say n colours, so we look at the set of all colourings that is our set Y. We are going to apply the group action on this colouring. Let G be the group of permutations of C that permute the colours cyclically, basically the rotations. We can see that, since there are 6 vertices you have exactly 6 rotations including the identity and you will see that this is actually isomorphic to $\mathbb{Z}_6$.

Now of course these permutations, the subgroup of all rotations can be generated by a single element you just rotate once and then that rotation you can apply that to itself again. So, if you take this single rotation forward like1 going 2, 2 into 3 etcetera. And you apply it again and again like for example if it is $\sigma,\ \sigma^2, \sigma^3$ etcetera you will get all the elements in the group.

So, it is generated by the single element, something that we will use maybe. So, now let us look a colouring of the vertices, so let us say r, g, y, b, b, w, so red, green, yellow, blue, blue and white. Now once I rotate this, what happens? I can get for example, red going to green, green going to yellow, yellow going to blue, blue going to blue and blue going to white and white going to red.

So, I will get w, r, g y, b, b and similarly I get b, w, r g, y, b et cetera by rotations. So, I can rotate and get this kind of colourings where I assume that 1 is now given the colour w after this 1 rotation, I will get 1 will be given the colour w, so w goes here and similarly red goes to 2 etcetera.

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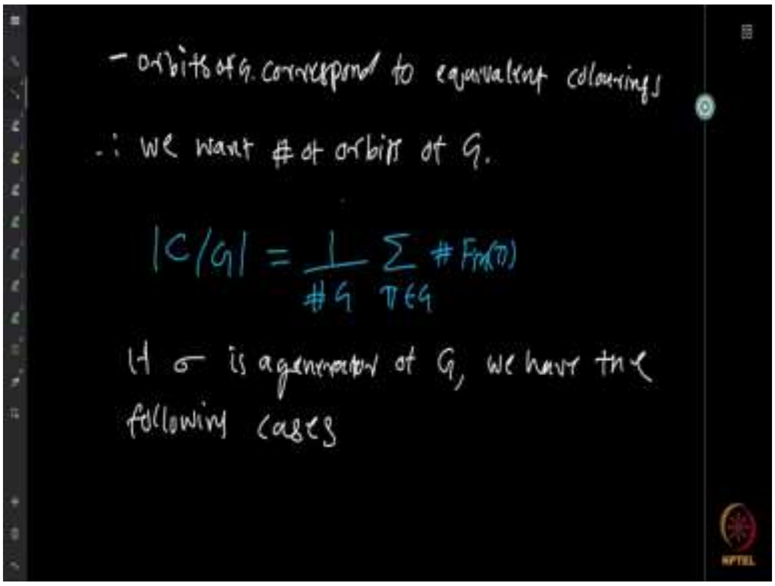

Now the orbits of the group G correspond to the equivalent colouring. So, we want to count the number of orbits because we want the inequivalent colouring. So, all the elements in the same orbit are basically equivalent. So, what we are seeing here is that these colourings are all equivalent because this is actually obtained from this by just rotation.

This can be obtained by rotation etcetera therefore elements in this now orbit of this colouring are all equivalent. So, we want to find how many such orbits are there so that we will get the number of inequivalent colourings. Now what is that? By Burnside's lemma

$$|C/G| = \frac{1}{\#G}\sum_{\pi \in G} \# \, Fix(\pi)$$

Now if sigma is a generator of the group G, we have the following case. So, we have either identity, $\sigma,\ \sigma^2, \sigma^3\ , \sigma^4, \ldots$.

