Combinatorics

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Lecture 48

Burnside's Lemma

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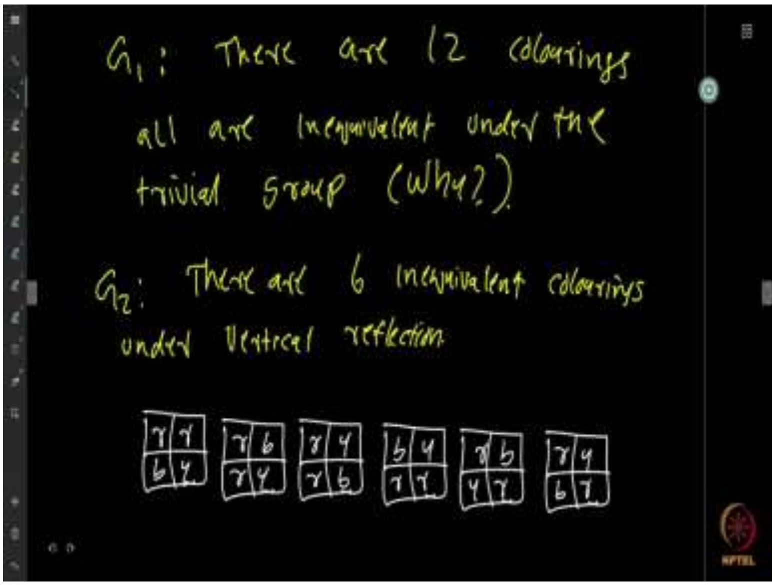

In case the group is G_1. So of course, there are 12 colourings there are 12 colourings when we are using two colours for red so two. So, there are four squares two of them are always going to be coloured red so it does not matter the first red or second red, there is no difference. Therefore, we need to just look at the all possible colourings. Basically, we have four factorial possibilities but two of them are the same by two factors so you have the multiset, this is the first thing that we learned in our course.

Therefore, we have 12 colourings and all are inequivalent under the trivial group. Because we are not allowing any changes to be made so once you fix the 12 different colourings they are all going to be inequivalent they are all different.

So, then G_2 is the group where we allow the vertical reflection. So, under vertical reflection we can see that there are six inequivalent colourings. So, why is this? For example you know under the vertical reflection you have r, r, b, y and it is basically like if you put a reflection and then you have r, r, y, b also.

So, r r b y and r r y b belong to this particular one class. Then similarly, you will see that under vertical reflection this one for example is equivalent to r, r you know here and b and y are interchanged. Similarly, in each of these you can find one another colouring to which this is actually identical under the reflection.

So, under reflection you take the reflection of this whatever you get they are both equivalent. So, you get six inequivalent colourings because these two are equivalent these two are equivalent these two are equivalent these two are equivalent etcetera. So, under the action of G_2 you have exactly six colourings.

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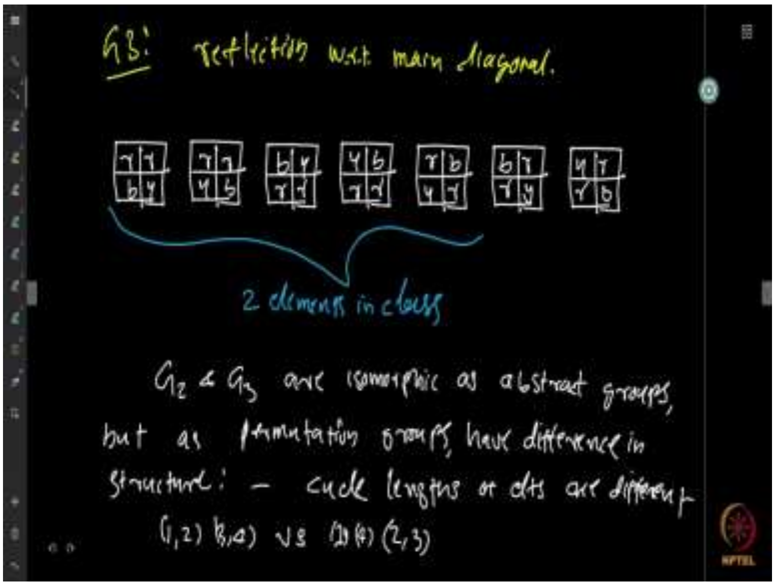

Now, under G_3, so reflection on the main diagonal you will see it is slightly different this is why I said G_2 and G_3 may be isomorphic as abstract sub groups. But, as permutation groups they are different. For example, the action of G_3 here is slightly different.

So, if you look at these colourings, each of these colouring under the reflection on the main diagonal, you will see another colouring so you will see that for example b here and r here these are different colouring. And they are inequivalent again you know y and r are swapped r and y are swapped here and similarly r and b are swapped here again y and b are swapped here.