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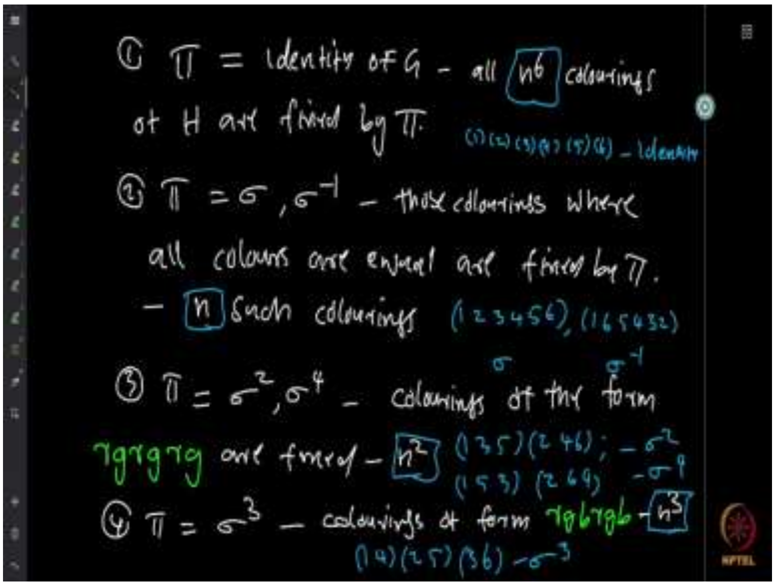

So, these are the generators, so let us look at each case. Now if you are looking at the identity of G, then all the colourings are fixed by π because it does not do anything. So, everything is fixed, so all the n^6 colouring because for every vertex I have n choices, so n^6 colourings of the hexagon are all fixed by π, the identity.

So, the identity permutation is written like this (1)(2)(3)(4)(5)(6) each one is a cycle. Now, on the other hand suppose we have the generator let us say $\sigma = (1\ 2\ 3\ 4\ 5\ 6)$, so which is the rotation by one. 1 going to 2, 2 going to 3, 3 going to 4, 4 going to 5 and 5 going to 6 and 6 going to 1. So, if sigma is a generator, then which kind of colouring sigma fixes?

Because it is rotating, you know the colours in the cyclic fashion, the 2 colourings are equal only if all the colours are the same because the colourings basically move around in circle and if I want these 2 colourings to be the same, I want all the colours in each of the vertex to be the same. Now if each vertex gets the same colour, since I have n colours, I have n different colourings possible where all the vertices get the same colour.

So, σ, for example, gives exactly n such colourings. Now what if you look at the σ^{-1} which is the backward rotation, that is 1 going to 6, 6 going to 5, 5 going to 4, 4 going to 3 and 3 going to 2 and 2 going to 1. That is, $\sigma^{-1} = (1\ 6\ 5\ 4\ \ 3\ 2)$, so that σ^{-1}again the same, so if I just rotate by 1, all the way backwards again I want all the vertices getting the same colour otherwise the colouring will not be fixed by π.

So, for each of these 2, I get n such colouring. Now what if π is σ^2 or σ^4, then what are these permutations if σ is this, then you can verify that σ^2 = (1 3 5) (2 4 6). Now what is this permutation like for example 1 going to 3, 3 going to 5 and 5 going to 1. Similarly, 2 going to 4, 4 going to 6 and 6 going into 2.

If the colourings of this alternate elements are the same, then such colourings are fixed by π because if I move twice. I just need to get the same colour. Similarly, for each element if it moves exactly twice in forward then that colours is the same, then the colouring is okay. So, therefore such colourings will be all fixed by σ^2 but how many are there?

Well, I can choose any 2 colourings for the first two elements then we know that the remaining must be all based on this. So, therefore I have $n \times n = n^2$ possible colourings and in n^2 of this colourings, this σ^2 will be fixing each of them and similarly σ^4 which is (1 5 3)(2 6 4) will also be fixing such colouring.

So, therefore n^2 colourings will be fixed by σ^2 and σ^4 . Now if $\pi = \sigma^3$, then σ^3 is precisely (1 4)(2 5)(3 6). Therefore, once you fix the first three then the remaining 3 are the same. So, therefore I have 3 choices, $n \times n \times n = n^3$ colourings are there and this n^3colouring will be fixed by permutations of the form σ^3.

So, this covers all possible elements, all the 6 elements in the group so therefore, we can now apply Burnside's Lemma.