So, they are all having two elements in the class so you will see that. But, on the other hand this one b, y, r, r under reflection this does not give anything else. It remains the same, so there is only one element in this class. Similarly, if you look at y, b and r, r, again there is only one element in this class.

So, these are different under the reflection with respect to the main diagonal. So therefore, instead of the six inequivalent colouring, we have seven inequivalent colourings here. So,

G_2 and G_3 are isomorphic as abstract groups but as permutation groups have different structure.

And if you think about it you will see that the difference comes from the difference in the cycle lengths of elements. For example, here (1, 2), (3, 4). (1, 2) is a cycle (3, 4) is a cycle there are two cycles of length two on the other hand under the reflection on the diagonal you have two one cycles and one two cycle.

And this is precisely what makes them different. So, they are the difference in their action is precisely because of the difference in the cycle structure. So, we will see more about this when we go further. In fact we are going to make use of precisely this observation and that is the essential part of our polya's theory.

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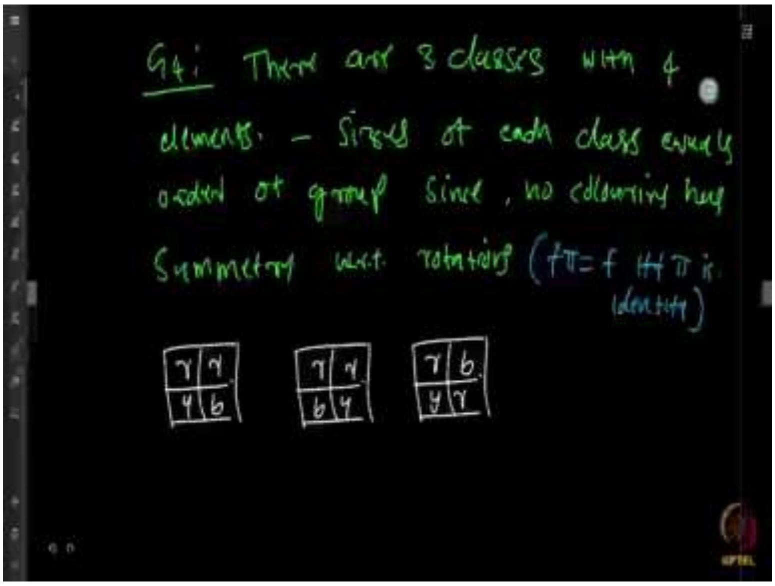

And then we have G_4, the group of rotations of X so all possible rotations and there are three classes with four elements. So, sizes of each class equals the order of the group. So, why is that? For example, what is the order of the group?

It is four; there are four rotations of this identity and then the three other rotations and why they are precisely 4 because you will see that, under rotation no colouring has symmetry with respect to this rotation. So, you take any colouring that we have.

If, after rotation, if it remains, so we have this 12 colouring that we started with we did not draw them but think of this you can take an example and draw them. So, take any of the 12 different colouring not any take all the 12 different colourings and see that if you apply rotation, they are always going to be inequivalent.

So, there are three rotated elements other than the identity, therefore every class has exactly four elements; identity as well as the other three permutations. They will all give different colourings and each of them are equivalent under rotations. So, you will see that if you rotate this you will get three more different colourings if you rotate this you will get another three if you rotate this you will get another three so therefore all the 12 colourings appear.

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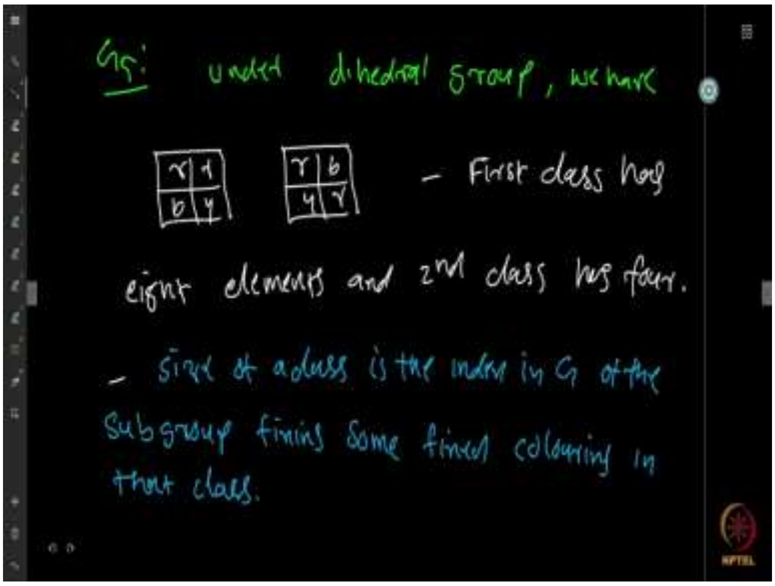

Now G_5 is the dihedral group. So, under the dihedral group we have only two elements so the first class has eight elements and second class has four elements. So, you will see they are different because you know under the reflection this r and r they will not change and therefore you have less number of elements in this class and you can verify that this has eight elements.

So, if you do all possible rotations of reflections you will see that this will lead to eight different objects. Now, another observation that we can make is that the size of a class is the index in G of the subgroup fixing some fixed colouring in that class. So, think of this and try to see why? So, size of class is the subgroup fixing some fixed colouring in that class for any action of a group this is a property that you can see.

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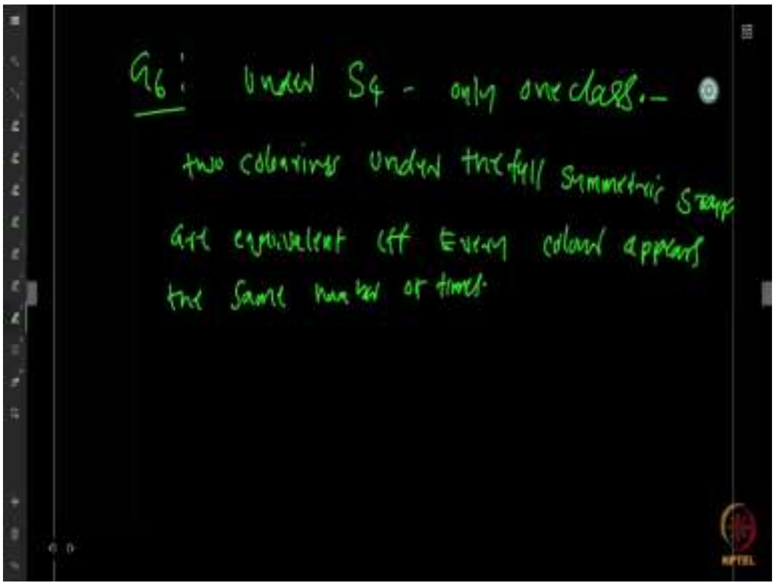

Then you have finally G_6 which is the set of all permutations there is only one class. Because, we are saying that no matter what permutation we take, they are all equivalent. So therefore, by permutation you can always get a colouring from one to other if the number of times each colour appears is the same.

So, if since we started with saying that now we have exactly two red squares and one blue and one yellow square, the number of cells in which red appears is always two and number of times blue appears is always 1, number of times yellow appears is always 1. So, we see that when you take permutations, permutation cannot change the number of times the colour appears.

We are just saying that this cell is mapped to the cell that cell goes to the cell etc. But the colourings remain same. So, the number of times each colour appears is always different. So, that is it. We have these six groups and each group acts on the set X and then you know this action will also give different colourings and then under this action of the permutations we can say that they are basically symmetries that we are looking at for the objects.

And these symmetries will tell you how many different inequivalent colouring are there under this action of symmetries. Now, I forgot to mention what is index of a subgroup? So, index of a subgroup is the number left cosets or equivalently number of right cosets. So, when I said that, the size of a class is the index in G what I said is that if you look at the subgroup of a group that we are looking at, so take a subgroup and then look at what is the number of cosets this subgroup has in G. That is called the index. So, now it should make more sense and then see why this is precisely the number of elements that appears in that class so this should give you a better idea. Now, this is why I said that we should also look at cosets.

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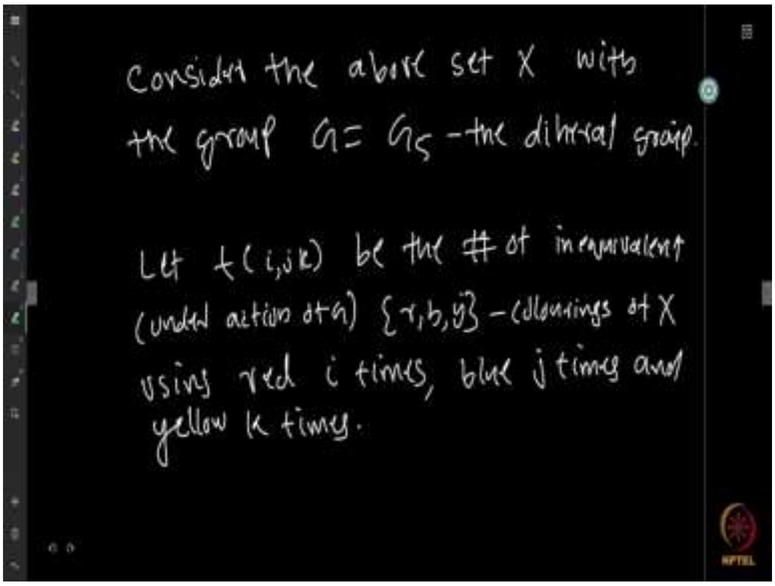

Now, consider the same set that we are looking at where we consider the group which is the dihedral group of all rotations and reflections. Let t(i, j, k)be the number of inequivalent colourings using the colours r b and y of X using the colour red i times colour blue j times and colour yellow k times.