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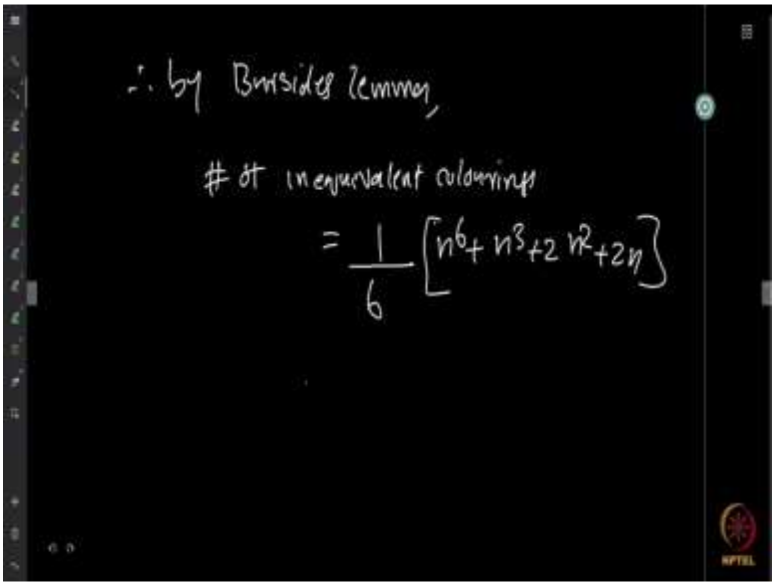

The number of inequivalent colourings = $\frac{1}{6}(n^6 + n^3 + 2n^2 + 2n)$. So, if we are using exactly n colours, we get this. If I am using only one colour, what you will get (1 + 1 + 2 + 2)/6 = 1 and for 2 you will see whatever it is.

So, this is something that we can do by using Burnside's Lemma. But now let us observe something else. So if you look at this count, that we have obtained here. We have $\frac{1}{6}(n^6 + n^3 + 2n^2 + 2n)$. So, let us see what is the exponent of n^6 here. So, n^6 comes from the identity of G and you will see that the identity has exactly 6 cycles in its cycle representation.

So, the identity permutation has exactly six 1-cycles. Now what about this 2n? So, the exponent of n is 1, now how many cycles are there, for example in σ. σ. has only 1 cycle which is with all the 6 elements there is 1 cycle, σ^{-1} also has only 1. So, you will see that the number of cycles in σ. here for example is the exponent of n here, in the count.

For each permutation you can verify that it is the case, for example when you are looking at the σ or σ^4 , σ^2 has 2, 2-cycles so the exponent is 2 for n. σ^4 also exactly the same and σ^3 has exactly 3 cycles. And therefore, we have exactly n^3 colourings.

Now why is this? Can you see why? This is something which is kind of easy once you think about it because let us say that, you are looking at a permutation and this permutation fixes some colourings. Now if a permutation is fixing the colourings, then look at any cycle of the permutation. So, if I look at the cycle of the permutation when you look at the permutation of the product of cycles, you take any factor which is a cycle.

All the elements in this cycle, precisely it is rotating like by 1 or one step or it forms a cyclic rotation. Now if I have this rotation, this rotation will be in this the cyclic rotation, will fix the colouring only if all the elements that corresponding elements which appear in the rotation get the same colour. If 1, 3 and 5 are all getting the same colour only the rotation of 1, 3 and 5, 1 going to 3, 3 going to 5 and 5 going to 1 will be also preserving or fixing the colours.

So, therefore the permutation will fix the colour only if all elements in that particular cycle will get the same look. So, therefore this has exactly n choices because I am using n colours. Now look at the second cycle, the second cycle also the same thing because all the elements in that cycle must be getting the same colour, so again it has n choices.

That is why the number of cycles determines the exponent of n. So, now it is clear, so we can even write it as a theorem, so we will go to that.