So, we have the set X you have the dihedral group acting on this set and we are looking at the number of inequivalent colourings under the action of the dihedral group where the red colour appears i times precisely, blue colour appears j time precisely and yellow colour appears k times. So, earlier we were looking at 2 red, 1 blue and 1 yellow. Now, we can say that instead of this let us say that we have i times red j times blue and k times yellow. Now, of course you know if X has exactly four elements we should be clear that, i + j + k = 4. Because, we are colouring all the cells and it can only change within this restriction.

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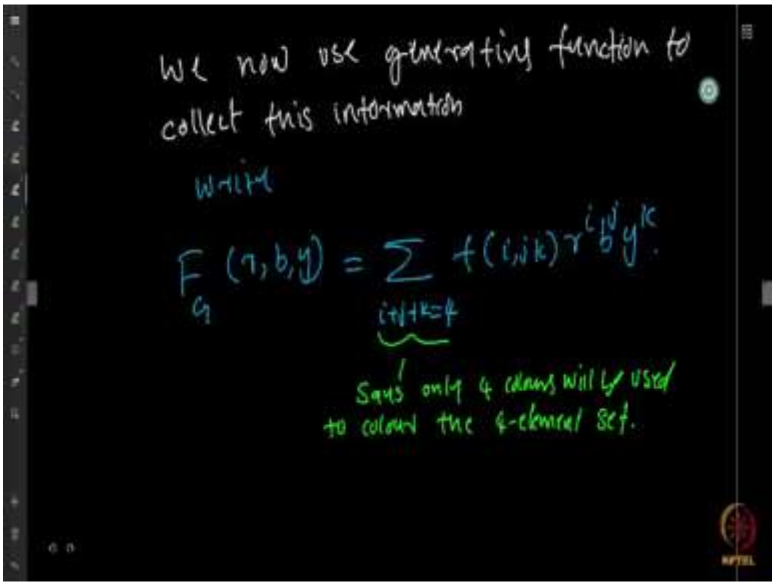

Then now we want to collect the information basically that counts t(i, j, k) for different values into a generating function. So, this is something that we are already familiar with. So let us write

$$F_G(r, b, y) = \sum_{i+kj+k=4} t(i, j, k) r^i b^j y^k$$

So, r, b, y can be considered as variables of this generating function so instead of one variable now we have several variables. So three variables here. If you look at the coefficient of $r^i b^j y^k$ it should precisely tell you how many inequivalent colourings are there under the action of G. So, we are putting it into the form of a generating function this is how we define the generating functions anyway. So, our generating function $F_G(r, b, y)$ is this one.

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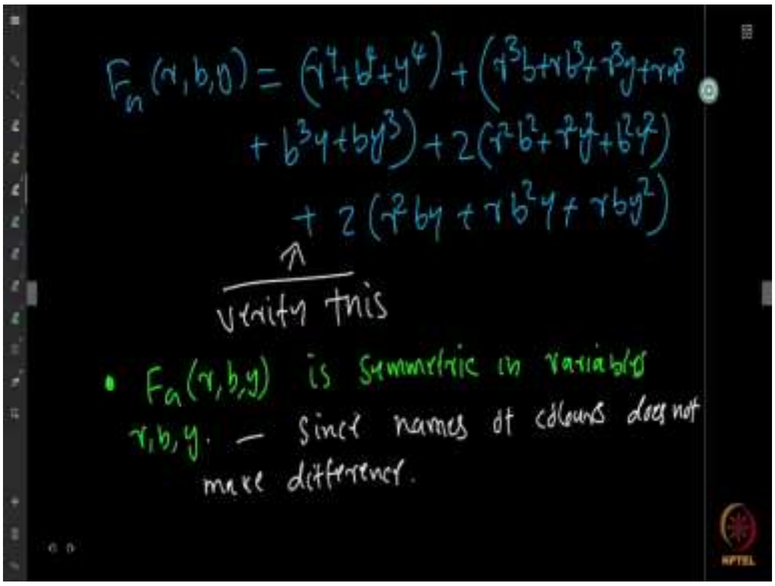

Now, what is the value? So, you work out the details you know looking at each colour appear and you see that how many are there and then apply the dihedral group and see how many inequivalent colourings are there.

$$F_G(r,b,y) = (r^4 + b^4 + y^4) + (r^3b + rb^3 + r^3y + ry^3 + b^3y + by^3) \\ + 2(r^2b^2 + r^2y^2 + b^2y^2) + 2(r^2by + rb^2y + rby^2)$$

So, you will see that this is actually going to be equal to $r^4 + b^4 + y^4$, because there is only one colouring where all the cells are coloured with the red. I mean whatever else you are going to do is going to be the same if you are using only red colour. Similarly, b^4 and y^4 is very clear.

Similarly, $r^3 b$. So all three coloured red and then one is blue of course you have different colourings but then under the action of dihedral group you will see that they are all the same. Because, by rotation you can get this.

Similarly, rb^3, r^3y, $r\,y^3$, b^3y, $b\,y^3$. So, if I am using only two colours one of them three times then you have this. But, on the other hand if you are using colour two times so r for example is appearing two times r^2 and b is appearing two times. Then you will see two different colouring.

So, even under the action of dihedral group you will see that there are two different colourings are possible. And similarly, for r^2y^2, b^2y^2. And then as we checked before in

r^2by, r is appearing twice b is appearing once and y appearing once this is precisely what we calculated in the previous example.

Where we are looking at this we say that there are two in this class G_5 has exactly two. These are the two representatives here. So, you will get two and for all the other symmetric cases rb^2y, rby^2. So, check this I mean it is quite easy and make sure that this is precisely the generating function that we are looking at.

Now, it is easy to observe and also see why that $F_G(r, b, y)$ is symmetric in all its variables r, b and y. Since, the names of colours does not make any difference I mean I can call red as blue and blue as red just renaming. So therefore, it should be symmetric. So, we have this.

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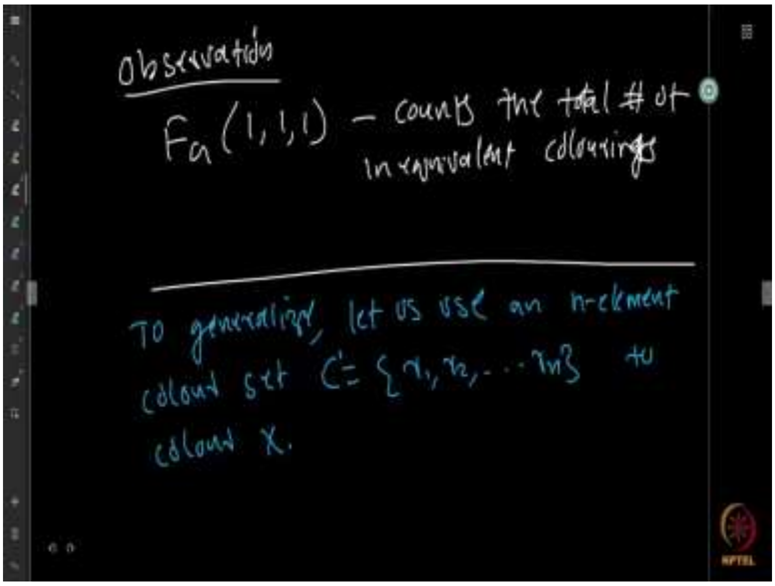

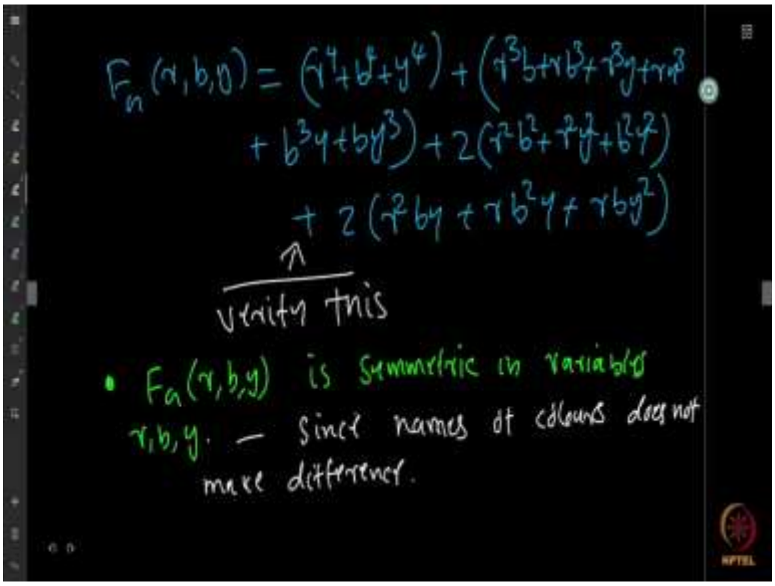

Now, let us look at some things from this so if you look at $F_G(r, b, y)$, I substitute with the value 1 so r becomes 1, b becomes 1 and y becomes 1 and what happens to this right hand side? So you get 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 2 (1 + 1 + 1) + 2(1 + 1 + 1).

So, all together what you will get? So, see what you will get there and you should get the total number of inequivalent colouring. I mean without the restriction on a specific number of times each colour appears you will see that if you are using the colouring using this colours on the four squares you will see that, this is precisely the number of colouring possible.

So, 21 will be your total number of inequivalent colourings so you can verify that. Now, to generalize let us now use an n element colour set instead of the 3-elements set. Let us say that we are using $r_1,\ r_2,\ r_3, \dots, r_n$ to colour the set X. So, we use more colours then what can we see.