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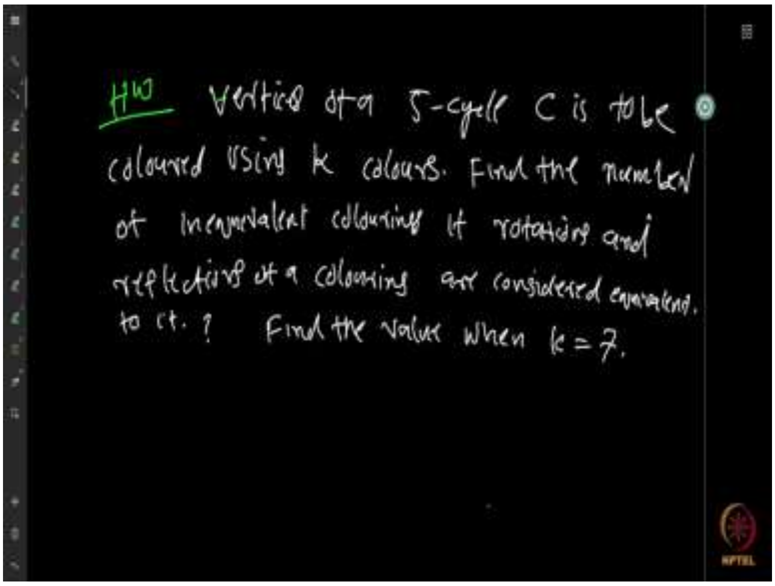

Before that let me set a homework. The vertices of a 5-cycle let us say C is to be coloured using k colours. Find the number of inequivalent colourings if rotations and reflections of colourings are considered equivalent. And once you do this formula again find the value when k is equal to 7, so that is a homework.

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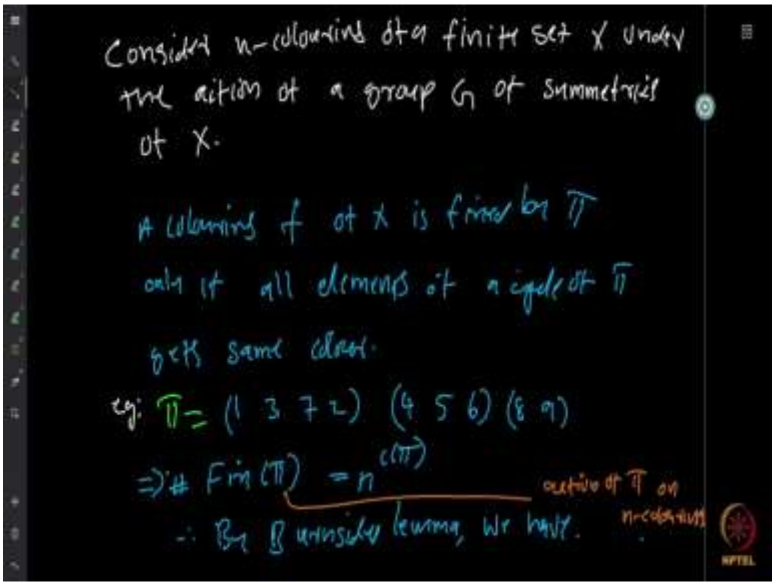

This is something that we already observed that if you look at an n-colouring of a finite set X under the action of a group G of the symmetries of X, then any colouring is fixed by a fixed permutation π, only if all the elements a cycle of π get the same colour, something we observed. So, therefore the cardinality of $Fix(\pi) = n^{c(\pi)}$, where $c(\pi)$ is the number of cycles in π.

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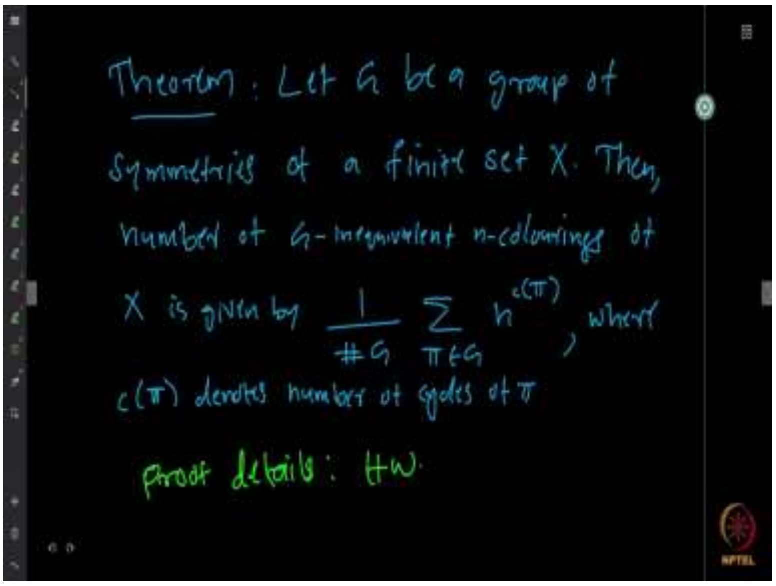