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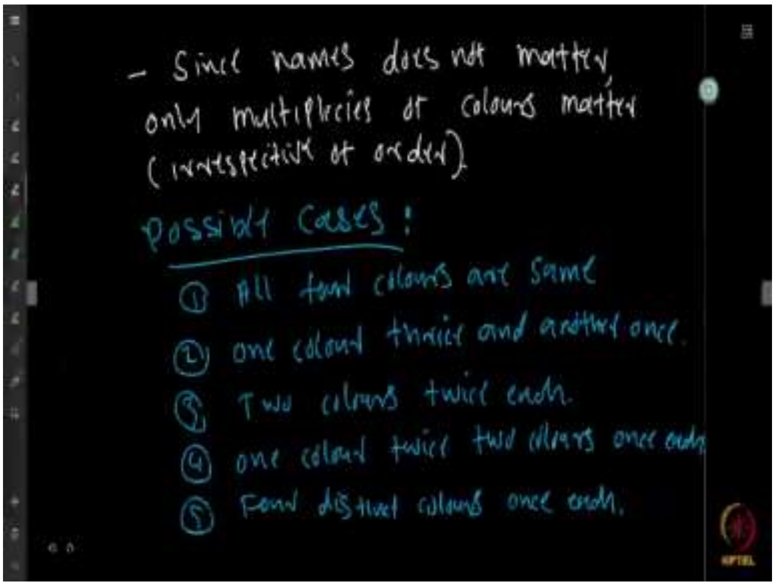

So, the names does not matter only the multiplicities of the colours matter. Multiplicities of the colours (how many times the colour appears) can make the difference. And again, the order of this colour in which this multiplicity occur also does not matter. So, what are the possible cases?

One possible case is that all 4 colours are the same we have r_1 is appearing in all vertices r_2 is appearing all the time or one colour appears thrice and another one appears once this is another possibility. We have only four squares to colour.

Then two squares are given the same colour and another two colours are given the same colour using two different colours. Then one colour appears twice and two colours are appearing only once. Then 4 colours are all 4 distinct colours. So, these are the possible cases and there is nothing else one can easily verify.

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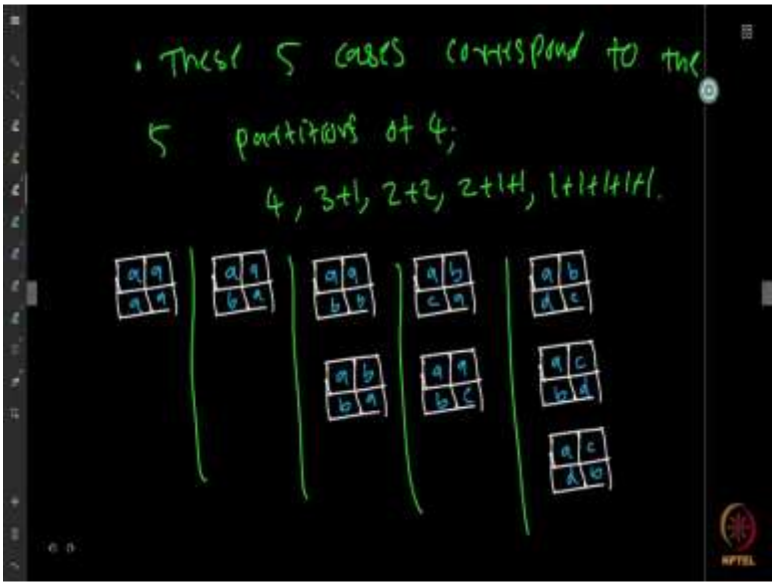

Now, these 5 cases correspond to the 5 partitions of 4 this is something that you should observe which may be not very easy but if you think about this it comes true. That is basically what are the partitions of 4? We have 4 by itself then you have 3 + 1, 2 + 2

Then you have 2 + 1 + 1 and 1 + 1 + 1 + 1. So, this basically corresponds to partitions of integers. If you have the total number of boxes to colour is four and if you have enough colours to colour from all it matters is how many times a colour appears and you know basically the multiplicities of the colours.

And since the order does not matter you will get this as exactly the partitions of four not combinations. So, with respect to the first one you will see that all colours are same a, a, a, a they are all identical then for 3 + 1 you have a, a, a and b then you have a, a, b, b or a and a in the diagonal or b and b in diagonal. They are different then you have a, a in the diagonal b and c are different so which is 2 + 1 + 1 and similarly you have a, a here and b, c again 2 + 1 + 1. Then you have all four different so a b c d, a c b d, or a c d b.

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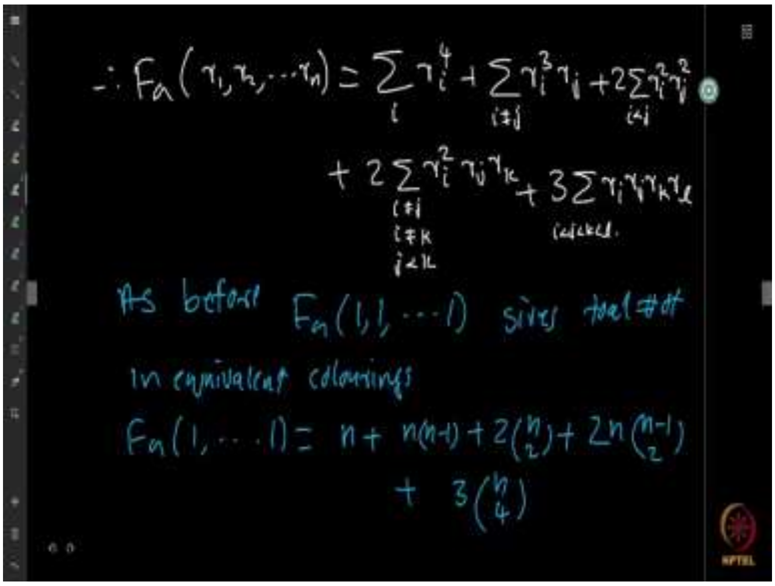

So once, you have this it is easy to find out the generating function. So what is $F_G(r_1, r_2, \dots, r_n)$?

$$F_G(r_1, r^2, \dots, r_n) = \sum_{i} r_i^4 + \sum_{i\neq j} r_i^3 r_j + 2\sum_{i<j} r_i^2 r_j^2 + 2\sum_{\substack{i\neq j\\ i\neq k\\ j<k}} r_i^3 r_j r_k + 3\sum_{i<j<k<l} r_i r_j r_k r_l$$

This is precisely the generating function that we have. And as we saw in the previous case if you substitute for all the variables to be one it will give you the total number of inequivalent colourings.

Where I use exactly n colours. So, you verify this and you can find out the numbers if you want. And what is the number $F_G(1,1,\dots,1)$?

$$F_G(1,1,\dots,1) = n + n(n-1) + 2\binom{n}{2} + 2n\binom{n-1}{2} + 3\binom{n}{4}$$

So, you will see all these things verify all these things as a homework you can think of this and try to verify these things.

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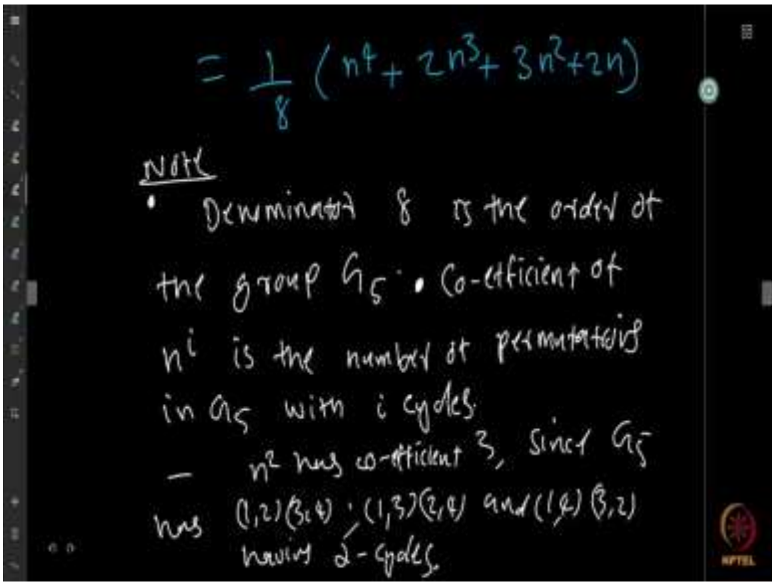

And what is this actually? This is more interesting part for this $F_G\ (1,1,\ldots,1) = \frac{1}{8}(n^4 + 2n^3 + 3n^2 + 2n)$.

And then what we observe from here is that the denominator 8 is actually equal to the order of the dihedral group. So, dihedral group G_5 has exactly eight elements and eight happens to be the denominator and it is not a coincidence we will see later. Now, the coefficient of n^i for example is the number of permutations in G_5 with i cycles.

So, how many permutations are there with 4 cycles there is precisely one. Therefore, I get coefficient of n^4 to be 1. So what are the permutations in G_5.

So, G_5 has this identity then you have (1, 2, 4, 3); (1, 4) (2, 3); (1, 3, 4, 2) and then you have (1, 2) (3, 4); (1, 3) (2, 4); (1) (4) (2, 3); and (2)(3)(1, 4). So, these are the dihedral groups and what we are saying is that, how many permutations are there with exactly three cycles.

So, we want to count that. So we will see that this one has three cycles this one has three cycles, anything else has having three cycles? Nothing. So we have coefficient of this is going to be two coefficient this is two here.

Similarly, you will see that permutations with exactly two cycles will be three. There will be three of them and one cycle there is two of them and that is it. So, this is an observation that we can make. This is one example that I had already written here.