So, therefore by Burnside's Lemma we can rewrite this observation as follows. If G is a group of symmetries of a finite set X and you are looking at the number of G-inequivalent n-colourings of the set X that is given by $\frac{1}{\#G}\sum_{\pi \in G} n^{c(\pi)}$, where $c(\pi)$ denote the number of cycles of π. So, I already explained the proof, so I want you to write it down formally with all the necessary arguments and that is the homework for you.

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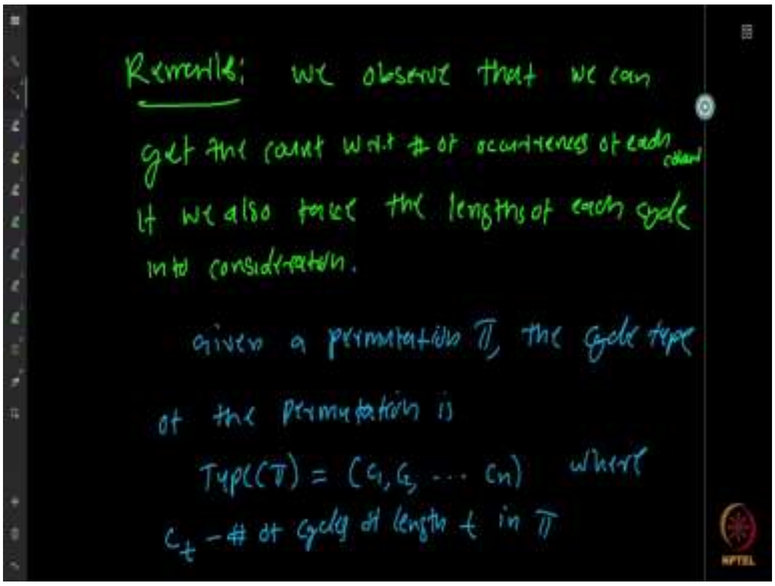

Now, so here is a remark, if you want to get the number of colourings with some additional information like for example you want to count with respect to the number of occurrences of each colour, so I want to say that, I want to count the number of colourings in G-in equivalent colourings, where the red colour appears 3 times, blue appears 5 times and green appears 7 times etcetera.

So, I can do that also, this is something which we can do if you put it into use the observation that we had in the previous result that using the cycle structure of the permutations, we can count this. So, if you include that also into the formula then we should be able to clearly tell how many inequivalent colourings are there where each colour occurs a specified number of times.

Now this becomes an even more clear when you observe that the permutations, the action is by the group G which is a subset of the symmetric group S_X. So, basically the action is by permutation so now the permutations of a colouring. Suppose I take a colouring and permute them whatever your permutations is, it is not going to change the number of times a given colour appears.

If I look at a colouring and I permute the elements then, I am going to get another colouring where the number of times each colour occurs is precisely the same. So, the set of all colourings is now can be divided into based on the number of times each colour occurs and that will be, so the action of the group will be any subgroup that we are looking at, will be always restricted to each of this subclass of the colouring.

So, with that observation we should be able to gain more insights, so that is what we are going to do in Polya's Theorem. So, continuing our observations, so let us write the cycle type of a permutation as the n-tuple, so we are looking at permutations of n elements let us say, then the cycle type of permutation is $Type(\pi) = (C_1, C_2, \ldots, C_n)$, where C_t is the number of cycles of length t in π. So, this could be like C_i could be 0, it is always a non-negative integer and you can look at some examples.