Now, we want to show that this is actually the case for all such situations. So, you will have, the coefficients of these terms are going to be precisely the number of times you can find a permutations or number of permutations in the group that we are looking at having exactly 3 cycles exactly 4 cycles exactly 2 cycles and exactly one cycle.

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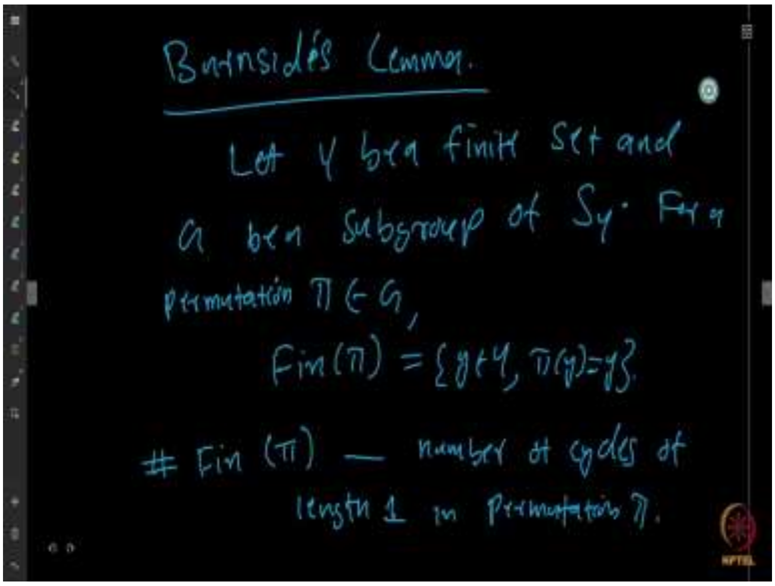

So, this can be generalized that is what it says. So, here we have the Burnside's Lemma Burnside's Lemma says the following. Let Y be a finite set and G be a subgroup of S_Y. Now, for some permutation in $\pi \in$ G, the set $Fix(\pi) = \{y \in Y, \pi(y) = y\}$.

That is $Fix(\pi)$ is the set of elements in Y that are unchanged under the action of π

So, if I take for example, the identity permutation does not change any element. Similarly, we have 1 going to itself 2 go into itself 3 and 4 are interchanged in the dihedral group that we looked at. That is another permutation where 1 and 2 are fixed.

So similarly, given a permutation we can see the set of all elements that are fixed by this permutation. So, that is the $Fix(\pi)$. Now, the cardinality of $Fix(\pi)$ is the number of cycles of length 1 in the permutation π, that is clear.

Because, you know basically the cycle representation of the permutation tells you which are the elements that are mapped to itself these are the unit cycles. When we put a single element in the bracket in the cycle notation and that tells you that these elements are the fixed elements. So, then the cardinality of $Fix(\pi)$ is the number of cycles of length one in the permutation π.

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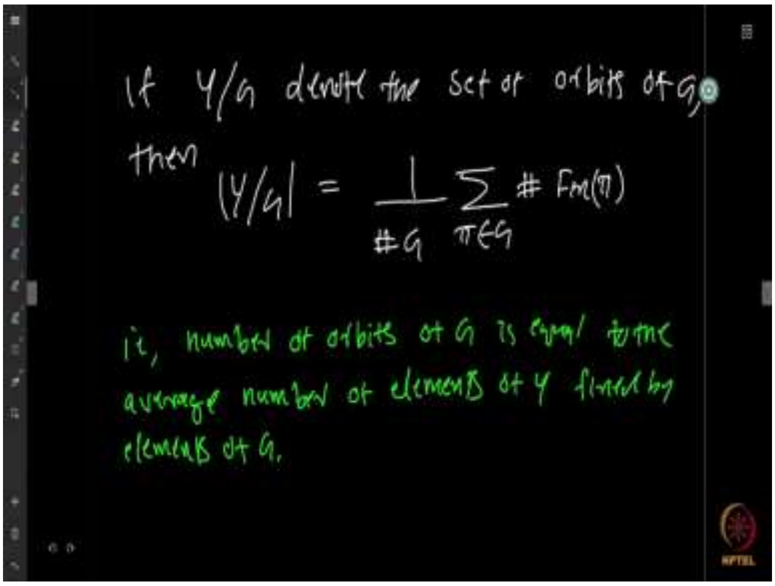

Now, suppose we denote by Y/G as the set of orbits of G. So, what is the orbit of a group? When the group acts on a set it basically moves the elements then we can look at the elements which come under a class.

So, we are saying about the classes. Some elements are basically moved to other elements, mapped to other elements and then this forms a subset. So, this is basically the orbit then you know the cardinality of $Y/G = \frac{1}{\#G}\sum_{\pi \in G} \#Fix(\pi)$.

And the number of orbits of G is equal to the average number of elements of Y fixed by the elements of G. This is what Burnside's lemma says. So we will see a proof for this in the next class.